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Research Article

Existence and Approximation of Fixed Points for Set-Valued Mappings

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Taking into account possibly inexact data, we study both existence and approximation of fixed points for certain set-valued mappings of contractive type. More precisely, we study the existence of convergent iterations in the presence of computational errors for two classes of set-valued mappings. The first class comprises certain mappings of contractive type, while the second one contains mappings satisfying a Caristi-type condition.

1. Introduction

The study of the convergence of iterations of mappings of contractive type has been an important topic in Nonlinear Functional Analysis since Banach's seminal paper [1] on the existence of a unique fixed point for a strict contraction [2–5]. Banach's celebrated theorem also yields convergence of iterates to the unique fixed point. During the last fifty years or so, many developments have taken place in this area. Interesting results have also been obtained regarding set-valued mappings, where the situation is more difficult and less understood. See, for example, [5–12] and the references cited therein. As already mentioned above, one of the methods used for proving the classical Banach theorem is to show the convergence of Picard iterations, which holds for any initial point. In the case of set-valued mappings, we do not have convergence of all trajectories of the dynamical system induced by the given mapping. Convergent trajectories have to be constructed in a special way. For instance, in [7], if at the moment $t = 0, 1, \dots$ we have reached a point x_t, then we choose an element of $T(x_t)$ (here T is the given mapping) such that x_{t+1} approximates the best approximation of x_t from $T(x_t)$. Since our mapping acts on a general complete metric space, we cannot, in general, choose x_{t+1} as the best approximation of x_t by elements of $T(x_t)$. Instead, we require x_{t+1} to approximate the best approximation up to a positive number ϵ_t, such that the sequence $\{\epsilon_t\}_{t=0}^{\infty}$ is summable. This method allowed Nadler [7] to obtain the existence of a fixed point of a strictly contractive set-valued mapping and the authors of [6] to obtain more general results. In view of this state of affairs, it is important to study convergence of the iterates of set-valued mappings in the presence of errors.

In this paper, we study the existence of convergent iterations in the presence of computational errors for two classes of set-valued mappings. The first class comprises certain mappings of contractive type, while the second one contains mappings satisfying a Caristi-type condition.

As we have already mentioned, the existence of a convergent iterative sequence for set-valued strict contractions was established by Nadler [7]. For a more general class of mappings satisfying a certain contractive condition, this was proved in [11]. In the present paper, we show that the existence result of [11] still holds even when possible computational errors are taken into account (see Theorems 2.2–2.4 below).

In Section 3, we obtain certain results regarding set-valued mappings satisfying a Caristi-type condition which complement the results in [6]. There we establish the existence of a fixed point for such mappings assuming that their graphs are closed. Here we first show that a set-valued mapping satisfies a Caristi-type condition if and only if there exists an iterative sequence $\{x_i\}_{i=1}^{\infty}$ such that the sum of the distances between x_i and x_{i+1}, when i runs from zero to infinity, is finite. Then we prove an analog of the Caristi-type result in [6], replacing the closedness of the graph of the mapping with a lower semicontinuity assumption as in Caristi's original theorem [13].

2. Set-Valued Mappings of Contractive Type

Let (X,ρ) be a complete metric space. For each $x \in X$ and each nonempty set $A \subset X$, set

$$\rho(x,A) = \inf\{\rho(x,y) : y \in A\}. \tag{2.1}$$

For each pair of nonempty sets $A, B \subset X$, put

$$H(A,B) = \max\left\{\sup_{x\in A}\rho(x,B), \sup_{x\in B}\rho(x,A)\right\}. \tag{2.2}$$

Let $T : X \to 2^X \setminus \{\emptyset\}$, $x_* \in X$ satisfy

$$x_* \in Tx_*, \tag{2.3}$$

let $\phi : [0,\infty) \to [0,1)$ be a decreasing function such that

$$\phi(t) < 1 \quad \forall t \in [0,\infty), \tag{2.4}$$

and assume that

$$H(T(x), T(x_*)) \le \phi(\rho(x,x_*))\rho(x,x_*) \quad \forall x \in X. \tag{2.5}$$

We begin with the following obvious fact.

Lemma 2.1. *Let $x_0 \in X$, $\delta > 0$, and let a sequence of mappings $T_i : X \to 2^X \setminus \{\emptyset\}$, $i = 0, 1, \dots$, satisfy*

$$H(T_i(x), T(x)) \leq \delta, \quad i = 0, 1, \dots, \ x \in X. \tag{2.6}$$

Then there exist sequences $\{x_i\}_{i=0}^{\infty} \subset X$ and $\{y_i\}_{i=1}^{\infty} \subset X$ such that for any integer $i \geq 0$,

$$\begin{gathered} y_{i+1} \in T(x_i), \\ \rho(x_*, y_{i+1}) \leq \rho(x_*, Tx_i) + \delta, \\ x_{i+1} \in T_i(x_i), \\ \rho(x_{i+1}, y_{i+1}) \leq \rho(y_{i+1}, T_i x_i) + \delta. \end{gathered} \tag{2.7}$$

Theorem 2.2. *Let ϵ and M be positive. Then there exist $\delta \in (0, \epsilon)$ and a natural number n_0 such that for each sequence of mappings $T_i : X \to 2^X \setminus \{\emptyset\}$, $i = 0, 1, \dots$, satisfying*

$$H(T_i(x), T(x)) \leq \delta, \quad i = 0, 1, \dots, \ x \in X, \tag{2.8}$$

and each $x \in X$ satisfying

$$\rho(x, x_*) \leq M, \tag{2.9}$$

there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_0 = x, \quad x_{i+1} \in T_i(x_i), \quad i = 0, 1, \dots, \tag{2.10}$$

and the inequality

$$\rho(x_i, x_*) \leq \epsilon \tag{2.11}$$

holds for all integers $i \geq n_0$.

This theorem is a consequence of Lemma 2.1 and our next result.

Theorem 2.3. *Let $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, \epsilon)$ so that for each $M > 0$, there is a natural number n_0 such that the following assertion holds.*

Assume that a sequence of mappings $T_i : X \to 2^X \setminus \{\emptyset\}$, $i = 0, 1, \dots$, satisfies

$$H(T_i(x), T(x)) \leq \delta, \quad i = 0, 1, \dots, \tag{2.12}$$

for all $x \in X$, a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$\rho(x_0, x_*) \leq M, \qquad x_{i+1} \in T_i(x_i), \quad i = 0, 1, \dots, \tag{2.13}$$

and that for each integer $i \geq 0$*, there is* $y_{i+1} \in T(x_i)$ *such that*

$$\rho(x_*, y_{i+1}) \leq \rho(x_*, Tx_i) + \delta, \tag{2.14}$$

$$\rho(x_{i+1}, y_{i+1}) \leq \rho(y_{i+1}, T_i x_i) + \delta. \tag{2.15}$$

Then $\rho(x_i, x_*) \leq \epsilon$ *for all integers* $i \geq n_0$*.*

Proof. Choose a positive number $\delta < \epsilon/12$ such that

$$6\delta \leq \left(\frac{\epsilon}{2}\right)\left(1 - \phi\left(\frac{\epsilon}{2}\right)\right). \tag{2.16}$$

Let $M > 0$. Fix a natural number n_0 such that

$$n_0 > 4 + \frac{M}{\delta}. \tag{2.17}$$

Assume that a sequence of mappings $T_i : X \to 2^X \setminus \{\emptyset\}$, $i = 0, 1, \ldots$, satisfies (2.12), $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (2.13), and that for each integer $i \geq 0$, there is

$$y_{i+1} \in T(x_i) \tag{2.18}$$

such that (2.14) and (2.15) hold.

Let $i \geq 0$ be an integer. By (2.3), (2.5), and (2.14),

$$\begin{aligned} \rho(x_*, y_{i+1}) &\leq \rho(x_*, T(x_i)) + \delta \leq H(T(x_*), T(x_i)) + \delta \\ &\leq \phi(\rho(x_i, x_*))\rho(x_i, x_*) \ + \delta. \end{aligned} \tag{2.19}$$

By (2.15), (2.18), and (2.12),

$$\rho(x_{i+1}, y_{i+1}) \leq \rho(y_{i+1}, T_i(x_i)) + \delta \leq H(T(x_i), T_i(x_i)) + \delta \leq 2\delta. \tag{2.20}$$

By (2.19) and (2.20),

$$\rho(x_*.x_{i+1}) \leq \rho(x_*, y_{i+1}) + \rho(y_{i+1}, x_{i+1}) \leq \phi(\rho(x_i, x_*))\rho(x_i, x_*) + 3\delta \tag{2.21}$$

for all integers $i \geq 0$.

We claim that there is an integer $j \in \{0, \ldots, n_0\}$ such that

$$\rho(x_j, x_*) \leq \frac{\epsilon}{2}. \tag{2.22}$$

Assume the contrary. Then

$$\rho(x_j, x_*) > \frac{\epsilon}{2}, \quad j = 0, \dots, n_0. \tag{2.23}$$

By (2.21), (2.23), and (2.16), we have, for all integers $j \in \{0, \dots, n_0 - 1\}$,

$$\begin{aligned}\rho(x_*, x_j) - \rho(x_*, x_{j+1}) &\geq \rho(x_*, x_j) - \phi(\rho(x_j, x_*))\rho(x_j, x_*) - 3\delta \\ &\geq \rho(x_*, x_j)(1 - \phi(\rho(x_j, x_*))) - 3\delta \\ &\geq \left(\frac{\epsilon}{2}\right)\left(1 - \phi\left(\frac{\epsilon}{2}\right)\right) - 3\delta \geq \delta.\end{aligned} \tag{2.24}$$

When combined with (2.13), this implies that

$$M \geq \rho(x_*, x_0) \geq \rho(x_*, x_0) - \rho(x_*, x_{n_0}) = \sum_{j=0}^{n_0-1} \left[\rho(x_*, x_j) - \rho(x_*, x_{j+1})\right] \geq n_0\delta. \tag{2.25}$$

This, however, contradicts (2.17).

Therefore, there is an integer $j \in \{0, \dots, n_0\}$ such that (2.22) holds. Next, we assert that

$$\rho(x_i, x_*) \leq \epsilon \quad \forall \text{ integers } i \geq j. \tag{2.26}$$

Assume the contrary. Then there is an integer $p > j$ such that

$$\rho(x_p, x_*) > \epsilon, \quad \rho(x_i, x_*) \leq \epsilon \quad \forall \text{ integers } i \text{ satisfying } j \leq i < p. \tag{2.27}$$

There are two cases: either

$$\rho(x_{p-1}, x_*) \leq \frac{\epsilon}{2} \tag{2.28}$$

or

$$\rho(x_{p-1}, x_*) > \frac{\epsilon}{2}. \tag{2.29}$$

Assume first that (2.28) holds. By (2.21), (2.28), and (2.16),

$$\rho(x_p, x_*) \leq \rho(x_{p-1}, x_*) + 3\delta \leq \frac{\epsilon}{2} + 3\delta \leq \epsilon. \tag{2.30}$$

Now assume that (2.29) holds. By (2.29), (2.21), (2.27), and (2.16),

$$\begin{aligned}\rho(x_p, x_*) &\le \phi(\rho(x_{p-1}, x_*))\rho(x_{p-1}, x_*) + 3\delta \\ &\le \phi\left(\frac{\epsilon}{2}\right)\epsilon + 3\delta \le \phi\left(\frac{\epsilon}{2}\right)\epsilon + \epsilon\left(1 - \phi\left(\frac{\epsilon}{2}\right)\right) = \epsilon.\end{aligned} \tag{2.31}$$

This contradicts (2.27).

The contradiction we have reached in both cases proves that (2.26) holds.

This completes the proof of Theorem 2.3. □

We end this section with another consequence of Theorem 2.3.

Theorem 2.4. *Let $\{\epsilon_i\}_{i=0}^{\infty} \subset (0,\infty)$, $T_i : X \to 2^X \setminus \{\emptyset\}$, $i = 0, 1, \ldots$, and assume that, for all $x \in X$,*

$$H(Tx, T_i(x)) \le \epsilon_i, \quad i = 1, 2, \ldots, \quad \lim_{i\to\infty} \epsilon_i = 0. \tag{2.32}$$

Then for each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_0 = x, \quad x_{i+1} \in T_i x_i, \quad i = 0, 1, \ldots, \quad \lim_{i\to\infty} x_i = x_*. \tag{2.33}$$

Proof. Let $x \in X$. Put $x_0 = x$ and define a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ by induction so that for each integer $i \ge 0$, there is $y_{i+1} \in T(x_i)$ satisfying

$$\begin{gathered}\rho(x_*, y_{i+1}) \le \rho(x_*, Tx_i) + \epsilon_i, \\ x_{i+1} \in T_i x_i, \quad \rho(y_{i+1}, x_{i+1}) \le \rho(y_{i+1}, T_i x_i) + \epsilon_i.\end{gathered} \tag{2.34}$$

In order to show that

$$\lim_{i\to\infty} \rho(x_i, x_*) = 0, \tag{2.35}$$

we let $\epsilon \in (0,1)$ and prove that for all sufficiently large natural numbers i,

$$\rho(x_i, x_*) \le \epsilon. \tag{2.36}$$

Let $\delta \in (0, \epsilon)$ be as guaranteed by Theorem 2.3.

There is a natural number n_1 such that

$$\epsilon_i < \delta \quad \forall \text{ integers } i \ge n_1. \tag{2.37}$$

Choose $M > 0$ such that

$$\rho(x_{n_1}, x_*) < M. \tag{2.38}$$

Let a natural number n_0 be as guaranteed by Theorem 2.3.

Then for all integers $i \geq n_0 + n_1$,

$$\rho(x_i, x_*) \leq \epsilon. \tag{2.39}$$

Theorem 2.4 is proved. □

3. Caristi-Type Theorems for Set-Valued Mappings

We begin this section by recalling [6, Theorem 5.3].

Theorem 3.1. *Assume that (X, ρ) is a complete metric space, $T : X \to 2^X \setminus \{\emptyset\}$, graph$(T) = \{(x, y) \in X \times X : y \in Tx\}$ is closed, $\phi : X \to R^1 \cup \{\infty\}$ is bounded from below, and that for each $x \in X$,*

$$\inf\{\phi(y) + \rho(x, y) : \ y \in Tx\} \leq \phi(x). \tag{3.1}$$

Let $\{\epsilon_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \epsilon_n < \infty$, and let $x_0 \in X$ satisfy $\phi(x_0) < \infty$. Assume that for each integer $n \geq 0$, $x_{n+1} \in T(x_n)$ and

$$\phi(x_{n+1}) + \rho(x_n, x_{n+1}) \leq \inf\{\phi(y) + \rho(x, y) : \ y \in T(x_n)\} + \epsilon_n. \tag{3.2}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of T.

In the proof of this theorem, we actually showed that $\sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) < \infty$.

It turns out that the existence of such a sequence is actually equivalent to the existence of a function $\phi : X \to R^1 \cup \{\infty\}$ which is bounded from below and such that for each $x \in X$,

$$\inf\{\phi(y) + \rho(x, y) : \ y \in Tx\} \leq \phi(x). \tag{3.3}$$

More precisely, we are going to prove the following result.

Theorem 3.2. *Let (X, ρ) be a complete metric space and $T : X \to 2^X \setminus \{\emptyset\}$. The following conditions are equivalent.*

(A) *There exists a function $\phi : X \to R^1 \cup \{\infty\}$, which is bounded from below and not identically ∞, such that for each $x \in X$, inequality* (3.3) *holds.*

(B) *There exists a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that $x_{n+1} \in T(x_n)$ for all integers n and $\sum_{n=0}^{\infty} \rho(x_n, x_{n+1}) < \infty$.*

Proof. The fact that (A) implies (B) was proved in [6]. To show that (B) implies (A), we define, for each $x \in X$,

$$\phi(x) = \inf\left\{ \sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) : \{x_i\}_{i=0}^{\infty} \subset X, x_0 = x, x_{i+1} \in T(x_i) \quad \forall \text{ integers } i \geq 0 \right\}. \tag{3.4}$$

Let $x \in X$. Note that $\phi(x) < \infty$ if and only if there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$, $i = 0, 1, \ldots$, and

$$\sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) < \infty. \tag{3.5}$$

It is sufficient to show that (3.3) holds for all $x \in X$. To this end, let $x \in X$. We may assume that $\phi(x) < \infty$. Let $\epsilon > 0$. There is $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$,

$$\begin{gathered} x_{t+1} \in T x_i, \quad i = 0, 1, \ldots, \\ \sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) \leq \phi(x) + \epsilon. \end{gathered} \tag{3.6}$$

Then

$$x_1 \in T(x), \quad \rho(x, x_1) + \phi(x_1) \leq \rho(x, x_1) + \sum_{i=1}^{\infty} \rho(x_i, x_{i+1}) \leq \phi(x) + \epsilon, \tag{3.7}$$

and so,

$$\inf\{\phi(y) + \rho(x, y) : \ y \in T(x)\} \leq \phi(x) + \epsilon. \tag{3.8}$$

Since ϵ is any positive number, we conclude that (3.3) holds. This completes the proof of Theorem 3.2. □

It should be mentioned that in Theorem 3.1 we pose an assumption on T without assuming that ϕ possesses lower semicontinuity properties, while in the original Caristi theorem no assumption was made on the mapping, but the function ϕ was assumed to be lower semicontinuous. In the following result, we obtain a simple analog of Theorem 3.1 for this situation.

Theorem 3.3. *Assume that (X, ρ) is a complete metric space, $T : X \to 2^X \setminus \{\emptyset\}$, Tx is closed for each $x \in X$, $\phi : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous function which is bounded from below and not identically ∞, and that for each $x \in X$, inequality* (3.3) *holds. Then T has a fixed point.*

Proof. Choose $x_0 \in X$ such that

$$\phi(x_0) \leq \inf\{\phi(z) : \ z \in X\} + \frac{1}{8}. \tag{3.9}$$

By Ekeland's variational principle [14], there is $x_1 \in X$ such that

$$\begin{gathered} \rho(x_0, x_1) \leq 1, \\ \phi(x_1) < \phi(z) + \left(\frac{1}{8}\right) \rho(z, x_1) \quad \forall z \in X \setminus \{x_1\}. \end{gathered} \tag{3.10}$$

Let $\epsilon > 0$. By (3.3) and (3.10), there exists $z_\epsilon \in Tx_1$ such that

$$\phi(z_\epsilon) + \rho(x_1, z_\epsilon) \le \phi(x_1) + \epsilon \le \epsilon + \phi(z_\epsilon) + \left(\frac{1}{8}\right)\rho(z_\epsilon, x_1),$$
$$\left(\frac{1}{2}\right)\rho(x_1, z_\epsilon) \le \epsilon, \tag{3.11}$$
$$\inf\{\rho(x_1, z) : \ z \in T(x_1)\} \le 2\epsilon.$$

Since ϵ is any positive number, it follows that $x_1 \in T(x_1)$. Theorem 3.3 is proved. □

Note added in proof

After our paper was accepted for publication, Pavel Semenov has kindly informed us that our Theorem 3.3 is almost identical with Corollary 1.7 on page 521 of Volume I of the *Handbook of Multivalued Analysis* by S. Hu and N. S. Papageorgiou, Kluwer Academic Publishers, Dordrecht, 1997. It may be of interest to note that the above authors deduce their Corollary from a set-valued version of Caristi's fixed point theorem, while we use Ekeland's variational principle (which is known to be equivalent to Caristi's fixed point theorem).

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