Research Article

On Equivalence of Some Iterations Convergence for Quasi-Contraction Maps in Convex Metric Spaces

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We show the equivalence of the convergence of Picard and Krasnoselskij, Mann, and Ishikawa iterations for the quasi-contraction mappings in convex metric spaces.

1. Introduction

Let (E, d) be a complete metric space and $I = [0, 1]$. Denote $E^2 = E \times E$, $I^2 = I \times I$. A continuous mapping $W : E^2 \times I^2 \to E$ is said to be a convex structure on E [1] if for all $u, z_1, z_2 \in E$, $\lambda_1, \lambda_2 \in I$ with $\lambda_1 + \lambda_2 = 1$ such that

$$d(u, W(z_1, z_2; \lambda_1, \lambda_2)) \leq \lambda_1 d(u, z_1) + \lambda_2 d(u, z_2); \tag{1.1}$$

$$W(z_1, z_2; 1, 0) = z_1, \qquad W(z_1, z_2; 0, 1) = z_2. \tag{1.2}$$

If (E, d) satisfies the conditions of convex structure, then (E, d) is called convex metric space that is denoted as (E, d, W).

In the following part, we will consider a few iteration sequences in convex metric space (E, d, W). Suppose that T is a self-map of E.

Picard iteration is as follows:

$$\forall p_0 \in E, \quad p_{n+1} = T p_n = T^{n+1} p_0, \quad n \geq 0. \tag{1.3}$$

Krasnoselskij iteration is as follows:

$$\forall v_0 \in E, \quad v_{n+1} = W(v_n, Tv_n; 1 - \lambda, \lambda), \quad n \geq 0, \tag{1.4}$$

where $\lambda \in [0, 1]$.

Mann iteration is as follows:

$$\forall u_0 \in E, \quad u_{n+1} = W(u_n, Tu_n; 1 - a_n, a_n), \quad n \geq 0, \tag{1.5}$$

where $a_n \in [0, 1]$.

Ishikawa iteration is as follows:

$$\forall x_0 \in E,$$
$$x_{n+1} = W(x_n, Ty_n; 1 - a_n, a_n), \quad n \geq 0, \tag{1.6}$$
$$y_n = W(x_n, Tx_n; 1 - b_n, b_n), \quad n \geq 0,$$

where $a_n, b_n \in [0, 1]$ for all $n \geq 0$.

A mapping $T : E \to E$ is called contractive if there exists $L \in (0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y), \tag{1.7}$$

for all $x, y \in E$.

The map T is called Kannan mapping [2] if there exists $b \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \tag{1.8}$$

for all $x, y \in E$.

A similar definition of mapping is due to the work Chatterjea [3] (that is called Chatterjea mapping), if there exists $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \tag{1.9}$$

for all $x, y \in E$.

Combining above three definitions, Zamfirescu [4] showed the following result.

Theorem 1.1. *Let (E, d) be a complete metric space and $T : E \to E$ a mapping for which there exist the real numbers a, b, and c satisfying $a \in (0, 1)$, $b, c \in (0, 1/2)$ such that, for any pair $x, y \in E$, at least one of the following conditions holds:*

(z1) $d(Tx, Ty) \leq ad(x, y)$;

(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;

(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point, and the Picard iteration converges to fixed point. This class mapping is called Zamfirescu mapping.

In 1974, Ćirić [5] introduced one of the most general contraction mappings and obtained that the unique fixed point can be approximated by Picard iteration. This mapping is called quasi-contractive if there exists $\delta \in (0, 1)$ such that

$$d(Tx, Ty) \leq \delta \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \qquad (1.10)$$

for any $x, y \in E$.

Clearly, every quasi-contraction mapping is the most general of above mappings.

Later on, in 1992, Xu [6] proved that Ishikawa iteration can also be used to approximate the fixed points of quasi-contraction mappings in real Banach spaces.

Theorem 1.2. *Let C be any nonempty closed convex subset of a Banach space X and $T : C \to C$ a quasi-contraction mapping. Suppose that $\alpha_n > 0$ for all n and $\sum \alpha_n = \infty$. Then the Ishikawa iteration sequence $\{x_n\}$ defined by (1)–(3) converges strongly to the unique fixed point x^* of T.*

In this paper, we will show the equivalence of the convergence of Picard and Krasnoselskij, Mann, and Ishikawa iterations for the quasi-contraction mappings in convex metric spaces.

Lemma 1.3. *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality*

$$\rho_{n+1} \leq (1 - \theta_n)\rho_n + \sigma_n, \quad n \geq 0, \qquad (1.11)$$

where $\theta_n \in (0, 1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, and $\sigma_n / \theta_n \to 0$ as $n \to \infty$. Then $\rho_n \to 0$ as $n \to \infty$, (see [7]).

2. Results for Quasi-Contraction Mappings

Theorem 2.1. *Let (E, d, W) be a convex metric space, $T : E \to E$ a quasi-contraction mapping with $F(T) \neq \emptyset$. Suppose that $\{p_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are defined by the iterative processes (1.3) and (1.4), respectively. Then, the following two assertions are equivalent:*

(i) *Picard iteration (1.3) converges strongly to the unique fixed point $q \in F(T)$;*

(ii) *Krasnoselskij iteration (1.4) converges strongly to the unique fixed point $q \in F(T)$.*

Proof. First, we show (i) $\Rightarrow$ (ii), that is, $d(p_n, q) \to 0$ as $n \to \infty \Rightarrow d(v_n, q) \to$ as $n \to \infty$.

From (1.3), (1.4), and (1.1), we can get

$$
\begin{aligned}
d(v_{n+1}, p_{n+1}) &= d(W(v_n, Tv_n; 1 - \lambda, \lambda), Tp_n) \\
&\leq (1 - \lambda)d(v_n, Tp_n) + \lambda d(Tv_n, Tp_n) \\
&\leq (1 - \lambda)d(v_n, p_n) + (1 - \lambda)d(p_n, Tp_n) + \lambda d(Tv_n, Tp_n) \\
&\leq (1 - \lambda)d(v_n, p_n) + \frac{1 - \lambda}{1 - \delta}d(p_n, q) + \lambda d(Tv_n, Tp_n).
\end{aligned}
\qquad (2.1)
$$

Next, we consider $d(Tv_n, Tp_n)$. Using (1.10) with $x = p_n$, $y = v_n$, to obtain

$$d(Tv_n, Tp_n) \leq \delta \cdot \max\{d(v_n, p_n), d(v_n, Tv_n), d(p_n, Tp_n), d(v_n, Tp_n), d(p_n, Tv_n)\}. \qquad (2.2)$$

Set

$$A_n = \{v_i\}_{i=0}^n \cup \{p_i\}_{i=0}^n \cup \{Tp_i\}_{i=0}^n \cup \{Tv_i\}_{i=0}^n, \qquad \gamma_n = \mathrm{diam}\{A_n\}. \qquad (2.3)$$

Then $\{A_n\}$ is bounded. Without loss of generality, we let $\gamma_n > 0$ for each n. Indeed, we will show this conclusion from the some following cases.

Case 1. Let $\gamma_n = d(Tp_i, Tv_j)$ for some $0 \leq i, j \leq n$. Then, from (1.10) and the above γ_n, we have

$$
\begin{aligned}
\gamma_n &= d(Tp_i, Tv_j) \\
&\leq \delta \cdot \max\{d(p_i, v_j), d(v_j, Tv_j), d(p_i, Tp_i), d(v_j, Tp_i), d(p_i, Tv_j)\} \\
&\leq \delta\gamma_n < \gamma_n,
\end{aligned} \qquad (2.4)
$$

and it leads to a contradiction. Thus, $\gamma_n \neq d(Tp_i, Tv_j)$. Similarity to $\gamma_n = d(Tp_i, Tp_j)$ or $\gamma_n = d(Tv_i, Tv_j)$ is also impossible.

Case 2. Let $\gamma_n = d(p_i, v_j)$ for some $0 \leq i, j \leq n$.

   (i) If $j = 0$, then $\gamma_n = d(p_i, v_0)$.
   (ii) If $j \geq 1$, $i = 0$, then, from (1.4) and (1.1)

$$
\begin{aligned}
\gamma_n &= d(p_0, v_j) \\
&= d(p_0, W(v_{j-1}, Tv_{j-1}; 1 - \lambda, \lambda)) \\
&\leq (1 - \lambda)d(p_0, v_{j-1}) + \lambda d(p_0, Tv_{j-1}) \\
&\leq (1 - \lambda)d(p_0, v_{j-1}) + \lambda\gamma_n,
\end{aligned} \qquad (2.5)
$$

that is, $\gamma_n = d(p_0, v_{j-1})$. By induction on j, we can obtain $\gamma_n = d(p_0, v_0)$.
   (iii) If $j \geq 1$, $i \geq 1$, from (1.4) and (1.1)

$$
\begin{aligned}
\gamma_n &= d(p_i, v_j) \\
&= d(p_i, W(v_{j-1}, Tv_{j-1}; 1 - \lambda, \lambda)) \\
&\leq (1 - \lambda)d(p_i, v_{j-1}) + \lambda d(p_i, Tv_{j-1}) \\
&\leq (1 - \lambda)d(p_i, v_{j-1}) + \lambda\gamma_n,
\end{aligned} \qquad (2.6)
$$

it implies that $\gamma_n = d(p_i, v_{j-1})$. By induction on j, we can get $\gamma_n = d(p_i, v_0)$.

Case 3. Let $\gamma_n = d(v_i, v_j)$ for some $0 \le i, j \le n$. Without loss of generality, we set $0 \le i < j \le n$. Then, from (1.4), (1.1)

$$
\begin{aligned}
\gamma_n &= d(v_i, v_j) \\
&\le d(v_i, W(v_{j-1}, Tv_{j-1}; 1 - \lambda, \lambda)) \\
&\le (1 - \lambda) d(v_i, v_{j-1}) + \lambda d(v_i, Tv_{j-1}) \\
&\le (1 - \lambda) d(v_i, v_{j-1}) + \lambda \gamma_n,
\end{aligned}
\tag{2.7}
$$

it implies that $\gamma_n = d(v_i, v_{j-1})$, and by induction on j, we may get $\gamma_n = d(v_i, v_i) = 0$, which is a contradiction.

Case 4. Let $\gamma_n = d(v_i, Tp_j)$ for some $0 \le i, j \le n$.

   (i) If $i = 0$, then $\gamma_n = d(v_0, Tp_j)$.

   (ii) If $i \ge 1$, from (1.4), (1.1), then

$$
\begin{aligned}
\gamma_n &= d(v_i, Tp_j) \\
&\le d(W(v_{i-1}, Tv_{i-1}; 1 - \lambda, \lambda), Tp_j) \\
&\le (1 - \lambda) d(v_{i-1}, Tp_j) + \lambda d(Tv_{i-1}, Tp_j) \\
&\le (1 - \lambda) d(v_{i-1}, Tp_j) \\
&\quad + \lambda \delta \cdot \max\{d(v_{i-1}, p_j), d(v_{i-1}, Tv_{i-1}), d(p_j, Tp_j), d(v_{i-1}, Tp_j), d(p_j, Tv_{i-1})\} \\
&\le (1 - \lambda) d(v_{i-1}, Tp_j) + \lambda \delta \gamma_n \\
&\le (1 - \lambda) d(v_{i-1}, Tp_j) + \lambda \gamma_n,
\end{aligned}
\tag{2.8}
$$

it implies that $\gamma_n = d(v_{i-1}, Tp_j)$ and by induction on i, then $\gamma_n = d(v_0, Tp_j)$.

Case 5. Let $\gamma_n = d(p_i, Tv_j)$ for some $0 \le i, j \le n$.

   (i) If $i = 0$, then $\gamma_n = d(p_0, Tv_j)$.

   (ii) If $i \ge 1$, then, from (1.3) and (1.10)

$$
\begin{aligned}
\gamma_n &= d(p_i, Tv_j) \\
&\le d(Tp_{i-1}, Tv_j) \\
&\le \lambda \delta \cdot \max\{d(p_{i-1}, v_j), d(p_{i-1}, Tp_{i-1}), d(v_j, Tv_j), d(p_{i-1}, Tv_j), d(v_j, Tp_{i-1})\} \\
&\le \lambda \delta \gamma_n,
\end{aligned}
\tag{2.9}
$$

this is a contradiction.

Case 6. let $\gamma_n = d(v_i, Tv_j)$ for some $0 \leq i, j \leq n$.

    (i) If $i = 0$, then $\gamma_n = d(v_0, Tv_j)$.

    (ii) If $i \geq 1$, then, from (1.4) and (1.10)

$$
\begin{aligned}
\gamma_n = d(v_i, Tv_j) & \\
& \leq d\big(W(v_{i-1}, Tv_{i-1}; 1-\lambda, \lambda), Tv_j\big) \\
& \leq (1-\lambda)d(v_{i-1}, Tv_j) + \lambda d(Tv_{i-1}, Tv_j) \\
& \leq (1-\lambda)d(v_{i-1}, Tv_j) \\
& \quad + \lambda\delta \cdot \max\{d(v_{i-1}, v_j), d(v_{i-1}, Tv_{i-1}), d(v_j, Tv_j), d(v_{i-1}, Tv_j), d(v_j, Tv_{i-1})\} \\
& \leq (1-\lambda)d(v_{i-1}, Tv_j) + \lambda\delta\gamma_n,
\end{aligned}
\tag{2.10}
$$

it implies that $\gamma_n = d(v_0, Tv_j)$.

Case 7. Let $\gamma_n = d(p_i, p_j)$ for some $0 \leq i, j \leq n$.

    (i) If $i = 0, \quad j > 0$, then $\gamma_n = d(p_0, p_j)$.

    (ii) If $i, j \geq 1$, then, from (1.3), (1.10)

$$
\begin{aligned}
\gamma_n = d(p_i, p_j) & \\
& \leq d(Tp_{i-1}, Tp_{j-1}) \\
& \leq \delta \cdot \max\Big\{d(p_{i-1}, p_{j-1}), d(p_{i-1}, Tp_{i-1}), d(p_{j-1}, Tp_{j-1}), d(p_{i-1}, Tp_{j-1}), d\big(p_{j-1}, Tp_{i-1}\big)\Big\} \\
& \leq \delta\gamma_n,
\end{aligned}
\tag{2.11}
$$

it is a contradiction.

Case 8. let $\gamma_n = d(p_i, Tp_j)$ for some $0 \leq i, j \leq n$.

    (i) If $i = 0$, then $\gamma_n = d(p_0, Tp_j)$.

    (ii) If $i \geq 1$, then, from (1.3) and (1.10)

$$
\begin{aligned}
\gamma_n = d(p_i, Tp_j) & \\
& \leq d(Tp_{i-1}, Tp_j) \\
& \leq \delta \cdot \max\{d(p_{i-1}, p_j), d(p_{i-1}, Tp_{i-1}), d(p_j, Tp_j), d(p_{i-1}, Tp_j), d(p_j, Tp_{i-1})\} \\
& \leq \delta\gamma_n,
\end{aligned}
\tag{2.12}
$$

which is a contradiction.

Set

$$
\begin{aligned}
\eta_n = \max\{&\max\{d(p_i, v_0) : 0 < i \le n\}, \max\{d(v_0, Tp_i) : 0 < i \le n\}, \\
&\max\{d(v_0, Tv_i) : 0 < i \le n\}, \max\{d(p_0, p_i) : 0 < i \le n\}, \\
&\max\{d(p_0, Tv_i) : 0 < i \le n\}, \max\{d(p_0, Tp_i) : 0 < i \le n\}, M\},
\end{aligned}
\tag{2.13}
$$

where $M = \max\{d(p_0, v_0), d(v_0, Tp_0), d(v_0, Tv_0), d(p_0, Tv_0), d(p_0, Tp_0)\}$.

In view of the above cases, then $\gamma_n = \eta_n$, and we obtain that $\{\gamma_n\}$ is bounded. Indeed, suppose that $\gamma_n = d(p_i, v_0)$ for some $0 < i \le n$. Then,

$$
\begin{aligned}
\gamma_n &= d(p_i, v_0) \\
&\le d(p_i, Tv_0) + d(Tv_0, v_0) \\
&= d(Tp_{i-1}, Tv_0) + d(Tv_0, v_0) \\
&\le d(Tp_{i-1}, Tv_0) + d(Tv_0, v_0) \\
&\le \delta \cdot \max\{d(p_{i-1}, v_0), d(v_0, Tv_0), d(p_{i-1}, Tp_{i-1}), d(v_0, Tp_{i-1}), d(p_{i-1}, Tv_0)\} \\
&\quad + d(Tv_0, v_0) \\
&\le \delta\gamma_n + d(Tv_0, v_0),
\end{aligned}
\tag{2.14}
$$

which implies that $\gamma_n \le (1/(1-\delta))d(Tv_0, v_0)$. Similarly, if $\gamma_n = d(v_0, Tp_i)$ or $\gamma_n = d(v_0, Tv_i)$, we also obtain $\gamma_n \le (1/(1-\delta))d(Tv_0, v_0)$.

On the other hand, suppose that $\gamma_n = d(p_0, p_i)$ for some $0 < i \le n$. Then,

$$
\begin{aligned}
\gamma_n &= d(p_0, p_i) \\
&\le d(p_0, Tp_0) + d(Tp_0, Tp_{i-1}) \\
&\le d(p_0, Tp_0) \\
&\quad + \delta \cdot \max\{d(p_0, p_{i-1}), d(p_0, Tp_0), d(p_{i-1}, Tp_{i-1}), d(p_0, Tp_{i-1}), d(p_{i-1}, Tp_0)\} \\
&\le d(p_0, Tp_0) + \delta\gamma_n,
\end{aligned}
\tag{2.15}
$$

which implies that $\gamma_n \le (1/(1-\delta))d(Tp_0, p_0)$. Similarly, if $\gamma_n = d(p_0, Tv_i)$ or $\gamma_n = d(p_0, Tp_i)$, we also obtain $\gamma_n \le (1/(1-\delta))d(Tp_0, p_0)$. Therefore, from the above results, we obtain that $\gamma_n \le (1/(1-\delta))M$, that is, $\{A_n\}$ is bounded.

For each $n \in \mathbb{N}$, define

$$
B_n = \{v_i\}_{i \ge n} \cup \{p_i\}_{i \ge n} \cup \{Tp_i\}_{i \ge n} \cup \{Tv_i\}_{i \ge n}, \qquad R_n = \mathrm{diam}(B_n).
\tag{2.16}
$$

Then, using the same proof above, it can be shown that

$$R_n = \text{diam}(B_n) = \max\{\sup\{d(p_i, v_n) : i \geq n\}, \sup\{d(v_n, Tp_i) : i \geq n\},$$
$$\sup\{d(p_n, Tv_i) : i \geq n\}, \sup\{d(v_n, Tv_i) : i \geq n\}, \quad (2.17)$$
$$\sup\{d(p_n, p_i) : i > n\}, \sup\{d(p_n, Tp_i) : i \geq n\}\}.$$

If $R_n = \sup\{d(p_i, v_n) : i \geq n\}$, and using (1.1) and (1.4), then

$$
\begin{aligned}
R_n &= \sup_{i \geq n} d(p_i, v_n) \\
&= \sup_{i \geq n} d(p_i, W(v_{n-1}, Tv_{n-1}; 1-\lambda, \lambda)) \\
&\leq \sup_{i \geq n}\{(1-\lambda)d(p_i, v_{n-1}) + \lambda d(p_i, Tv_{n-1})\} \\
&\leq \sup_{i \geq n}\{(1-\lambda)R_{n-1} + \lambda d(Tp_{i-1}, Tv_{n-1})\} \\
&\leq \sup_{i \geq n}\{(1-\lambda)R_{n-1} + \lambda\delta \\
&\qquad \cdot \max\{d(p_{i-1}, v_{n-1}), d(v_{n-1}, Tv_{n-1}), d(p_{i-1}, Tp_{i-1}), d(v_{n-1}, Tp_{i-1}), d(p_{i-1}, Tv_{n-1})\}\} \\
&\leq (1-\lambda)R_{n-1} + \lambda\delta R_{n-1} \\
&= (1 - \lambda(1-\delta))R_{n-1} \\
&\leq \cdots \\
&\leq (1 - \lambda(1-\delta))^n R_0 \\
&\longrightarrow 0
\end{aligned}
$$
$$(2.18)$$

as $n \to \infty$. Since $d(Tv_n, Tp_n) \leq R_n$, hence $d(Tv_n, Tp_n) \to 0$ as $n \to \infty$. Similarly, if $R_n = \sup\{d(v_n, Tp_i) : i \geq n\}$ or $R_n = \sup\{d(v_n, Tv_i) : i \geq n\}$, $R_n = \sup\{d(p_n, Tv_i) : i \geq n\}$, $R_n = \sup\{d(v_n, Tv_i) : i \geq n\}$, $R_n = \sup\{d(p_n, p_i) : i > n\}$, $R_n = \sup\{d(p_n, Tp_i) : i \geq n\}$, we may obtain the similar results. Therefore, from (2.1), we get

$$d(v_{n+1}, p_{n+1}) \leq (1-\lambda)d(v_n, p_n) + \sigma_n, \quad (2.19)$$

where $\sigma_n = ((1-\lambda)/(1-\delta))d(p_n, q) + \lambda d(Tv_n, Tp_n)$.

In (2.19), set $\rho_n = d(v_n, p_n)$. Then (2.19) is as follows:

$$\rho_{n+1} \leq (1-\lambda)\rho_n + \sigma_n. \quad (2.20)$$

By Lemma 1.3, we have $d(v_n, p_n) \to 0$ as $n \to \infty$. From the inequality $0 \leq d(v_n, q) \leq d(v_n, p_n) + d(p_n, q)$, we have $\lim_{n\to\infty} d(v_n, q) = 0$.

Conversely, we will prove that (ii) $\Rightarrow$ (i). If $\lambda = 1$, then $v_{n+1} = W(v_n, Tv_n; 0, 1) = Tv_n$ is Picard iteration. □

Theorem 2.2. *Let $(E, d, W), T, F(T)$ be as in Theorem 2.1. Suppose that $\{u_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}$ are defined by the iterative processes (1.5) and (1.6), respectively, and $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} a_n = \infty$. Then, the following two assertions are equivalent:*

(i) *Mann iteration (1.5) converges strongly to the unique fixed point $q \in F(T)$;*

(ii) *Ishikawa iteration (1.6) converges strongly to the unique fixed point $q \in F(T)$.*

Proof. If the Ishikawa iteration (1.6) converges strongly to q, then setting $b_n = 0$, for all $n \geq 0$, in (1.6), we can get the convergence of Mann iteration (1.5). Conversely, we will show that (i) $\Rightarrow$ (ii). Letting $\lim_{n\to\infty} d(u_n, q) = 0$, we want to prove $\lim_{n\to\infty} d(x_n, q) = 0$.

From (1.5) and (1.6),

$$
\begin{aligned}
d(x_{n+1}, &u_{n+1}) \\
&= d(W(x_n, Ty_n; 1 - a_n, a_n), W(u_n, Tu_n; 1 - a_n, a_n)) \\
&\leq (1 - a_n)d(x_n, W(u_n, Tu_n; 1 - a_n, a_n)) + a_n d(Ty_n, W(u_n, Tu_n; 1 - a_n, a_n)) \\
&\leq (1 - a_n)^2 d(x_n, u_n) + a_n(1 - a_n)d(x_n, Tu_n) \\
&\quad + (1 - a_n)a_n d(Ty_n, u_n) + a_n^2 d(Ty_n, Tu_n) \\
&\leq (1 - a_n)^2 d(x_n, u_n) + a_n(1 - a_n)d(x_n, u_n) + a_n(1 - a_n)d(u_n, Tu_n) \\
&\quad + (1 - a_n)a_n d(Ty_n, Tu_n) + (1 - a_n)a_n d(Tu_n, u_n) + a_n^2 d(Ty_n, Tu_n) \\
&= (1 - a_n)d(x_n, u_n) + 2a_n(1 - a_n)d(u_n, Tu_n) + a_n d(Ty_n, Tu_n) \\
&\leq (1 - a_n)d(x_n, u_n) + 2a_n(1 - a_n)d(u_n, q) \\
&\quad + 2a_n(1 - a_n)d(Tu_n, Tq) + a_n d(Ty_n, Tu_n) \\
&\leq (1 - a_n)d(x_n, u_n) + 2a_n \frac{1 - a_n}{1 - \delta} d(u_n, q) + a_n d(Ty_n, Tu_n).
\end{aligned}
\tag{2.21}
$$

Using (1.10) with $x = y_n, \quad y = u_n$, to obtain

$$
d(Ty_n, Tu_n) \leq \delta \cdot \max\{d(y_n, u_n), d(u_n, Tu_n), d(y_n, Ty_n), d(y_n, Tu_n), d(u_n, Ty_n)\}, \tag{2.22}
$$

set

$$
\begin{aligned}
A_{nn} = \{u_i\}_{i=0}^n &\cup \{y_i\}_{i=0}^n \cup \{x_i\}_{i=0}^n \cup \{Tu_i\}_{i=0}^n \cup \{Ty_i\}_{i=0}^n \cup \{Tx_i\}_{i=0}^n, \\
&\gamma_{nn} = \operatorname{diam}(A_{nn}).
\end{aligned}
\tag{2.23}
$$

Applying the similar proof methods of Theorem 2.1, we obtain that $\{A_{nn}\}$ is also bounded. The other proof is the same as that of Theorem 2.1 and is here omitted. $\qquad \square$

Acknowledgments

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