Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 182539, 11 pages
doi:10.1155/2010/182539

Research Article

Existence of Solutions and Algorithm for a System of Variational Inequalities

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Received 13 May 2009; Revised 10 November 2009; Accepted 24 January 2010

Academic Editor: Nanjing Huang

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We obtain some existence results for a system of variational inequalities (for short, denoted by SVI) by Brouwer fixed point theorem. We also establish the existence and uniqueness theorem using the projection technique for the SVI and suggest an iterative algorithm and analysis convergence of the algorithm.

1. Questions under Consideration in This Paper

Suppose that X is a nonempty closed and convex subset of R^n; $F : R^n \to R^n$ is a vector-valued mapping. Variational inequality problem (for short, $\mathrm{VI}(X, F)$) is to find an $x \in X$, such that

$$\langle F(x), u - x\rangle \geq 0, \quad \forall u \in X. \tag{1.1}$$

We denote the solution set for $\mathrm{VI}(X, F)$ by $\mathrm{sol}(X, F)$. In this paper, we suggest and study the following SVI: find $(x, y) \in A \times B$, such that

$$\begin{aligned} &\langle F(x, y), u - x\rangle \geq 0, \quad \forall u \in A, \\ &\langle G(x, y), v - y\rangle \geq 0, \quad \forall v \in B, \end{aligned} \tag{1.2}$$

where $F : A \times B \to R^n, G : A \times B \to R^m$ are vector-valued mappings, and $A \subset R^n, B \subset R^m$. The above SVI can be described as

$$\begin{gathered}\mathrm{VI}(A, F(\cdot, y)),\\ \mathrm{VI}(B, G(x, \cdot)).\end{gathered} \tag{1.3}$$

2. Existence and Uniqueness of Solutions for SVI

In this paper, otherwise specification, R^n is a n-dimensional Euclidean space, for all $x, y \in R^n$, $\langle x, y\rangle$ denotes the inner product between x and y, $\|x\|$ denotes norm of x, that is, $\|x\| = \sqrt{\langle x, x\rangle}$.

In order to obtain our main results, we recall the following definitions and lemmas.

Definition 2.1. Let $X \subset R^n$ be a nonempty subset, and let $F : X \to R^n$ be a vector-valued mapping.

(i) F is said to be monotone if, for all $x, y \in X$, $\langle F(x) - F(y), x - y\rangle \geq 0$.

(ii) F is said to be strictly monotone if, for all $x, y \in X$, $x \neq y$, $\langle F(x) - F(y), x - y\rangle > 0$.

(iii) F is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in X. \tag{2.1}$$

(iv) F is said to be coercive if there exists an $x_0 \in X$ and a constant $R > 0$ such that $\|x_0\| < R$ and

$$\langle F(x), x - x_0\rangle \geq 0, \quad \forall x \in X,\ \|x\| = R. \tag{2.2}$$

(v) F is said to be Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in X. \tag{2.3}$$

Remark 2.2. It is easy to see that

$$F \text{ is strongly monotone} \Longrightarrow \begin{cases} F \textit{ is strictly monotone} \Longrightarrow F \textit{ is monotone} \\ F \textit{ is coercive.} \end{cases} \tag{2.4}$$

Based on the above all kinds of monotonicity, we have the following existence results for $\mathrm{VI}(X, F)$.

Lemma 2.3 (see [1]). *Let $X \subset R^n$ be nonempty compact and convex set, and let $F : X \to R^n$ be continuous mapping. Then* $\mathrm{VI}(X, F)$ *must has solution.*

Lemma 2.4 (see [2]). *Let $X \subset R^n$ be nonempty closed and convex set, and let $F : X \to R^n$ be continuous mapping.*

(i) *If F is strictly monotone, then* $\mathrm{VI}(X, F)$ *has at most one solution,*

(ii) *If F is coercive, then* $\mathrm{VI}(X, F)$ *must has solution,*

(iii) *If F is strongly monotone, then* $\mathrm{VI}(X, F)$ *has a unique solution.*

In order to obtain the existence results for SVI, one needs to study parametric variational inequalities $\mathrm{VI}(A, F(\cdot, y))$ *and* $\mathrm{VI}(B, G(x, \cdot))$ *in SVI.*

Setting $\Omega = \{(x, y) \in R^n \times R^m \mid x \in A,\ y \in B\} \neq \emptyset$ and Ω is the feasible region of SVI, $A \subset R^n$, $B \subset R^m$ are nonempty subset, and $F : A \times B \to R^n$, $G : A \times B \to R^m$ are two continuous mappings. At first, one considers $\mathrm{VI}(B, G(x, \cdot))$, *which is a parametric variational inequality with respect to x in SVI.*

Theorem 2.5. *In* $\mathrm{VI}(B, G(x, \cdot))$, *assume that $A \subset R^n$ and $B \subset R^m$ are two compact and convex sets, $G : A \times B \to R^m$ is continuous, and G is strictly monotone in y. Then, for any given $x \in A$,* $\mathrm{VI}(B, G(x, \cdot))$ *has a unique solution and for all $x \in A$, there exists an implicit function $y = \varphi(x)$ which is the unique solution to* $\mathrm{VI}(B, G(x, \cdot))$. *In addition, the implicit function $y = \varphi(x)$ determined by* $\mathrm{VI}(B, G(x, \cdot))$ *is continuous on A.*

Proof. (i) For any given $x \in A$, since $B \subset R^m$ is compact and convex and G is continuous on $A \times B$, then by Lemma 2.3, parametric variational inequality $\mathrm{VI}(B, G(x, \cdot))$ has solutions. In terms of strict monotonicity of the mapping G in y and Lemma 2.4, we know that $\mathrm{VI}(B, G(x, \cdot))$ has a unique solution. So, for all $x \in A$, the implicit function $y = \varphi(x)$ determined by $\mathrm{VI}(B, G(x, \cdot))$ is well defined.

(ii) We claim that $y = \varphi(x)$ is continuous on A. In fact, for any given $x_0 \in A$, $\{x_n\} \subset A$, $x_n \to x_0$ as $n \to \infty$, by (i), we know that for all n, there exists $y_n \in B$, such that $y_n = \varphi(x_n)$. That is,

$$\langle G(x_n, y_n), y - y_n \rangle \geq 0, \quad \forall y \in B. \tag{2.5}$$

Since $\{y_n\} \subset B$ is bounded, then there exists convergent subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y_0$ as $n \to \infty$, and $y_0 \in B$. In the following, we prove that y_0 is a solution to $\mathrm{VI}(B, G(x, \cdot))$ in x_0. For given $\overline{y} \in B$, there exists sequence $\{\overline{y}_{n_k}\}$ such that $\overline{y}_{n_k} \in B$, $k = 1, 2, \ldots$ and $\overline{y}_{n_k} \to \overline{y}$ as $k \to \infty$ in view of the closedness of B. Letting $y = \overline{y}_{n_k}$ in (2.5), we have

$$\langle G(x_{n_k}, y_{n_k}), \overline{y}_{n_k} - y_{n_k} \rangle \geq 0, \quad k = 1, 2, \ldots, \tag{2.6}$$

that is,

$$\langle G(x_{n_k}, y_{n_k}), \overline{y} - y_{n_k} \rangle \geq \langle G(x_{n_k}, y_{n_k}), \overline{y} - \overline{y}_{n_k} \rangle, \quad k = 1, 2, \ldots, \tag{2.7}$$

observe that G is continuous, and letting $k \to \infty$ in (2.7), we have

$$\langle G(x_0, y_0), \overline{y} - y_0 \rangle \geq 0. \tag{2.8}$$

For $\overline{y} \in B$ is arbitrary, then y_0 is a solution to $\mathrm{VI}(B, G(x, \cdot))$, implying $y_0 = \varphi(x_0)$. In order to explain that φ is continuous at x_0, we only need to know that the sequence $\{y_n\}$ satisfies $y_n \to y_0$ as $n \to \infty$. Let $\{y_{n_l}\}$ be any subsequence of $\{y_n\}$. Since $\{y_{n_l}\}$ is bounded, there exists a subsequence $\{y_{n_{l_k}}\} \subset \{y_{n_l}\}$ such that $y_{n_{l_k}} \to y_0'$ as $k \to \infty$. Using the method appeared in Theorem 2.5, we can show that

$$\langle G(x_0, y_0'), \overline{y} - y_0' \rangle \geq 0, \quad \forall \overline{y} \in B. \tag{2.9}$$

Thus, by the uniqueness of the solution to the problem $\mathrm{VI}(B, G(x_0, \cdot))$, we conclude that $y_0' = y_0$. Since, $\{y_{n_l}\} \subset \{y_n\}$ is arbitrary, we can conclude that $y_n \to y_0$ as $n \to \infty$, which means that implicit function $y = \varphi(x)$ is continuous at x_0. For $x_0 \in A$ is arbitrary, we know that $y = \varphi(x)$ is continuous on A.

From Theorem 2.5, we see that in order to ensure that $y = \varphi(x)(x \in A)$ is well defined, the condition that G is strictly monotone on $A \times B$ is necessary, but the boundedness of B is a strong condition. As usual, B is unbounded (e.g., inequality constraint set $B = \{y \in R^m \mid g(x, y) \geq 0\}$, where $g : R^n \times R^m \to R^l$, is always unbounded). So, we try to weaken the boundedness of B. For this, we introduce the concept uniform coercivity of G in $\mathrm{VI}(B, G(x, \cdot))$.

Definition 2.6. In $\mathrm{VI}(B, G(x, \cdot))$, let $x_0 \in A$; G is said to be uniformly coercive near x_0, if there exists some neighbourhood V of x_0, $y_0 \in B$ and $R > 0$ such that $\|y_0\| < R$ and for all $x \in V$,

$$\langle G(x, y), y - y_0 \rangle \geq 0, \tag{2.10}$$

where $y \in B$ and $\|y\| = R$.

If for each $x \in A$, G is uniformly coercive near x, then G is said to be uniformly coercive on A.

Lemma 2.7. *In* $\mathrm{VI}(B, G(x, \cdot))$*, let* $B \subset R^m$ *be nonempty closed and convex set, and let* G *be uniformly coercive near* $x_0 \in A$*, then there exists some neighbourhood* $V \subset A$ *of* x_0*, such that* $\bigcup_{x \in V} \mathrm{Sol}(B, G(x, \cdot))$ *is bounded set.*

Proof. For given $x_0 \in V$, by the definition of the uniform coercivity of G near x_0, there exists some neighbourhood $V \subset A$ of x_0, $y_0 \in B$, and $R > 0$ such that $\|y_0\| < R$ and for all $x \in V$,

$$\langle G(x, y), y - y_0 \rangle \geq 0, \quad \forall y \in B, \ \|y\| = R. \tag{2.11}$$

Let $B_R := \{y \in R^m \mid \|y\| \leq R\} \cap B$. It is obvious that B_R is a nonempty bounded closed convex subset of R^m. In view of Lemma 2.3, we know that $\mathrm{VI}(B_R, G(x_0, y))$ must have solution. That is, there exists an $\overline{y}_0 \in B_R$ such that

$$\langle G(x_0, \overline{y}_0), y - \overline{y}_0 \rangle \geq 0, \quad \forall y \in R_R. \tag{2.12}$$

Now, we state that $\overline{y}_0 \in \mathrm{Sol}(B, G(x_0, \cdot))$ and $\|\overline{y}_0\| \le R$. In fact, if $\|\overline{y}_0\| < R$, for all $y \in B$, $y \notin B_R$, join $\overline{y}_0$ and y into $\lambda\overline{y}_0 + (1-\lambda)y$ with $0 \le \lambda \le 1$. Then take small enough $\overline{\lambda} > 0$ such that $\overline{y} = \overline{\lambda}y + (1-\overline{\lambda})\overline{y}_0 \in B_R$. Substituting y with $\overline{y}$ in (2.12), we have

$$\langle G(x_0, \overline{y}_0), y - \overline{y}_0 \rangle \ge 0, \tag{2.13}$$

implying that $\overline{y}_0 \in \mathrm{Sol}(B, G(x_0, \cdot))$ and $\|\overline{y}_0\| < R$. On the other hand, if $\|\overline{y}_0\| = R$, substituting y with $\overline{y}_0$ in (2.11), we have

$$\langle G(x_0, \overline{y}_0), \overline{y}_0 - y_0 \rangle \ge 0, \tag{2.14}$$

which, by plus (2.12), we get

$$\langle G(x_0, \overline{y}_0), y - y_0 \rangle \ge 0, \quad \forall y \in B_R, \ \|y_0\| < R. \tag{2.15}$$

For all $y \in B$, $y \notin B_R$, consider the connection of y and y_0; following the same argument, we have that $\overline{y}_0 \in \mathrm{Sol}(B, G(x_0, \cdot))$ and $\|\overline{y}_0\| = R$. Therefore, $\overline{y}_0 \in \mathrm{Sol}(B, G(x_0, \cdot))$ and $\|\overline{y}_0\| \le R$. That is, $\mathrm{Sol}(B, G(x_0, \cdot))$ is bounded. For $x_0 \in V$ is arbitrary, the conclusion holds. This completes the proof. □

If the boundedness of B is replaced by the uniform coercivity of G in Theorem 2.5, then we have the following result.

Theorem 2.8. *In* $\mathrm{VI}(B, G(x, \cdot))$, *let* $B \subset R^m$ *be nonempty closed and convex set, and let* G *be uniformly coercive on* A *with respect to* B *and strict monotone in* y. *Then for each* $x \in A$, $\mathrm{VI}(B, G(x, \cdot))$ *has a unique solution, and for all* $x \in A$, *the implicit function* $y = \varphi(x)$ *determined by* $\mathrm{VI}(B, G(x, \cdot))$ *is continuous on* A.

Proof. (i) For given $x \in A$, by Lemma 2.4 and the coercivity of G on B, we know that $\mathrm{VI}(B, G(x, \cdot))$ has solution. Noting that G is strictly monotone in y, $\mathrm{VI}(B, G(x, \cdot))$ has a unique solution, and so the implicit function $y = \varphi(x)$ is well defined.

(ii) For given $x_0 \in A$, $\{x_n\} \subset A$, satisfying $x_n \to x_0$ as $n \to \infty$. By Lemma 2.7, there exists some neighbourhood $U \subset A$ of x_0 and bounded open set V, such that $\bigcup_{x \in U} \mathrm{Sol}(B, G(x, \cdot)) \subset V$, that is, the solution set of $\mathrm{VI}(B, G(x, \cdot))$ denoted by $\{y \in B \mid y = \varphi(x),\ x \in U\} \subset V$. $\{x_n\} \subset A$ such that $x_n \to x_0$ as $n \to \infty$. Let $\{x_n\} \subset U$ without generality, then $\{y_n\} \subset V$ is bounded; the following argument is similar to Theorem 2.5, so it is omitted, and this completes the proof. □

Set $\Lambda = \{y \in B \mid y = \varphi(x),\ x \in A\}$; it is to see that $\Lambda \subset B$. We will investigate the parametric variational inequality $\mathrm{VI}(A, F(\cdot, y))$ with respect to y in SVI.

Corollary 2.9. *In* $\mathrm{VI}(A, F(\cdot, y))$, *let* $A \subset R^n$, $B \subset R^m$ *be two nonempty compact and convex subsets,* $F : A \times B \to R^n$ *be continuous and strict monotone in* x. *Then for each given* $y \in \Lambda$, $\mathrm{VI}(A, F(\cdot, y))$ *has a unique solution, and for all* $y \in \Lambda$, *the implicit function* $x = \psi(y)$ *determined by* $\mathrm{VI}(A, F(\cdot, y))$ *is continuous on* Λ.

Proof. The conclusion holds directly from Theorem 2.5. □

Lemma 2.10 (see [3, (Brouwer fixed point theorem)]). *Let $X \subset R^n$ be nonempty compact and convex set, and let $T : X \to X$ be continuous. Then there exists an x_0, such that $x_0 = T(x_0)$.*

Theorem 2.11. *In SVI, let $A \subset R^n, B \subset R^m$ be two compact and convex subset, and let $F : A \times B \to R^n$ and $G : A \times B \to R^m$ be two continuous mappings and strict monotone in x and y, respectively. Then SVI has solution.*

Proof. By the given conditions of Theorems 2.11 and 2.5, we know that there exists continuous implicit function $y = \varphi(x)$ $(x \in A)$ determined by parametric variational inequality $\mathrm{VI}(B, G(x,\cdot))$ with respect to x in SVI. Also denoted the range of $y = \varphi(x)$ by Λ. By Corollary 2.9, there exists continuous implicit function $x = \psi(y)$ determined by parametric variational inequality $\mathrm{VI}(A, F(\cdot, y))$ with respect to y in SVI such that for all $y \in \Lambda$, $x = \psi(y)$ is the unique solution to $\mathrm{VI}(A, F(\cdot, y))$. Let $\Phi(x) = \psi(\varphi(x))$, for all $x \in A$. Making use of Brouwer fixed point theorem (Lemma 2.10), we have that there exists $x_0 \in A$, such that $x_0 = \Phi(x_0) = \psi(\varphi(x_0))$. Setting $y_0 = \varphi(x_0)$, by the definitions of φ and ψ, we know that (x_0, y_0) is a solution of SVI. □

Corollary 2.12. *In SVI, let $A \subset R^n$ be nonempty compact and convex subset, let $B \subset R^m$ be nonempty closed and convex subset, let $F : A \times B \to R^n$ and $G : A \times B \to R^m$ be two continuous mappings, and F, G be strict monotone in x and y, respectively. Let G be uniformly coercive on A with respect to B. Then SVI has solution.*

Proof. By Theorem 2.8 and similar argument in Theorem 2.11, our conclusion holds. □

Now, we give the definition of uniformly strong monotonicity, which is stronger condition than the uniformly coercivity.

Definition 2.13. Let $F : R^n \times R^m \to R^n$ be vector-valued mapping; if there exists $\alpha > 0$, such that for all $y \in R^m$,

$$\langle F(x_1, y) - F(x_2, y), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in R^n, \tag{2.16}$$

then F is said to be uniformly strongly monotone in x.

Lemma 2.14. *In* $\mathrm{VI}(A, F(\cdot, y))$*, let F be uniformly strongly monotone, then F is uniformly coercive.*

Proof. For given $y_0 \in B$, we only need to prove that F is uniformly coercive at y_0. Let us consider $\mathrm{VI}(A, F(\cdot, y))$. Assume that $x_0 \in A$ is a solution to $\mathrm{VI}(A, F(\cdot, y))$, and $\|x_0\| < R$, $(R > 0)$. Since F is strongly monotone, then there exists $\alpha > 0$, such that for all $y \in B$,

$$\langle F(x, z) - F(z, y), x - z \rangle \geq \alpha \|x - z\|^2, \quad \forall x, z \in A. \tag{2.17}$$

Letting $z = x_0$, $y = y_0$ in (2.17), we have

$$\langle F(x, y_0), x - x_0 \rangle \geq \alpha \|x - x_0\|^2 + \langle F(x_0, y_0), x - x_0 \rangle, \quad \forall x \in A. \tag{2.18}$$

Noting that $x_0 \in A$ is a solution to $\mathrm{VI}(A, F(\cdot, y))$, we have

$$\langle F(x_0, y_0), x - x_0\rangle \geq 0, \quad \forall x \in A. \tag{2.19}$$

Combining (2.18), we obtain

$$\langle F(x, y_0), x - x_0\rangle \geq 0, \quad \forall x \in A,\ \|x\| = R. \tag{2.20}$$

Since F is continuous, then there exists some neighbourhood U of y_0, such that for all $y \in U$,

$$\langle F(x, y), x - x_0\rangle \geq 0, \quad \forall x \in A,\ \|x\| = R, \tag{2.21}$$

which implies that F is uniform coercive at y_0.

Using the uniformly strong monotonicity of F, we can obtain the following existence result for SVI under the condition that A is a nonempty closed and convex subset of R^n. □

Corollary 2.15. *In SVI, let $A \subset R^n$ be a nonempty closed and convex set, let $B \subset R^m$ be a compact and convex set, let F, G be two continuous mapping, let F be uniformly strongly monotone in x, and let G be strictly monotone in y. Then SVI has solution.*

Proof. By Lemma 2.14 and Corollary 2.12, it is easy to see that the conclusion holds. □

Corollary 2.16. *In SVI, Let A and B be nonempty closed and convex subsets, let F, G be two continuous mappings, and let F be uniformly strongly monotone in x, and let G be uniformly coercive and strict monotone in y. Then SVI has solution.*

Furthermore, If F and G are Lipschitz continuous in x and y, respectively, one can obtain the following existence and uniqueness result for SVI.

Theorem 2.17. *In SVI, let A, B be nonempty compact and convex subsets, let F be uniformly strongly monotone with constant $\alpha_1 > 0$ and Lipschitz continuous with Lipschitz constant $\beta_1 > 0$ in x, and Lipschitz continuous with constant $\gamma_1 > 0$ in y, and let G be uniformly strongly monotone with constant $\alpha_2 > 0$ and Lipschitz continuous with constant $\beta_2 > 0$ in y, and Lipschitz continuous with constant $\gamma_2 > 0$ in x. If there exists constants $\rho_1, \rho_2 > 0$ such that*

$$\max\left\{\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2} + \rho_2\gamma_2, \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} + \rho_1\gamma_1\right\} < 1, \tag{2.22}$$

then SVI has a unique solution.

In order to prove Theorem 2.17, we need the following lemma.

Lemma 2.18 (see [4]). *In SVI, let A, B be nonempty closed and convex subsets, and let F, G be two continuous mappings. SVI has solution (x, y) if and only if (x, y) satisfies*

$$\begin{aligned} x &= P_A(x - \rho_1 F(x, y)), \\ y &= P_B(y - \rho_2 G(x, y)). \end{aligned} \tag{2.23}$$

where $P_A(\cdot), P_B(\cdot)$ denote the projection from R^n and R^m to A and B, respectively; furthermore, projection operator is nonexpansive and $\rho_1, \rho_2 > 0$ are constants.

The Proof of Theorem 2.17

For arbitrary given constant $\rho_1, \rho_2 > 0$, define $T_{\rho_1} : A \times B \to A$ and $T_{\rho_2} : A \times B \to B$ by

$$\begin{aligned} T_{\rho_1}(x, y) &= P_A(x - \rho_1 F(x, y)), \\ T_{\rho_2}(x, y) &= P_B(y - \rho_2 G(x, y)), \quad \forall (x, y) \in A \times B. \end{aligned} \tag{2.24}$$

For any $(x_1, y_1), (x_2, y_2) \in A \times B$, it follows from (2.24) and Lemma 2.18 that

$$\begin{aligned} &\|T_{\rho_1}(x_1, y_1) - T_{\rho_1}(x_2, y_2)\| \\ &\quad \le \|x_1 - x_2 - \rho_1(F(x_1, y_1) - F(x_2, y_2))\| \\ &\quad \le \|x_1 - x_2 - \rho_1(F(x_1, y_1) - F(x_2, y_1))\| + \rho_1\|F(x_2, y_1) - F(x_2, y_2)\| \\ &\quad \le \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2}\|x_1 - x_2\| + \rho_1\gamma_1\|y_1 - y_2\|. \end{aligned} \tag{2.25}$$

We have used the strong monotonicity and Lipschitz continuity of F in x and Lipschitz continuity of F in y. Similarly, we have

$$\begin{aligned} &\|T_{\rho_2}(x_1, y_1) - T_{\rho_2}(x_2, y_2)\| \\ &\quad \le \|y_1 - y_2 - \rho_2(G(x_1, y_1) - G(x_2, y_2))\| \\ &\quad \le \|y_1 - y_2 - \rho_2(G(x_1, y_1) - G(x_2, y_1))\| + \rho_2\|G(x_2, y_1) - G(x_2, y_2)\| \\ &\quad \le \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}\|y_1 - y_2\| + \rho_2\gamma_2\|x_1 - x_2\|. \end{aligned} \tag{2.26}$$

It follows from (2.25) and (2.26) that

$$\begin{aligned} &\|T_{\rho_1}(x_1, y_1) - T_{\rho_1}(x_2, y_2)\| + \|T_{\rho_2}(x_1, y_1) - T_{\rho_2}(x_2, y_2)\| \\ &\quad \le \left(\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2} + \rho_2\gamma_2\right)\|x_1 - x_2\| + \left(\sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} + \rho_1\gamma_1\right)\|y_1 - y_2\| \\ &\quad \le k(\|x_1 - x_2\| + \|y_1 - y_2\|), \end{aligned} \tag{2.27}$$

where

$$k = \max\left\{\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2} + \rho_2\gamma_2, \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} + \rho_1\gamma_1\right\}. \tag{2.28}$$

Define $\|\cdot\|_1$ on $A \times B$ by

$$\|(x, y)\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in A \times B. \tag{2.29}$$

It is easy to see that $(A \times B, \|\cdot\|_1)$ is a Banach space. For any given $\rho_1, \rho_2 > 0$, define $S_{\rho_1,\rho_2} : A \times B \to A \times B$ by

$$S_{\rho_1,\rho_2}(x,y) = (T_{\rho_1}(x,y), T_{\rho_2}(x,y)), \quad \forall (x,y) \in A \times B. \tag{2.30}$$

By assumption, we know that $0 < k < 1$. It follows from (2.27) that

$$\|S_{\rho_1,\rho_2}(x_1,y_1) - S_{\rho_1,\rho_2}(x_2,y_2)\|_1 \leq k\|(x_1,y_1) - (x_2,y_2)\|, \tag{2.31}$$

which implies that $S_{\rho_1,\rho_2} : A \times B \to A \times B$ is a contraction operator. Hence, there exists a unique $(x^*, y^*) \in A \times B$, such that

$$S_{\rho_1,\rho_2}(x^*,y^*) = (x^*,y^*). \tag{2.32}$$

That is,

$$\begin{aligned} x^* &= P_A(x^* - \rho_1 F(x^*,y^*)), \\ y^* &= P_B(y^* - \rho_2 G(x^*,y^*)). \end{aligned} \tag{2.33}$$

By Lemma 2.18, (x^*, y^*) is the unique solution of SVI. □

3. Iterative Algorithm and Convergence

In this section, we will construct an iterative algorithm for approximating the unique solution of SVI and discuss the convergence analysis of the algorithm.

Lemma 3.1 (see, [5]). *Let $\{c_n\}$ and $\{k_n\}$ be two real sequence of nonnegative numbers that satisfy the following conditions.*

(i) $0 \leq k_n < 1$, $n = 0, 1, 2, \ldots$ *and* $\limsup_n k_n < 1$,

(ii) $c_{n+1} \leq k_n c_n$, $n = 0, 1, 2, \ldots$.

Then, c_n converges to 0 as $n \to \infty$.

Algorithm 3.2. Let A, B, F, G, ρ_1, and ρ_2 be the same as in Theorem 2.17. For any given $(x_0, y_0) \in A \times B$, define iterative sequence $\{(x_n, y_n)\}$ by

$$\begin{aligned} x_{n+1} &= a_n x_n + (1 - a_n) P_A(x_n - \rho_1 F(x_n, y_n)), \quad n = 0, 1, 2, \ldots, \\ y_{n+1} &= a_n y_n + (1 - a_n) P_B(x_n - \rho_2 G(x_n, y_n)), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.1}$$

where

$$0 \leq a_n < 1, \quad \limsup_n a_n < 1. \tag{3.2}$$

Theorem 3.3. *Let A, B, F, G, ρ_1, and ρ_2 be the same as in Theorem 2.17. Assume that all the conditions of Theorem 2.17 hold. Then, (x_n, y_n) generated by Algorithm 3.2 converges to the unique solution (x^*, y^*) of SVI and there exists $d \in [0,1)$, such that*

$$\|x_n - x^*\| + \|y_n - y^*\| \leq d^n(\|x_0 - x^*\| + \|y_0 - y^*\|), \quad \forall n \geq 0. \tag{3.3}$$

Proof. By Theorem 2.17, SVI admits a unique solution (x^*, y^*). It follows from Lemma 2.18 that

$$\begin{aligned} x^* &= a_n x^* + (1-a_n)P_A(x^* - \rho_1 F(x^*, y^*)), \\ y^* &= a_n y^* + (1-a_n)P_B(y^* - \rho_2 G(x^*, y^*)). \end{aligned} \tag{3.4}$$

It follows from (3.1) and (3.4) that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq a_n\|x_n - x^*\| + (1-a_n)\|P_A(x_n - \rho_1 F(x_n, y_n)) - P_A(x^* - \rho_1 F(x^*, y^*))\| \\ &\leq a_n\|x_n - x^*\| + (1-a_n)\|x_n - x^* + \rho_1(F(x_n, y_n) - F(x^*, y^*))\| \\ &\leq a_n\|x_n - x^*\| + (1-a_n)\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2}\|x_n - x^*\| + (1-a_n)\rho_1\gamma_1\|y_n - y^*\|, \\ \|y_{n+1} - y^*\| &\leq a_n\|y_n - y^*\| + (1-a_n)\|P_B(y_n - \rho_2 G(x_n, y_n)) - P_B(y^* - \rho_2 G(x^*, y^*))\| \\ &\leq a_n\|y_n - y^*\| + (1-a_n)\|y_n - y^* + \rho_2(G(x_n, y_n) - G(x^*, y^*))\| \\ &\leq a_n\|y_n - y^*\| + (1-a_n)\sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}\|y_n - y^*\| + (1-a_n)\rho_2\gamma_2\|x_n - x^*\|. \end{aligned} \tag{3.5}$$

By (3.5), we get

$$\begin{aligned} \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| &\leq a_n(\|x_n - x^*\| + \|y_n - y^*\|) + (1-a_n)k(\|x_n - x^*\| + \|y_n - y^*\|) \\ &= (k + (1-k)a_n)(\|x_n - x^*\| + \|y_n - y^*\|), \end{aligned} \tag{3.6}$$

where $0 \leq k < 1$ is defined by

$$k = \max\left\{\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2} + \rho_2\gamma_2, \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} + \rho_1\gamma_1\right\}. \tag{3.7}$$

Set

$$c_n = \|x_n - x^*\| + \|y_n - y^*\|, \qquad k_n = k + (1-k)a_n. \tag{3.8}$$

Then (3.6) can be rewritten as

$$c_{n+1} \leq k_n c_n, \quad n = 0, 1, 2, \ldots. \tag{3.9}$$

By (3.2), we know that $\limsup_n k_n < 1$. It follows from Lemma 3.1 that $0 \le k_n \le d < 1$ and that

$$\|x_n - x^*\| + \|y_n - y^*\| \le d^n \left(\|x_0 - x^*\| + \|y_0 - y^*\|\right), \quad \forall n \ge 0 \tag{3.10}$$

for all $n \ge 0$. Therefore, (x_n, y_n) converges geometrically to the unique solution (x^*, y^*) of SVI. This completes the proof. □

Acknowledgment

The authors are grateful to the referee for his/her valuable suggestions and comments that help to present the paper in the present form. This work was supported by the Doctoral Initiating Foundation of Liaoning Province (20071097).

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