Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 362829, 13 pages
doi:10.1155/2010/362829

Research Article

The Fixed Point Property of Unital Abelian Banach Algebras

W. Fupinwong and S. Dhompongsa

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to S. Dhompongsa, sompongd@chiangmai.ac.th

Received 28 October 2009; Accepted 22 January 2010

Academic Editor: Anthony To Ming Lau

We give a general condition for infinite dimensional unital Abelian Banach algebras to fail the fixed point property. Examples of those algebras are given including the algebras of continuous functions on compact sets.

1. Introduction

Let X be a Banach space. A mapping $T : E \subset X \to X$ is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for each $x, y \in E$. The *fixed point set* of T is $\mathrm{Fix}(T) = \{x \in E : Tx = x\}$. We say that the space X has the *fixed point property* (or *weak* fixed point property) if for every nonempty bounded closed convex (or weakly compact convex, resp.) subset E of X and every nonexpansive mapping $T : E \to E$ we have $\mathrm{Fix}(T) \neq \emptyset$. One of the central goals in fixed point theory is to solve the problem: which Banach spaces have the (weak) fixed point property?

For weak fixed point property, Alspach [1] exhibited a weakly compact convex subset E of the Lebesgue space $L_1[0,1]$ and an isometry $T : E \to E$ without a fixed point, proving that the space $L_1[0,1]$ does not have the weak fixed point property. Lau et al. [2] proved the following results.

Theorem 1.1. *Let X be a locally compact Hausdorff space. If $C_0(X)$ has the weak fixed point property, then X is dispersed.*

Corollary 1.2. *Let G be a locally compact group. Then the C^*-algebra $C_0(G)$ has the weak fixed point property if and only if G is discrete.*

Corollary 1.3. *A von Neumann algebra $\mathcal{M}$ has the weak fixed point property if and only if $\mathcal{M}$ is finite dimensional.*

Continuing in this direction, Benavides and Pineda [3] developed the concept of ω-almost weak orthogonality in the Banach lattice $C(K)$ and obtained the results.

Theorem 1.4. *Let X be a ω-almost weakly orthogonal closed subspace of $C(K)$ where K is a metrizable compact space. Then X has the weak fixed point property.*

Theorem 1.5. *Let K be a metrizable compact space. Then, the following conditions are all equivalent:*

(1) *$C(K)$ is ω-almost weakly orthogonal,*

(2) *$C(K)$ is ω-weakly orthogonal,*

(3) *$K^{(\omega)} = \emptyset$.*

Corollary 1.6. *Let K be a compact set with $K^{(\omega)} = \emptyset$. Then $C(K)$ has the weak fixed point property.*

As for the fixed point property, Dhompongsa et al. [4] showed that a C^*-algebra has the fixed point property if and only if it is finite dimensional. In this paper, we approach the question on the fixed point property from the opposite direction by identifying unital abelian Banach algebras which fail to have the fixed point property. As consequences, we obtain results on the algebra of continuous functions $C(S)$, where S is a compact set, and there is a unital abelian subalgebra of the algebra $l_\infty(\mathbb{N})$ which does not have the fixed point property and does not contain the space c_0.

2. Preliminaries and Lemmas

The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{F}$ are called *scalars*.

An element x in a unital algebra X is said to be *invertible* if there is an element y in X such that

$$xy = yx = 1. \tag{2.1}$$

In this case y is unique and written x^{-1}.

We define the *spectrum* of an element x of a unital algebra X over $\mathbb{F}$ to be the set

$$\sigma(x) = \{\lambda \in \mathbb{F} : \lambda 1 - x \text{ is not invertible}\}. \tag{2.2}$$

The *spectral radius* of x is defined to be

$$r(x) = \sup_{\lambda\in\sigma(x)} |\lambda|. \tag{2.3}$$

We note that a subalgbra of a normed algebra is itself a normed algebra with the norm got by restriction. The closure of a subalgebra is a subalgebra. A closed subalgebra of a Banach algebra is a Banach algebra. If $(B_\alpha)_{\alpha\in\Lambda}$ is a family of subalgebras of an algebra X, then $\bigcap_{\alpha\in\Lambda} B_\alpha$ is a subalgebra also. Hence, for any subset E of X, there is the smallest subalgebra $A(E)$ of X containing E. This algebra is called the subalgebra of X *generated* by E. If E is the singleton $\{x\}$, then $A(E)$ is the linear span of all powers x^n of x. If X is a normed algebra, the closed algebra $B(E)$ *generated* by a set E is the smallest closed subalgebra containing E. We can see that $B(E) = \overline{A(E)}$.

We denote by $C_{\mathbb{F}}(S)$ the Banach algebra of continuous functions from a topological space S to $\mathbb{F}$, with the sup-norm

$$\|f\|_\infty = \sup_{x\in S}|f(x)|. \tag{2.4}$$

The following theorems are known as the Stone-Weierstrass approximation theorem for $C_{\mathbb{R}}(S)$ and $C_{\mathbb{C}}(S)$, respectively. For the details, the readers are referred to [5].

Theorem 2.1. *Let A be a subalgebra of $C_{\mathbb{R}}(S)$ such that*

(1°) *A separates the points of S,*

(2°) *A annihilates no point of S.*

Then A is dense in $C_{\mathbb{R}}(S)$.

Theorem 2.2. *Let S be a compact space, A a subalgebra of $C_{\mathbb{C}}(S)$ such that*

(1°) *A separates the points of S,*

(2°) *A annihilates no point of S,*

(3°) *$f \in A$ implies that the conjugate $\overline{f}$ of f is in A.*

Then A is dense in $C_{\mathbb{C}}(S)$.

A *character* on a unital algebra X over $\mathbb{F}$ is a nonzero homomorphism $\tau : X \to \mathbb{F}$. We denote by $\Omega(X)$ the set of characters on X. Note that if X is a unital abelian complex Banach algebra, then

$$\sigma(x) = \{\tau(x) : \tau \in \Omega(X)\} \tag{2.5}$$

for each $x \in X$ (see [6]).

Remark 2.3. It is unknown if (2.5) is valid whenever $\Omega(X) \neq \emptyset$. Equation (2.5) obviously does not hold for a space X with $\Omega(X) = \emptyset$ as the following example shows.

Example 2.4. Let $X = \mathbb{C}$ be considered as a real unital abelian Banach algebra under ordinary complex multiplication and whose norm is the absolute value. We have $\Omega(X) = \emptyset$. Indeed, assume to the contrary that there is a non-zero homomorphism τ_0 on X and $\tau_0(i) = \lambda \in \mathbb{R}$, so

$$\begin{gathered} \tau_0((1+i)i) = \tau_0(i-1) = \lambda - 1, \\ \tau_0(1+i)\tau_0(i) = (1+\lambda)\lambda = \lambda + \lambda^2. \end{gathered} \tag{2.6}$$

Thus $\lambda - 1 = \lambda + \lambda^2$; so λ is not a real number, which is a contradiction.

Since $\Omega(X) = \emptyset$, so $\{\tau(1) : \tau \in \Omega(X)\} = \emptyset$ but $\sigma(1) = \{1\}$.

We consider throughout this paper on Banach algebras X for which $\Omega(X) \neq \emptyset$ and satisfy (2.5).

If X is a unital abelian Banach algebra, it follows from Proposition 2.5 that $\Omega(X)$ is contained in the closed unit ball of X^*. We endow $\Omega(X)$ with the relative weak* topology and call the topological space $\Omega(X)$ the *character space* of X.

Detailed proofs of the following propositions can be found in [6].

Proposition 2.5. *Let X be a unital abelian Banach algebra. If $\tau \in \Omega(X)$, then $\|\tau\| = 1$.*

Proposition 2.6. *If X is a unital Banach algebra, then $\Omega(X)$ is compact.*

If X is a unital abelian Banach algebra, and $x \in X$, we define a continuous function $\widehat{x}$ by

$$\widehat{x} : \Omega(X) \longrightarrow \mathbb{F}, \qquad \tau \longmapsto \tau(x). \tag{2.7}$$

We call $\widehat{x}$ the *Gelfand transform* of x, and the homomorphism

$$\varphi : X \longrightarrow C_{\mathbb{F}}(\Omega(X)), \quad x \mapsto \widehat{x} \tag{2.8}$$

is called the *Gelfand representation.*

The following two lemmas, Lemmas 2.7 and 2.10, will be used to prove our main theorem.

Lemma 2.7. *Let X be a unital abelian real Banach algebra with*

$$\inf\{r(x) : x \in X, \ \|x\| = 1\} > 0. \tag{2.9}$$

Then one has the following:

(i) *the Gelfand representation φ is a bounded isomorphism,*

(ii) *the inverse φ^{-1} is also a bounded isomorphism.*

Proof. (i) φ is injective since $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$ implies $\ker(\varphi) = 0$. It is easily checked that φ is a bounded homomorphism, and $\varphi(X)$ is a subalgebra of $C_{\mathbb{R}}(\Omega(X))$ separating the points of $\Omega(X)$, and having the property that for any $\tau \in \Omega(X)$ there is an element $x \in X$ such that $\widehat{x}(\tau) \neq 0$. In order to use the Stone-Weierstrass theorem to show that $\varphi(X) = C_{\mathbb{R}}(\Omega(X))$, we shall show that $\varphi(X)$ is closed. We show that $\varphi(X)$ is closed by showing that $\varphi(X)$ is complete. Let $\{\widehat{x_n}\}$ be a Cauchy sequence in $\varphi(X)$. First, we show that the sequence $\{x_n\} = \{\varphi^{-1}(\widehat{x_n})\}$ is Cauchy. Assume on the contrary that $\{x_n\}$ is not Cauchy. Thus there exists $\varepsilon_0 > 0$ and subsequences $\{z_n\}$ and $\{z'_n\}$ of $\{x_n\}$ such that

$$\left\| z_n - z'_n \right\| \geq \varepsilon_0 \tag{2.10}$$

for each $n \in \mathbb{N}$. Write $y_n = (z_n - z'_n)/\varepsilon_0$, then $\|y_n\| \geq 1$, for each $n \in \mathbb{N}$. But $\{\widehat{x_n}\}$ is Cauchy, and so we have $\widehat{y_n} \to 0$. Thus

$$0 < \inf\{r(x) : x \in X,\ \|x\| = 1\} \leq \inf_{n\in\mathbb{N}} r\left(\frac{y_n}{\|y_n\|}\right) = \inf_{n\in\mathbb{N}} \left\| \frac{\widehat{y_n}}{\|y_n\|} \right\|_\infty = 0, \tag{2.11}$$

which is a contradiction. Hence $\{x_n\}$ must be Cauchy and so $x_n \to x_0$, for some $x_0 \in X$. Since $\|\widehat{x}\|_\infty = \|\varphi(x)\|_\infty \leq \|x\|$, for each $x \in X$, so $\widehat{x_n} \to \widehat{x_0}$. Thus $\varphi(X)$ is complete. The Stone-Weierstrass theorem can be applied to conclude that φ is surjective.

(ii) follows from the open mapping theorem. □

Remark 2.8. (i) Lemma 2.7 tells us that if X is a unital abelian real Banach algebra with property

$$\inf\{r(x) : x \in X,\ \|x\| = 1\} > 0, \tag{2.12}$$

then X and $C_{\mathbb{R}}(\Omega(X))$ are homeomorphic and isomorphic under φ. Hence if we would like to consider the convergence of a sequence $\{x_n\}$ in X, we could look at the convergence of the corresponding sequence $\{\widehat{x_n}\}$.

(ii) Property (2.12) clearly implies the semisimplicity property ($r(x) \Leftrightarrow x = 0$) but the following example shows that it is stronger.

Example 2.9. Let $l_1(\mathbb{Z})$ denote the Banach algebra of complex-valued absolutely summable functions on the group of integers $\mathbb{Z}$ under convolution regarded as a real Banach algebra and let X be the real subalgebra of $l_1(\mathbb{Z})$ consisting of those functions that satisfy $f(-n) = \overline{f}(n),\ n \in \mathbb{Z}$. Then the maximal ideal space of X equals $T = R/Z$ and the Gelfand transform is precisely the Fourier transform which maps X into the real Banach algebra $C_{\mathbb{R}}(T)$ of continuous real-valued functions on $C_{\mathbb{R}}(T)$ under pointwise multiplication and maximum norm. Although the image of the Fourier transform is dense, it is clearly not all of $C_{\mathbb{R}}(T)$ since it is simply the real-valued functions in the Wiener space which consists of complex-valued functions whose Fourier series are absolutely summable. Therefore Lemma 2.7 shows that X does not have Property (2.12).

Lemma 2.10. *Let X be an infinite dimensional unital abelian real Banach algebra with*

$$\inf\{r(x) : x \in X,\ \|x\| = 1\} > 0. \tag{2.13}$$

Then one has the following:

(i) $\Omega(X)$ *is an infinite set,*

(ii) *if there exists a bounded sequence* $\{x_n\}$ *in* X *which contains no convergent subsequences and such that* $\{\tau(x_n) : \tau \in \Omega(X)\}$ *is finite for each* $n \in \mathbb{N}$*, then there is an element* $x_0 \in X$ *with*

$$\{\tau(x_0) : \tau \in \Omega(X)\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}, \tag{2.14}$$

(iii) *there is an element* $x_0 \in X$ *such that* $\{\tau(x_0) : \tau \in \Omega(X)\}$ *is an infinite set,*

(iv) *there exists a sequence* $\{x_n\}$ *in* X *such that* $\{\tau(x_n) : \tau \in \Omega(X)\} \subset [0,1]$, *for each* $n \in \mathbb{N}$, *and* $\{(\widehat{x_n})^{-1}\{1\}\}$ *is a sequence of nonempty pairwise disjoint subsets of* $\Omega(X)$.

Proof. Let X be an infinite dimensional unital abelian real Banach algebra with

$$\inf\{r(x) : x \in X, \ \|x\| = 1\} > 0. \tag{2.15}$$

(i) If suffices to show that if $\Omega(X)$ is a finite set, for then the closed unit ball B_X of X is compact, and this will lead to us a contradiction. Let $\Omega(X)$ be a finite set, say $\{\tau_1, \tau_2, \ldots, \tau_m\}$, and let $\{x_n\}$ be a sequence in B_X.

Since the sequences $\{\tau_p(x_n)\}$, $p = 1,2,\ldots,m$, are bounded, we can choose a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\tau_p(x_{n_m}) \to \lambda_p$, for each $p = 1,2,\ldots,m$.

Define $\varphi : \Omega(X) \to \mathbb{R}$ by $\varphi(\tau_p) = \lambda_p$. Thus there exists $x \in X$ such that $\varphi = \widehat{x}$, and consequently, $\widehat{x_{n_m}} \to \widehat{x}$ since

$$\|\widehat{x_{n_m}} - \widehat{x}\|_\infty = \sup_{\tau\in\Omega(X)} |\widehat{x_{n_m}}(\tau) - \widehat{x}(\tau)| = \max_{1\le p\le m} \left|\widehat{x_{n_m}}(\tau_p) - \widehat{x}(\tau_p)\right|, \tag{2.16}$$

and $\tau_p(x_{n_m}) \to \widehat{x}(\tau_p)$, for each $p = 1,2,\ldots,m$.

So $\{\widehat{x_{n_m}}\}$ is a subsequence of $\{\widehat{x_n}\}$ such that $\widehat{x_{n_m}} \to \widehat{x}$. By Remark 2.8, $x_{n_m} \to x$, where $x = \varphi^{-1}(\widehat{x})$. Thus B_X is compact.

(ii) Let $\{x_n\}$ be a bounded sequence in X which has no convergent subsequences and suppose that the set $\{\tau(x_n) : \tau \in \Omega(X)\}$ is finite for each $n \in \mathbb{N}$. By Remark 2.8, we will consider $\{x_n\}$ as a sequence of Gelfand transforms $\{\widehat{x_n}\}$.

First, we show that we can write

$$\Omega(X) = \left(\bigcup_{n\in\mathbb{N}} G_n\right) \cup F, \tag{2.17}$$

where F is closed, G_n are all closed and open, and $\{F, G_1, G_2, \ldots\}$ is a partition of $\Omega(X)$. For each $n \in \mathbb{N}$, write

$$\begin{gathered} \{\tau(x_n) : \tau \in \Omega(X)\} = \left\{\lambda_{(n,i)} : i = 1,2,\ldots,m_n\right\}, \\ \mathcal{L}_n = \left\{(\widehat{x_n})^{-1}\left\{\lambda_{(n,i)}\right\} : i = 1,2,\ldots,m_n\right\}. \end{gathered} \tag{2.18}$$

Define

$$\mathcal{L} = \left\{\bigcap_{k\in\mathbb{N}} A_k : A_k \in \mathcal{L}_k\right\} \setminus \{\emptyset\}. \tag{2.19}$$

Note that $(\widehat{x_n})^{-1}\{\lambda_{(n,i)}\}$ are all closed and open. Since $\mathcal{L}_n$ is a partition of $\Omega(X)$ for each $n \in \mathbb{N}$, $\mathcal{L}$ is a partition of $\Omega(X)$. There are two cases to be considered.

Case 1 ($\mathcal{L}$ is infinite). Thus there exists i_1 such that

$$\left\{ (\widehat{x_1})^{-1}\{\lambda_{(1,i_1)}\} \cap \left(\bigcap_k A_k \right) : A_k \in \mathcal{L}_k,\ k \geq 2 \right\} \tag{2.20}$$

is an infinite set. Similarly, there exists i_2 such that

$$\left\{ (\widehat{x_1})^{-1}\{\lambda_{(1,i_1)}\} \cap (\widehat{x_2})^{-1}\{\lambda_{(2,i_2)}\} \cap \left(\bigcap_k A_k \right) : A_k \in \mathcal{L}_k,\ k \geq 3 \right\} \tag{2.21}$$

is an infinite set. Continuing in this process we obtain a sequence of the sets $(\widehat{x_n})^{-1}\{\lambda_{(n,i_n)}\} \in \mathcal{L}_n$ such that

$$\left\{ \bigcap_{j=1}^{n} \left((\widehat{x}_j)^{-1} \left\{ \lambda_{(j,i_j)} \right\} \right) \cap \left(\bigcap_k A_k \right) : A_k \in \mathcal{L}_k,\ k \geq n+1 \right\} \tag{2.22}$$

is an infinite set, for each $n \in \mathbb{N}$.

Write

$$\begin{gathered}
H_1 = \bigcup_{i \neq i_1} (\widehat{x_1})^{-1}\{\lambda_{(1,i)}\}, \\
H_2 = (\widehat{x_1})^{-1}\{\lambda_{(1,i_1)}\} \cap \left(\bigcup_{i \neq i_2} (\widehat{x_2})^{-1}\{\lambda_{(2,i)}\} \right), \\
H_3 = (\widehat{x_1})^{-1}\{\lambda_{(1,i_1)}\} \cap (\widehat{x_2})^{-1}\{\lambda_{(2,i_2)}\} \cap \left(\bigcup_{i \neq i_3} (\widehat{x_3})^{-1}\{\lambda_{(3,i)}\} \right), \ldots.
\end{gathered} \tag{2.23}$$

Thus H_n are all closed and open, and

$$\Omega(X) = \left(\bigcup_{n \in \mathbb{N}} H_n \right) \cup \left(\Omega(X) \setminus \bigcup_{n \in \mathbb{N}} H_n \right), \tag{2.24}$$

where $\Omega(X) \setminus \bigcup_{n\in\mathbb{N}} H_n$ is a nonempty closed set since $\Omega(X)$ is compact. And since $\mathcal{L}$ has infinite elements, we can see that there exists a subsequence $\{G_n\}$ of $\{H_n\}$ such that $\bigcup_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} H_n$ and $G_n \neq \emptyset$, for each $n \in \mathbb{N}$.

Hence we have

$$\Omega(X) = \left(\bigcup_{n \in \mathbb{N}} G_n \right) \cup \left(\Omega(X) \setminus \bigcup_{n \in \mathbb{N}} G_n \right), \tag{2.25}$$

and $\{(\Omega(X) \setminus \bigcup_{n\in\mathbb{N}} G_n), G_1, G_2, \ldots\}$ is a partition of $\Omega(X)$.

Case 2 ($\mathcal{L} = \{L_i : i = 1, 2, \dots, m\}$). To show that this case leads to a contradiction, we first observe that if $\tau,\ \tau'$ are in the same $L_i \in \mathcal{L}$, then

$$\tau(x_n) = \tau'(x_n) \tag{2.26}$$

for each $n \in \mathbb{N}$. Write $\alpha_{(n,i)} = \tau(x_n)$, if $\tau \in L_i$. There exists a subsequence $\{(\alpha_{(n',1)}, \alpha_{(n',2)}, \dots, \alpha_{(n',m)})\}_{n' \in \mathbb{N}}$ of $\{(\alpha_{(n,1)}, \alpha_{(n,2)}, \dots, \alpha_{(n,m)})\}_{n \in \mathbb{N}}$ such that for each $i = 1, 2, \dots, m$,

$$\alpha_{(n',i)} \longrightarrow \alpha_i \tag{2.27}$$

for some $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m$. Define a Gelfand transform $\widehat{x} : \Omega(X) \to \mathbb{R}$ by $\widehat{x}(\tau) = \alpha_i$, if $\tau \in L_i$. Since

$$\|\widehat{x_{n'}} - \widehat{x}\|_\infty = \sup_{\tau \in \Omega(X)} |\widehat{x_{n'}}(\tau) - \widehat{x}(\tau)| = \max_{1 \le i \le m} \left|\alpha_{(n',i)} - \alpha_i\right|, \tag{2.28}$$

so $\widehat{x_{n'}} \to \widehat{x}$, which is a contradiction.

Now we conclude that

$$\Omega(X) = \left(\bigcup_{n \in \mathbb{N}} G_n\right) \cup F, \tag{2.29}$$

where F is closed, G_n is closed and open for each $n \in \mathbb{N}$, and $\{F, G_1, G_2, \dots\}$ is a partition of $\Omega(X)$. Define a map $\psi : \Omega(X) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} \dfrac{n}{n+1} & \text{if } \tau \in G_n, \\ 1 & \text{if } \tau \in F. \end{cases} \tag{2.30}$$

We can check that the inverse image of each closed set in $\psi(\Omega(X))$ is closed. Therefore, $\varphi^{-1}(\psi)$ is an element in X, say x_0, with

$$\{\tau(x_0) : \tau \in \Omega(X)\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}. \tag{2.31}$$

(iii) Assume to the contrary that $\{\tau(x) : \tau \in \Omega(X)\}$ is finite for each $x \in X$. Since X is infinite dimensional, so, as B_X is noncompact, there exists a bounded sequence $\{x_n\}$ in X which has no convergent subsequences. Hence $\{\tau(x_n) : \tau \in \Omega(X)\}$ is finite for each $n \in \mathbb{N}$. It follows from (ii) that

$$\{\tau(x_0) : \tau \in \Omega(X)\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\} \tag{2.32}$$

for some $x_0 \in X$, which is the contradiction.

(iv) From (iii), there is an element $x_1 \in X$ such that $\{\tau(x_1) : \tau \in \Omega(X)\}$ is an infinite set. We can choose x_1 so that there exists a strictly decreasing sequence $\{a_n\}$ such that

$$\{a_n\} \subset \widehat{x_1}(\Omega(X)) \subset [0,1], \quad a_1 < 1, \tag{2.33}$$

and $\tau(x_1) = 1$ for some $\tau \in \Omega(X)$. Define a continuous function $g_1 : [0,1] \to [0,1]$ to be linear on $[0, a_1]$ and on $[a_1, 1]$ joining the points $(0,0)$ and $(a_1, 1)$, and $g(1) \subset (g_1(a_2), 1)$. Put $\widehat{x_2} = g_1 \circ \widehat{x_1}$, for some $x_2 \in X$, and define a continuous function $g_2 : [0,1] \to [0,1]$ similar to the way we construct g_1. The left part of g_2 is the line joining the point $(0,0)$ and $(g_1(a_2), 1)$ and $g_2(1) \in (g_2(g_1(a_2)), 1)$. Then put $\widehat{x_3} = g_2 \circ \widehat{x_2}$, for some $x_3 \in X$. Continuing in this process we obtain a sequence of points $\{x_n\}$ such that $\{\tau(x_n) : \tau \in \Omega(X)\} \subset [0,1]$, for each $n \in \mathbb{N}$, and $\{(\widehat{x_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(X)$. We then obtain the required result. □

3. Main Theorem

Now we prove our main theorem.

Theorem 3.1. *Let X be an infinite dimensional unital abelian real Banach algebra satisfying each of the following:*

(i) *if $x, y \in X$ is such that $|\tau(x)| \le |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \le \|y\|$,*

(ii) $\inf\{r(x) : x \in X,\ \|x\| = 1\} > 0$.

Then X does not have fixed point property.

Proof. Let X be an infinite dimensional unital abelian real Banach algebra satisfying (i) and (ii). Assume to the contrary that X has fixed point property. From Lemma 2.10(iv), there exists a sequence $\{x_n\}$ in X such that

$$\{\tau(x_n) : \tau \in \Omega(X)\} \subset [0,1] \tag{3.1}$$

for each $n \in \mathbb{N}$, and $\{(\widehat{x_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(X)$. Write $A_n = (\widehat{x_n})^{-1}\{1\}$, and define $T_n : E_n \to E_n$ by

$$x \longmapsto x_n x, \tag{3.2}$$

where

$$E_n = \{x \in X : 0 \le \tau(x) \le 1 \text{ for each } \tau \in \Omega(X), \text{and } \tau(x) = 1 \text{ if } \tau \in A_n\}. \tag{3.3}$$

It follows from (i) that $T_n : E_n \to E_n$ is a nonexpansive mapping on the bounded closed convex set E_n, for each $n \in \mathbb{N}$. Indeed, E_n is bounded since

$$0 < \inf\{r(x) : x \in X,\ \|x\| = 1\} \le r\left(\frac{x}{\|x\|}\right) = \sup_{\tau\in\Omega(X)}\left|\tau\left(\frac{x}{\|x\|}\right)\right| = \frac{1}{\|x\|}\sup_{\tau\in\Omega(X)}|\tau(x)| \tag{3.4}$$

for each $x \in X$. It follows that T_n has a fixed point, say y_n, for each $n \in \mathbb{N}$. Since $y_n = x_n y_n$, thus $\widehat{y_n} = \widehat{x_n}\widehat{y_n}$, and then

$$\widehat{y_n}(\tau) = \begin{cases} 0 & \text{if } \tau \text{ is not in } A_n, \\ 1 & \text{if } \tau \text{ is in } A_n, \end{cases} \tag{3.5}$$

for each $n \in \mathbb{N}$. Since $A_1, A_2, A_3, \ldots$ are pairwise disjoint, so $\|\widehat{y_m} - \widehat{y_n}\| = 1$, if $m \neq n$. Hence $\{\widehat{y_n}\}$ has no convergent subsequences. From Lemma 2.7, $\{y_n\}$ has no convergent subsequences too. It follows from the existence of $\{y_n\}$ and Lemma 2.10 (ii) that there exists an element x_0 in X with

$$\{\tau(x_0) : \tau \in \Omega(X)\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}. \tag{3.6}$$

Write $A_0 = (\widehat{x_0})^{-1}\{1\}$. Define $T_0 : E_0 \rightarrow E_0$ by

$$x \longmapsto x_0 x, \tag{3.7}$$

where

$$E_0 = \{x \in X : \ 0 \leq \tau(x) \leq 1 \text{ for each } \tau \in \Omega(X), \text{and } \tau(x) = 1 \text{ if } \tau \in A_0\}. \tag{3.8}$$

By (i), T_0 is a nonexpansive mapping on the bounded closed convex set E_0. Thus T_0 has a fixed point, say y_0, that is, $y_0 = x_0 y_0$. Thus $\widehat{y_0} = \widehat{x_0}\widehat{y_0}$. Consequently,

$$\widehat{y_0}(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } (\widehat{x_0})^{-1}\{1\}, \\ 1, & \text{if } \tau \text{ is in } (\widehat{x_0})^{-1}\{1\}. \end{cases} \tag{3.9}$$

The set $(\widehat{x_0})^{-1}\{1\} = (\widehat{y_0})^{-1}\{1\}$ is open in $\Omega(X)$, since $\widehat{y_0}$ is continuous. Also the set $(\widehat{x_0})^{-1}\{n/(n+1)\}$ is open in $\Omega(X)$ for each $n \in \mathbb{N}$, since $\widehat{x_0}$ is continuous. Thus,

$$\left\{(\widehat{x_0})^{-1}\left\{\frac{n}{n+1}\right\} : n \in \mathbb{N}\right\} \cup \left\{(\widehat{x_0})^{-1}\{1\}\right\} \tag{3.10}$$

is an open covering of $\Omega(X)$. This leads to a contradiction, since $\Omega(X)$ is compact. □

From the above theorem we have the following.

Corollary 3.2. *Let S be a compact Hausdorff topological space. If $C_{\mathbb{R}}(S)$ is infinite dimensional, then $C_{\mathbb{R}}(S)$ fails to have the fixed point property.*

Proof. $C_{\mathbb{R}}(S)$ satisfies (i), (ii) in Theorem 3.1. Indeed, if $x, y \in C_{\mathbb{R}}(S)$ is such that $|\tau(x)| \le |\tau(y)|$, for each $\tau \in \Omega\ (C_{\mathbb{R}}(S))$, then $|x(s)| \le |y(s)|$, for each $s \in S$. Hence $\|x\| \le \|y\|$. And since

$$r(x) = \sup_{\lambda\in\sigma(x)} |\lambda| = \sup_{s\in S}|x(s)| = \|x\|, \tag{3.11}$$

so $\inf\{r(x) : x \in X,\ \|x\| = 1\} = 1 > 0$. □

Let $\ell_\infty(\mathbb{N})$ denote the Banach algebra of all real bounded sequences with the sup-norm. The following two propositions tell us that there is a subalgebra of $\ell_\infty(\mathbb{N})$ which does not contain c_0 but fails to have the fixed point property.

Proposition 3.3. *If E is a subset of $\ell_\infty(\mathbb{N})$ which contains an infinite bounded sequence and the identity, then the Banach subalgebra $B(E)$ of $\ell_\infty(\mathbb{N})$ generated by E fails to have the fixed point property.*

Proof. Let E be a subset of $\ell_\infty(\mathbb{N})$ which contains an infinite bounded sequence $\{z_n\}$ and the identity. It follows that $B(E)$ is unital and abelian. $B(E)$ is infinite dimensional, since the set $\{(\{z_n\})^n : n \in \mathbb{N}\}$ is a linearly independent subset of $B(E)$. Next, we show that $B(E)$ satisfies (i) and (ii) in Theorem 3.1.

Let $a = \{a_1, a_2, a_3, \ldots\}$, $\ b = \{b_1, b_2, b_3, \ldots\} \in B(E)$ be such that $a \ne b$ and $|\tau(a)| \le |\tau(b)|$, for each $\tau \in \Omega(X)$. Define $\tau_n : B(E) \to \mathbb{R}$ by

$$\tau_n(\{x_1, x_2, x_3, \ldots\}) = x_n \tag{3.12}$$

for each $n \in \mathbb{N}$. Hence $\tau_n \in \Omega\ (B(E))$ for each $n \in \mathbb{N}$, and thus

$$|a_n| = |\tau_n(a)| \le |\tau_n(b)| = |b_n| \tag{3.13}$$

for each $n \in \mathbb{N}$. Clearly, $\|a\| \le \|b\|$.

Since for each $x \in B(E)$ we have

$$\|x\| \ge r(x) = \sup_{\lambda\in\sigma(x)} |\lambda| = \sup_{\tau\in\Omega(B(E))} |\tau(x)| \ge \sup_{n\in\mathbb{N}}|\tau_n(x)| = \|x\|, \tag{3.14}$$

so $\inf\{r(x) : x \in X,\ \|x\| = 1\} = 1 > 0$. Now it follows from Theorem 3.1 that $B(E)$ doesn't have the fixed point property. □

Proposition 3.4. *Let $z = \{1/p, 1/p^2, 1/p^3, \ldots\}$ with $p > 1$. Then the Banach subalgebra $B(\{1, z\})$ of $\ell_\infty(\mathbb{N})$ generated by the identity and z does not contain the space c_0.*

Proof. We have

$$A(\{1, z\}) = \left\{ \sum_{i=0}^{n} \alpha_i (z)^i : \alpha_i \in \mathbb{R},\ n \in \mathbb{N} \right\}. \tag{3.15}$$

If $a = \{a_1, a_2, a_3, \ldots\} \in A(\{1, z\})$, then $a = \sum_{i=0}^{N} \alpha_i (z)^i$, for some $N \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}$. It follows that $a_n = \sum_{i=0}^{N} \alpha_i (1/p^n)^i$, for each $n \in \mathbb{N}$. Write $M = \max_{i=1,\ldots,N} |\alpha_i|$. Hence

$$\alpha_0 - M\left(\frac{1}{p^n - 1}\right) \leq a_n \leq \alpha_0 + M\left(\frac{1}{p^n - 1}\right) \tag{3.16}$$

for each $n \in \mathbb{N}$. From the above inequality, and since a is arbitrary, we can see that the sequence $\{1, 1/2, 1/3, \ldots\}$ does not lie in $\overline{A(\{1, z\})} = B(\{1, z\})$. □

4. Results on Complex Banach Algebras

Let X be a unital abelian complex Banach algebra. Consider the following condition.

(A) For each $x \in X$, there exists an element $y \in X$ such that $\tau(y) = \overline{\tau(x)}$, for each $\tau \in \Omega(X)$.

If X satisfies condition (A), then $\varphi(X)$ is a subspace of $C_{\mathbb{C}}(\Omega(X))$ which is closed under the complex conjugation. By using the Stone-Weierstrass theorem for the complex Banach algebra $C_{\mathbb{C}}(S)$ and following the proof of Lemma 2.7, we obtain the following result.

Lemma 4.1. *Let X be a unital abelian complex Banach algebra satisfying (A) and*

$$\inf\{r(x) : x \in X, \ \|x\| = 1\} > 0. \tag{4.1}$$

Then one has the following:

(i) *the Gelfand representation φ is a bounded isomorphism,*

(ii) *the inverse φ^{-1} is also a bounded isomorphism.*

Using Lemma 4.1 we obtain the complex counterpart of Lemma 2.10.

Lemma 4.2. *Let X be an infinite dimensional unital abelian complex Banach algebra satisfying (A) and*

$$\inf\{r(x) : x \in X, \ \|x\| = 1\} > 0. \tag{4.2}$$

Then one has the following:

(i) $\Omega(X)$ *is an infinite set,*

(ii) *if there exists a bounded sequence $\{x_n\}$ in X which contains no convergent subsequences and such that $\{\tau(x_n) : \tau \in \Omega(X)\}$ is finite for each $n \in \mathbb{N}$, then there is an element $x_0 \in X$ with*

$$\{\tau(x_0) : \tau \in \Omega(X)\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}, \tag{4.3}$$

(iii) *there is an element* $x_0 \in X$ *such that* $\{\tau(x_0) : \tau \in \Omega(X)\}$ *is an infinite set,*

(iv) *there exists a sequence* $\{x_n\}$ *in* X *such that* $\{\tau(x_n) : \tau \in \Omega(X)\} \subset [0,1]$, *for each* $n \in \mathbb{N}$, *and* $\{(\widehat{x_n})^{-1}\{1\}\}$ *is a sequence of nonempty pairwise disjoint subsets of* $\Omega(X)$.

By using Lemmas 4.1 and 4.2, and by following the proof of Theorem 3.1, we get the following theorem.

Theorem 4.3. *Let* X *be an infinite dimensional unital abelian complex Banach algebra satisfying (A) and each of the following:*

(i) *if* $x, y \in X$ *is such that* $|\tau(x)| \leq |\tau(y)|$, *for each* $\tau \in \Omega(X)$, *then* $\|x\| \leq \|y\|$,

(ii) $\inf\{r(x) : x \in X,\ \|x\| = 1\} > 0$.

Then X *does not have the fixed point property.*

Acknowledgments

The authors would like to express their thanks to the referees for valuable comments, especially, to whom that provides them Remark 2.8(ii) and Example 2.9 for completeness. This work was supported by the Thailand Research Fund, grant BRG50800016.

References

[1] D. E. Alspach, “A fixed point free nonexpansive map,” *Proceedings of the American Mathematical Society*, vol. 82, no. 3, pp. 423–424, 1981.
[2] A. T.-M. Lau, P. F. Mah, and A. Ülger, “Fixed point property and normal structure for Banach spaces associated to locally compact groups,” *Proceedings of the American Mathematical Society*, vol. 125, no. 7, pp. 2021–2027, 1997.
[3] T. Domínguez Benavides and M. A. Japón Pineda, “Fixed points of nonexpansive mappings in spaces of continuous functions,” *Proceedings of the American Mathematical Society*, vol. 133, no. 10, pp. 3037–3046, 2005.
[4] S. Dhompongsa, W. Fupinwong, and W. Lawton, “Fixed point properties of C^*-algebras,” submitted.
[5] S. K. Berberian, *Fundamentals of Real Analysis*, Springer, New York, NY, USA, 1996.
[6] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, Mass, USA, 1990.