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Research Article

Comparison of the Rate of Convergence among Picard, Mann, Ishikawa, and Noor Iterations Applied to Quasicontractive Maps

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We provide sufficient conditions for Picard iteration to converge faster than Krasnoselskij, Mann, Ishikawa, or Noor iteration for quasicontractive operators. We also compare the rates of convergence between Krasnoselskij and Mann iterations for Zamfirescu operators.

1. Introduction

Let (X, d) be a complete metric space, and let T be a self-map of X. If T has a unique fixed point, which can be obtained as the limit of the sequence $\{p_n\}$, where $p_n = T^n p_0, p_0$ any point of X, then T is called a Picard operator (see, e.g., [1]), and the iteration defined by $\{p_n\}$ is called Picard iteration.

One of the most general contractive conditions for which a map T is a Picard operator is that of Ćirić [2] (see also [3]). A self-map T is called quasicontractive if it satisfies

$$d(Tx, Ty) \leq \delta \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.1}$$

for each $x, y \in X$, where δ is a real number satisfying $0 \leq \delta < 1$.

Not every map which has a unique fixed point enjoys the Picard property. For example, let $X = [0, 1]$ with the absolute value metric, $T : X \to X$ defined by $Tx = 1 - x$. Then, T has a unique fixed point at $x = 1/2$, but if one chooses as a starting point $x_0 = a$ for any $a \neq 1/2$, then successive function iterations generate the bounded divergent sequence $\{a, 1 - a, a, 1 - a, \ldots\}$.

To obtain fixed points for some maps for which Picard iteration fails, a number of fixed point iteration procedures have been developed. Let X be a Banach space, the corresponding quasicontractive mapping $T : X \to X$ is defined by

$$\|Tx - Ty\| \leq \delta \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}. \tag{1.2}$$

In this paper, we will consider the following four iterations.

Krasnoselskij:

$$\forall v_0 \in X, \quad v_{n+1} = (1 - \lambda)v_n + \lambda T v_n, \quad n \geq 0, \tag{1.3}$$

where $0 < \lambda < 1$.

Mann:

$$\forall u_0 \in X, \quad u_{n+1} = (1 - a_n)u_n + a_n T u_n, \quad n \geq 0, \tag{1.4}$$

where $0 < a_n \leq 1$ for $n \geq 0$, and $\sum_{n=0}^{\infty} a_n = \infty$.

Ishikawa:

$$\begin{gathered} \forall x_0 \in X, \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \quad n \geq 0, \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \geq 0, \end{gathered} \tag{1.5}$$

where $\{a_n\} \subset (0, 1]$, $\{b_n\} \subset [0, 1]$.

Noor:

$$\begin{gathered} \forall w_0 \in X, \\ z_n = (1 - c_n)w_n + c_n T w_n, \quad n \geq 0, \\ y_n = (1 - b_n)w_n + b_n T z_n, \quad n \geq 0, \\ w_{n+1} = (1 - a_n)w_n + a_n T y_n, \quad n \geq 0, \end{gathered} \tag{1.6}$$

where $\{a_n\} \subset (0, 1]$, $\{b_n\}, \{c_n\} \subset [0, 1]$.

Three of these iteration schemes have also been used to obtain fixed points for some Picard maps. Consequently, it is reasonable to try to determine which process converges the fastest.

In this paper, we will discuss this question for the above quasicontractions and for Zamfirescu operators. For this, we will need the following result, which is a special case of the Theorem in [4].

Theorem 1.1. *Let C be any nonempty closed convex subset of a Banach space X, and let T be a quasicontractive self-map of C. Let $\{x_n\}$ be the Ishikawa iteration process defined by* (1.5)*, where each $a_n > 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. then $\{x_n\}$ converges strongly to the fixed point of T.*

2. Results for Quasicontractive Operators

To avoid trivialities, we shall always assume that $p_0 \neq q$, where q denotes the fixed point of the map T.

Let $\{f_n\}$, $\{g_n\}$ be two convergent sequences with the same limit q, then $\{f_n\}$ is said to converge faster than $\{g_n\}$ (see, e.g., [5]) if

$$\lim_{n\to\infty} \frac{\|f_n - q\|}{\|g_n - q\|} = 0. \tag{2.1}$$

Theorem 2.1. *Let E be a Banach space, D a closed convex subset of E, and T a quasicontractive self-map of D, then, for $0 < \lambda < (1-\delta)^2$, Picard iteration converges faster than Krasnoselskij iteration.*

Proof. From Theorem 1 of [2] and (1.2),

$$\begin{aligned} \|p_{n+1} - q\| &= \left\|T^{n+1}p_0 - q\right\| \\ &\leq \frac{\delta^{n+1}}{1-\delta}\|Tp_0 - p_0\| \\ &\leq \frac{\delta^{n+1}}{1-\delta}(\|Tp_0 - Tq\| + \|p_0 - q\|) \\ &\leq \frac{\delta^{n+1}}{1-\delta}(\delta \max\{\|p_0 - q\|, \|p_0 - q\| + \|Tp_0 - Tq\|\} + \|p_0 - q\|) \\ &\leq \frac{\delta^{n+1}}{1-\delta}\left(\delta\left(\|p_0 - q\| + \frac{\delta}{1-\delta}\|p_0 - q\|\right) + \|p_0 - q\|\right) \\ &\leq \frac{\delta^{n+1}}{(1-\delta)^2}\|p_0 - q\|, \end{aligned} \tag{2.2}$$

where q is the fixed point of T.

From (1.3), with $v_0 \neq q$,

$$\begin{aligned} \|v_{n+1} - q\| &\geq (1-\lambda)\|v_n - q\| - \lambda\|Tv_n - Tq\| \\ &\geq \left(1 - \frac{\lambda}{1-\delta}\right)\|v_n - q\| \\ &\geq \cdots \\ &\geq \left(1 - \frac{\lambda}{1-\delta}\right)^{n+1}\|v_0 - q\|. \end{aligned} \tag{2.3}$$

By setting each $\beta_n = 0$ and each $\alpha_n = \lambda$, it follows from Theorem 1.1 that $\{v_n\}$ converges to q.

Therefore,

$$\frac{\|p_{n+1}-q\|}{\|v_{n+1}-q\|} \le \left(\frac{\delta}{1-\delta-\lambda}\right)^{n+1}(1-\delta)^{n-1}\frac{\|p_0-q\|}{\|v_0-q\|} \longrightarrow 0, \tag{2.4}$$

as $n \to \infty$, since $\lambda < (1-\delta)^2$. □

Theorem 2.2. *Let E, D, and T be as in Theorem 2.1. And let $0 < a_n < \theta(1-\delta), b_n, c_n \in [0,1]$ for all $n > 0$.*

(A) *If the constant $0 < \theta < 1-\delta$, then Picard iteration converges faster than Mann iteration.*

(B) *If the constant $0 < \theta < (1-\delta)^2/(1-\delta+\delta^2)$, then Picard iteration converges faster than Ishikawa iteration.*

(C) *If the constant $0 < \theta < (1-\delta)^3/(1-2\delta+2\delta^2)$, then Picard iteration converges faster than Noor iteration.*

Proof. We have the following cases

Case A (Mann Iteration). Using Theorem 1.1 with each $\beta_n = 0$, $\{u_n\}$ converges to q. Using (1.4),

$$\begin{aligned}\|u_{n+1}-q\| &\ge (1-a_n)\|u_n-q\| - a_n\|Tu_n-Tq\| \\ &\ge \left(1-\frac{a_n}{1-\delta}\right)\|u_n-q\| \ge \cdots \ge \prod_{i=0}^{n}\left(1-\frac{a_i}{1-\delta}\right)\|u_0-q\|.\end{aligned} \tag{2.5}$$

Therefore,

$$\frac{\|p_{n+1}-q\|}{\|u_{n+1}-q\|} \le \frac{\delta^{n+1}\|p_0-q\|}{(1-\delta)^2\prod_{i=0}^{n}(1-a_i/(1-\delta))\|u_0-q\|} \longrightarrow 0, \tag{2.6}$$

as $n \to \infty$, since $a_n < \theta(1-\delta)$ for each $n > 0$.

Case B (Ishikawa Iteration). From Theorem 1.1, $\{x_n\}$ converges to q. Using (1.5),

$$\begin{aligned}\|x_{n+1}-q\| &\ge (1-a_n)\|x_n-q\| - a_n\|Ty_n-Tq\| \\ &\ge (1-a_n)\|x_n-q\| - \frac{a_n\delta}{1-\delta}\|y_n-q\| \\ &\ge (1-a_n)\|x_n-q\| - \frac{a_n\delta}{1-\delta}(\|x_n-q\| + b_n\|Tx_n-Tq\|)\end{aligned}$$

$$
\begin{aligned}
&\geq \left(1-a_n-\frac{a_n\delta}{1-\delta}\right)\|x_n-q\|-\frac{a_nb_n\delta^2}{(1-\delta)^2}\|x_n-q\| \\
&\geq \left(1-a_n-\frac{a_n\delta}{1-\delta}-\frac{a_n\delta^2}{(1-\delta)^2}\right)\|x_n-q\| \\
&\geq \cdots \\
&\geq \prod_{i=0}^{n}\left(1-a_i-\frac{a_i\delta}{1-\delta}-\frac{a_i\delta^2}{(1-\delta)^2}\right)\|x_0-q\|.
\end{aligned}
\tag{2.7}
$$

Hence,

$$
\frac{\|p_{n+1}-q\|}{\|x_{n+1}-q\|} \leq \frac{\delta^{n+1}\|p_0-q\|}{(1-\delta)^2\prod_{i=0}^{n}\left(1-a_i-a_i\delta/(1-\delta)-a_i\delta^2/(1-\delta)^2\right)\|x_0-q\|} \longrightarrow 0, \tag{2.8}
$$

as $n \to \infty$, since $a_n < \theta(1-\delta)$ for each $n > 0$.

Case C (Noor Iteration). First we must show that $\{w_n\}$ converges to q. The proof will follow along the lines of that of Theorem 1.1. □

Lemma 2.3. *Define*

$$
\begin{aligned}
A_n &= \{z_i\}_{i=0}^{n} \cup \{y_i\}_{i=0}^{n} \cup \{w_i\}_{i=0}^{n} \cup \{Tz_i\}_{i=0}^{n} \cup \{Ty_i\}_{i=0}^{n} \cup \{Tw_i\}_{i=0}^{n}, \\
\alpha_n &= \operatorname{diam}(A_n), \\
\beta_n &= \max\{\max\{\|w_0-Tw_i\| : 0 \leq i \leq n\}, \max\{\|w_0-Ty_i\| : 0 \leq i \leq n\}, \\
&\qquad \max\{\|w_0-Tz_i\| : 0 \leq i \leq n\}\},
\end{aligned}
\tag{2.9}
$$

then $\{A_n\}$ *is bounded.*

Proof.

Case 1. Suppose that $\alpha_n = \|Tz_i - Tz_j\|$ for some $0 \leq i, j \leq n$, then, from (1.2) and the definition of α_n,

$$
\begin{aligned}
\alpha_n &= \|Tz_i - Tz_j\| \\
&\leq \delta \max\{\|z_i-z_j\|, \|z_i-Tz_i\|, \|z_j-Tz_j\|, \|z_i-Tz_j\|, \|z_j-Tz_i\|\} \\
&\leq \delta\alpha_n,
\end{aligned}
\tag{2.10}
$$

a contradiction, since $\delta < 1$.

Similarly, $\alpha_n \neq \|Ty_i - Ty_j\|$, $\alpha_n \neq \|Tw_i - Tw_j\|$, $\alpha_n \neq \|Tz_i - Ty_j\|$, $\alpha_n \neq \|Tz_i - Tw_j\|$, and $\alpha_n \neq \|Ty_i - Tw_j\|$ for any $0 \leq i, j \leq n$.

Case 2. Suppose that $\alpha_n = \|w_i - w_j\|$, without loss of generality we let $0 \le i < j \le n$. Then, from (1.6),

$$\begin{aligned}\alpha_n &= \left\|w_i - w_j\right\| \\ &\le \left(1 - a_{j-1}\right)\left\|w_i - w_{j-1}\right\| + a_{j-1}\left\|w_i - Ty_{j-1}\right\| \\ &\le \left(1 - a_{j-1}\right)\left\|w_i - w_{j-1}\right\| + a_{j-1}\alpha_n.\end{aligned} \tag{2.11}$$

Hence, $\alpha_n \le \|w_i - w_{j-1}\| \le \alpha_n$, that is, $\alpha_n = \|w_i - w_{j-1}\|$. By induction on j, we obtain $\alpha_n = \|w_i - w_i\| = 0$, a contradiction.

Case 3. Suppose that $\alpha_n = \|w_i - Tw_j\|$ for some $0 \le i, j \le n$. If $i > 0$, then we have, using (1.6),

$$\begin{aligned}\alpha_n &= \left\|w_i - Tw_j\right\| \\ &\le (1 - a_{i-1})\left\|w_{i-1} - Tw_j\right\| + a_{i-1}\left\|Ty_{i-1} - Tw_j\right\| \\ &\le (1 - a_{i-1})\left\|w_{i-1} - Tw_j\right\| + a_{i-1}\alpha_n,\end{aligned} \tag{2.12}$$

which implies that $\alpha_n \le \|w_{i-1} - Tw_j\|$, and by induction on i, we get $\alpha_n = \|w_0 - Tw_j\|$.

Case 4. Suppose that $\alpha_n = \|w_i - z_j\|$ or $\alpha_n = \|z_i - z_j\|, \|y_i - z_j\|, \|z_i - Ty_j\|, \|y_i - y_j\|$ for some $0 \le i, j \le n$, then

$$\begin{aligned}\alpha_n &= \left\|w_i - z_j\right\| \\ &\le \left(1 - c_j\right)\left\|w_i - w_j\right\| + c_j\left\|w_i - Tw_j\right\| \\ &\le \max\left\{\left\|w_i - w_j\right\|, \left\|w_i - Tw_j\right\|\right\}.\end{aligned} \tag{2.13}$$

From Cases 2 and 3, $\|w_i - w_j\| < \alpha_n$, and $\|w_i - Tw_j\| \le \|w_0 - Tw_m\|$ for some $m \le j$, that is, $\alpha_n = \|w_0 - Tw_m\|$. If $\alpha_n = \|z_i - z_j\|$, we obtain that $\alpha_n \le \|w_i - z_j\|$. Therefore; $\alpha_n = \|w_0 - Tw_m\|$, other cases, omitting.

Case 5. Suppose that $\alpha_n = \|w_i - Tz_j\|$ or $\alpha_n = \|z_i - Tz_j\|, \|w_i - y_j\|, \|y_i - Tz_j\|$ for some $0 \le i, j \le n$, then if $i > 0$,

$$\begin{aligned}\alpha_n &= \left\|w_i - Tz_j\right\| \\ &\le (1 - a_{i-1})\left\|w_{i-1} - Tz_j\right\| + a_{i-1}\left\|Ty_{i-1} - Tz_j\right\| \\ &\le (1 - a_{i-1})\left\|w_{i-1} - Tz_j\right\| + a_{i-1}\alpha_n,\end{aligned} \tag{2.14}$$

it leads to $\alpha_n \le \|w_{i-1} - Tz_j\|$. Again by induction on i, we have $\alpha_n = \|w_0 - Tz_j\|$. Similarly, if $\alpha_n = \|z_i - Tz_j\|$ or, $\alpha_n = \|w_i - y_j\|$, we also get $\alpha_n = \|w_0 - Tz_j\|$; other cases, omitting.

Case 6. Suppose that $\alpha_n = \|z_i - Tw_j\|$ or $\alpha_n = \|y_i - Tw_j\|$ for some $0 \leq i, j \leq n$, then, using Case 1,

$$\begin{aligned}\alpha_n &= \left\|z_i - Tw_j\right\| \\ &\leq (1-c_i)\left\|w_i - Tw_j\right\| + c_i\left\|Tw_i - Tw_j\right\| \\ &\leq (1-c_i)\left\|w_i - Tw_j\right\| + c_i\alpha_n,\end{aligned} \tag{2.15}$$

or

$$\begin{aligned}\alpha_n &= \left\|y_i - Tw_j\right\| \\ &\leq (1-b_i)\left\|w_i - Tw_j\right\| + b_i\left\|Tz_i - Tw_j\right\| \\ &\leq (1-b_i)\left\|w_i - Tw_j\right\| + b_i\alpha_n,\end{aligned} \tag{2.16}$$

these imply that $\alpha_n \leq \|w_i - Tw_j\|$. By Case 3, we obtain that $\alpha_n = \|w_0 - Tw_j\|$.

Case 7. Suppose that $\alpha_n = \|w_i - Ty_j\|$ or $\alpha_n = \|y_i - Ty_j\|$ for some $0 \leq i, j \leq n$, then if $i > 0$, using Case 2,

$$\begin{aligned}\alpha_n &= \left\|w_i - Ty_j\right\| \\ &\leq (1-a_{i-1})\left\|w_{i-1} - Ty_j\right\| + a_{i-1}\left\|Ty_{i-1} - Ty_j\right\| \\ &\leq (1-a_{i-1})\left\|w_{i-1} - Ty_j\right\| + a_{i-1}\alpha_n,\end{aligned} \tag{2.17}$$

which implies that $\alpha_n \leq \|w_{i-1} - Ty_j\|$. Using induction on i, we have $\alpha_n = \|w_0 - Ty_j\|$.

In view of the above cases, so we have shown that $\alpha_n = \beta_n$. It remains to show that $\{\alpha_n\}$ is bounded.

Indeed, suppose that $\alpha_n = \|w_0 - Tw_j\|$ for some $0 \leq j \leq n$, then, using Case 1,

$$\begin{aligned}\alpha_n &= \left\|w_0 - Tw_j\right\| \\ &\leq \|w_0 - Tw_0\| + \left\|Tw_0 - Tw_j\right\| \\ &\leq B + \delta\alpha_n,\end{aligned} \tag{2.18}$$

where $B := \|w_0 - Tw_0\|$, then $\alpha_n \leq B/(1-\delta)$.

Similarly, if $\alpha_n = \|w_0 - Ty_j\|$, or $\alpha_n = \|w_0 - Tz_j\|$ we again get $\alpha_n \leq B/(1-\delta)$. Hence, $\{\alpha_n\}$ is bounded, that is, $\{A_n\}$ is bounded. □

Lemma 2.4. *Let E, D, and T be as in Theorem 2.1, and that $\sum a_n = \infty$, then $\{w_n\}$, as defined by* (1.6)*, converges strongly to the unique fixed point q of T.*

Proof. From Ćirić [2], T has a unique fixed point q. For each $n \in \mathbb{N}$, define

$$B_n = \{w_i\}_{i\geq n} \cup \{y_i\}_{i\geq n} \cup \{z_i\}_{i\geq n} \cup \{Tw_i\}_{i\geq n} \cup \{Ty_i\}_{i\geq n} \cup \{Tz_i\}_{i\geq n}. \tag{2.19}$$

Then, using the same proof as that of Lemma 2.3, it can be shown that

$$\begin{aligned} r_n &:= \operatorname{diam}(B_n) \\ &= \max\{\sup\{\|w_n - Tw_j\| : j \geq n\}, \sup\{\|w_n - Ty_j\| : j \geq n\}, \sup\{\|w_n - Tz_j\| : j \geq n\}\}. \end{aligned} \tag{2.20}$$

Using (1.2) and (1.6),

$$\begin{aligned} r_n &= \|w_n - Tw_j\| \\ &\leq (1 - a_{n-1})\|w_{n-1} - Tw_j\| + a_{n-1}\|Tw_{n-1} - Tw_j\| \\ &\leq (1 - a_{n-1})r_{n-1} + a_{n-1}\delta r_{n-1} \\ &= (1 - a_{n-1}(1 - \delta))r_{n-1} \\ &\leq \cdots \\ &\leq r_0 \prod_{i=0}^{n-1}(1 - (1 - \delta)a_i), \end{aligned} \tag{2.21}$$

$\lim r_n = 0$, since $\sum a_n = \infty$.

For any $m, n > 0$ with $j \geq 0$,

$$\begin{aligned} \|w_n - w_m\| &\leq \|w_n - Tw_j\| + \|Tw_j - w_m\| \\ &= r_n + r_m, \end{aligned} \tag{2.22}$$

and $\{w_n\}$ is Cauchy sequence. Since D is closed, there exists $w_\infty \in D$ such that $\lim w_n = w_\infty$. Also, $\lim \|w_n - Tw_n\| = 0$.

Using (1.2),

$$\begin{aligned} \|Tw_\infty - w_\infty\| &= \|Tw_\infty - Tw_n + Tw_n - w_n + w_n - w_\infty\| \\ &= \lim\|Tw_\infty - Tw_n\| \\ &\leq \limsup \delta \max\{\|w_\infty - w_n\|, \|w_\infty - Tw_\infty\|, \|w_n - Tw_n\|, \\ &\qquad \|w_\infty - Tw_n\|, \|w_n - Tw_\infty\|\} \\ &= \delta\|w_\infty - Tw_\infty\|. \end{aligned} \tag{2.23}$$

Since $\delta < 1$, it follows that $w_\infty = Tw_\infty$, and w_∞ is a fixed point of T. But the fixed point is unique. Therefore, $w_\infty = q$.

Returning to the proof of Case C, from (1.6),

$$\begin{aligned}
\|w_{n+1}-q\| &\geq (1-a_n)\|w_n-q\| - a_n\|Ty_n - Tq\| \\
&\geq (1-a_n)\|w_n-q\| - \frac{a_n\delta}{1-\delta}\|y_n-q\| \\
&\geq (1-a_n)\|w_n-q\| - \frac{a_n\delta}{1-\delta}\left(\|w_n-q\| + b_n\|\mathrm{T}z_n - Tq\|\right) \\
&\geq \left(1-a_n-\frac{a_n\delta}{1-\delta}\right)\|w_n-q\| - \frac{a_n\delta^2}{(1-\delta)^2}\|z_n-q\| \\
&\geq \left(1-a_n-\frac{a_n\delta}{1-\delta}-\frac{a_n\delta^2}{(1-\delta)^2}-\frac{a_n\delta^3}{(1-\delta)^3}\right)\|w_n-q\| \\
&\geq \cdots \\
&\geq \prod_{i=0}^{n}\left(1-a_i-\frac{a_i\delta}{1-\delta}-\frac{a_i\delta^2}{(1-\delta)^2}-\frac{a_i\delta^3}{(1-\delta)^3}\right)\|w_0-q\|.
\end{aligned} \tag{2.24}$$

So,

$$\begin{aligned}
&\frac{\|p_{n+1}-q\|}{\|w_{n+1}-q\|} \\
&\quad\leq \frac{\delta^{n+1}\|p_0-q\|}{(1-\delta)^2\prod_{i=0}^{n}\left(1-a_i-a_i\delta/1-\delta-a_i\delta^2/(1-\delta)^2-a_i\delta^3/(1-\delta)^3\right)\|w_0-q\|} \longrightarrow 0,
\end{aligned} \tag{2.25}$$

as $n \to \infty$, since $a_n < \theta(1-\delta)$ for $n > 0$. □

It is not possible to compare the rates of convergence between the Krasnoselskij, Mann, and Noor iterations for quasicontractive maps. However, if one considers Zamfirescu maps, then some comparisons can be made.

3. Zamfirescu Maps

A selfmap T is called a Zamfirescu operator if there exist real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < 1/2$ such that, for each $x, y \in X$ at least one of the following conditions is true:

(1) $d(Tx, Ty) \leq ad(x, y)$,

(2) $d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty))$,

(3) $d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx))$.

In [6] it was shown that the above set of conditions is equivalent to

$$d(Tx,Ty) \le \delta \max\left\{ d(x,y), \frac{[d(x,Tx)+d(y,Ty)]}{2}, \frac{[d(x,Ty)+d(y,Tx)]}{2} \right\}, \tag{3.1}$$

for some $0 < \delta < 1$.

In the following results, we shall use the representation (3.1).

Theorem 3.1. *Let E, and D be as in Theorem 2.1, T a Zamfirescu selfmap of D, then if $a_n < \lambda(1-\delta)\theta/(1+\delta)$ with the constant $0 < \theta < 1+\delta$ for each $n > 0$, Krasnoselskij iteration converges faster than Mann, Ishikawa, or Noor iteration.*

Proof. Since Zamfirescu maps are special cases of quasicontractive maps, from Theorem 1.1 $\{v_n\}, \{x_n\}$, and $\{w_n\}$ converge to the unique fixed point of T, which we will call q.

Using (1.2),

$$\|v_{n+1} - q\| \le (1-\lambda)\|v_n - q\| + \lambda\|Tv_n - q\|. \tag{3.2}$$

Using (3.1),

$$\begin{aligned} \|Tv_n - q\| &\le \delta \max\left\{ \|v_n - q\|, \frac{[\|v_n - Tv_n\| + 0]}{2}, \frac{[\|v_n - q\| + \|q - Tv_n\|]}{2} \right\} \\ &= \delta\|v_n - q\|. \end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned} \|v_{n+1} - q\| &\le (1-\lambda(1-\delta))\|v_n - q\| \\ &\le \cdots \\ &\le (1-\lambda(1-\delta))^{n+1}\|v_0 - q\|, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|u_{n+1} - q\| &\ge (1-a_n(1+\delta))\|u_n - q\| \\ &\ge \cdots \\ &\ge \prod_{i=0}^{n}(1-a_i(1+\delta))\|u_0 - q\|. \end{aligned} \tag{3.5}$$

Thus,

$$\frac{\|v_{n+1} - q\|}{\|u_{n+1} - q\|} \le \frac{(1-\lambda(1-\delta))^{n+1}\|v_0 - q\|}{\prod_{i=0}^{n}(1-a_i(1+\delta))\|u_0 - q\|} \longrightarrow 0, \tag{3.6}$$

as $n \to \infty$, since $a_n < \lambda(1-\delta)$.

The proofs for Ishikawa and Noor iterations are similar. □

Theorem 3.2. *Let E, D, and T be as in Theorem 3.1, then if $\lambda(1+\delta)\theta/(1-\delta) < a_n < 1$ with the constant $0 < \theta < 1-\delta$ for any n, Mann iteration converges faster than Krasnoselskij iteration.*

Proof. Using (1.4) and (3.1),

$$\begin{aligned} \|u_{n+1} - q\| &\leq (1 - a_n)\|u_n - q\| + a_n\|Tu_n - q\| \\ &\leq (1 - a_n(1-\delta))\|u_n - q\| \\ &\leq \cdots \\ &\leq \prod_{i=0}^{n}(1 - a_i(1-\delta))\|u_0 - q\|. \end{aligned} \tag{3.7}$$

And again using (1.3), (3.1), we have

$$\begin{aligned} \|v_{n+1} - q\| &\geq (1-\lambda)\|v_n - q\| - \lambda\|Tv_n - Tq\| \\ &\geq (1 - \lambda(1+\delta))\|v_n - q\| \\ &\geq \cdots \\ &\geq (1 - \lambda(1+\delta))^{n+1}\|u_0 - q\|. \end{aligned} \tag{3.8}$$

Thus,

$$\frac{\|u_{n+1} - q\|}{\|v_{n+1} - q\|} \leq \frac{\prod_{i=0}^{n}(1 - a_i(1-\delta))\|u_0 - q\|}{(1-\lambda(1+\delta))^{n+1}\|v_0 - q\|} \longrightarrow 0, \tag{3.9}$$

as $n \to \infty$, since $\lambda(1+\delta)\theta/(1-\delta) < a_n < 1$. □

It is not possible to compare the rates of convergence for Mann, Ishikawa, and Noor iterations, even for Zamfirescu maps.

Remark 3.3. It has been noted in [7] that the principal result in [8] is incorrect.

Remark 3.4. Krasnoselskij and Mann iterations were developed to obtain fixed point iteration methods which converge for some operators, such as nonexpansive ones, for which Picard iteration fails. Ishikawa iteration was invented to obtain a convergent fixed point iteration procedure for continuous pseudocontractive maps, for which Mann iteration failed. To date, there is no example of any operator that requires Noor iteration; that is, no example of an operator for which Noor iteration converges, but for which neither Mann nor Ishikawa converges.

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