Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 367274, 11 pages
doi:10.1155/2010/367274

Research Article

Approximating Fixed Points of Nonexpansive Nonself Mappings in CAT(0) Spaces

W. Laowang and B. Panyanak

Department of Mathematics, Faculty of Science, Chaing Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to B. Panyanak, banchap@chiangmai.ac.th

Received 23 July 2009; Accepted 30 November 2009

Academic Editor: Tomonari Suzuki

Suppose that K is a nonempty closed convex subset of a complete CAT(0) space X with the nearest point projection P from X onto K. Let $T : K \to X$ be a nonexpansive nonself mapping with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Suppose that $\{x_n\}$ is generated iteratively by $x_1 \in K$, $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$, $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ Δ-converges to some point x^* in $F(T)$. This is an analog of a result in Banach spaces of Shahzad (2005) and extends a result of Dhompongsa and Panyanak (2008) to the case of nonself mappings.

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [3]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. The work by Burago et al. [4] contains a somewhat more elementary treatment, and by Gromov [5] a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [6, 7]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared (see, e.g., [8–19]).

In 2008, Kirk and Panyanak [20] used the concept of Δ-convergence introduced by Lim [21] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence, and Dhompongsa and Panyanak [22] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.

The purpose of this paper is to study the iterative scheme defined as follows. Let K is a nonempty closed convex subset of a complete CAT(0) space X with the nearest point projection P from X onto K. If $T : K \to X$ is a nonexpansive mapping with nonempty fixed point set, and if $\{x_n\}$ is generated iteratively by

$$x_1 \in K, \quad x_{n+1} = P\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big]\big), \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, we show that the sequence $\{x_n\}$ defined by (1.1) Δ-converges to a fixed point of T. This is an analog of a result in Banach spaces of Shahzad [23] and also extends a result of Dhompongsa and Panyanak [22] to the case of nonself mappings. It is worth mentioning that our result immediately applies to any CAT(κ) space with $\kappa \le 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \ge \kappa$ (see [1, page 165]).

2. Preliminaries and Lemmas

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x,y_0)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [24]. In fact (cf. [1, page 163]), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

We now collect some elementary facts about CAT(0) spaces which will be used frequently in the proofs of our main results.

Lemma 2.1. *Let (X,d) be a CAT(0) space.*

(i) *[1, Proposition 2.4] Let K be a convex subset of X which is complete in the induced metric. Then, for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x,P(x)) = \inf\{d(x,y) : y \in K\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from X onto K.*

(ii) *[22, Lemma 2.1(iv)] For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{2.2}$$

one uses the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.2).

(iii) *[22, Lemma 2.4] For $x,y,z \in X$ and $t \in [0,1]$, one has*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td(y,z). \tag{2.3}$$

(iv) *[22, Lemma 2.5] For $x,y,z \in X$ and $t \in [0,1]$, one has*

$$d\big((1-t)x \oplus ty, z\big)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2. \tag{2.4}$$

Let K be a nonempty subset of a CAT(0) space X and let $T : K \to X$ be a mapping. T is called *nonexpansive* if for each $x,y \in K$,

$$d(Tx,Ty) \le d(x,y). \tag{2.5}$$

A point $x \in K$ is called a fixed point of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. The existence of fixed points for nonexpansive nonself mappings in a CAT(0) space was proved by Kirk [6] as follows.

Theorem 2.2. *Let K be a bounded closed convex subset of a complete CAT(0) space X. Suppose that $T : K \to X$ is a nonexpansive mapping for which*

$$\inf\{d(x,T(x)) : x \in K\} = 0. \tag{2.6}$$

Then T has a fixed point in K.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n). \tag{2.7}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}, \tag{2.8}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.9}$$

It is known (see, e.g., [12, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of Δ-convergence.

Definition 2.3 (see [20, 21]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one writes $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

The following lemma was proved by Dhompongsa and Panyanak (see [22, Lemma 2.10]).

Lemma 2.4. *Let K be a closed convex subset of a complete CAT(0) space X, and let $T : K \to X$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in K such that $\lim_n d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

We now turn to a wider class of spaces, namely, the class of hyperbolic spaces, which contains the class of CAT(0) spaces (see Lemma 2.8).

Definition 2.5 (see [16]). A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is such that

(W1) $d(z, W(x, y, \alpha)) \le (1-\alpha)d(z, x) + \alpha d(z, y)$;

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \le (1-\alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X, \alpha, \beta \in [0,1]$.

It follows from (W1) that for each $x, y \in X$ and $\alpha \in [0,1]$,

$$d(x, W(x, y, \alpha)) \le \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) \le (1-\alpha)d(x, y). \tag{2.10}$$

In fact, we have

$$d(x, W(x, y, \alpha)) = \alpha d(x, y), \qquad d(y, W(x, y, \alpha)) = (1-\alpha)d(x, y), \tag{2.11}$$

since if

$$d(x, W(x, y, \alpha)) < \alpha d(x, y) \quad \text{or} \quad d(y, W(x, y, \alpha)) < (1-\alpha)d(x, y), \tag{2.12}$$

we get

$$\begin{aligned} d(x,y) &\le d(x,W(x,y,\alpha)) + d(W(x,y,\alpha),y) \\ &< \alpha d(x,y) + (1-\alpha)d(x,y) \\ &= d(x,y), \end{aligned} \tag{2.13}$$

which is a contradiction. By comparing between (2.2) and (2.11), we can also use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x,y,\alpha)$ in a hyperbolic space (X,d,W).

Definition 2.6 (see [16]). The hyperbolic space (X,d,W) is called uniformly convex if for any $r > 0$, and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$\left.\begin{aligned} d(x,a) &\le r \\ d(y,a) &\le r \\ d(x,y) &\ge \varepsilon r \end{aligned}\right\} \Longrightarrow d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1-\delta)r. \tag{2.14}$$

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\delta := \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity.

Lemma 2.7 (see [16, Lemma 7]). *Let (X,d,W) be a uniformly convex hyperbolic with modulus of uniform convexity η. For any $r > 0$, $\varepsilon \in (0,2]$, $\lambda \in [0,1]$ and $a,x,y \in X$,*

$$\left.\begin{aligned} d(x,a) &\le r \\ d(y,a) &\le r \\ d(x,y) &\ge \varepsilon r \end{aligned}\right\} \Longrightarrow d((1-\lambda)x \oplus \lambda y, a) \le (1 - 2\lambda(1-\lambda)\eta(r,\varepsilon))r. \tag{2.15}$$

Lemma 2.8 (see [16, Proposition 8]). *Assume that X is a CAT(0) space. Then X is uniformly convex, and*

$$\eta(r,\varepsilon) = \frac{\varepsilon^2}{8} \tag{2.16}$$

is a modulus of uniform convexity.

The following result is a characterization of uniformly convex hyperbolic spaces which is an analog of Lemma 1.3 of Schu [25]. It can be applied to a CAT(0) space as well.

Lemma 2.9. *Let (X,d,W) be a uniformly convex hyperbolic space with modulus of convexity η, and let $x \in X$. Suppose that η increases with r (for a fixed ε) and suppose that $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_n d(x_n,x) \le r$, $\limsup_n d(y_n,x) \le r$, and $\lim_n d((1-t_n)x_n \oplus t_n y_n, x) = r$ for some $r \ge 0$. Then*

$$\lim_{n\to\infty} d(x_n,y_n) = 0. \tag{2.17}$$

Proof. The case $r = 0$ is trivial. Now suppose $r > 0$. If it is not the case that $d(x_n, y_n) \to 0$ as $n \to \infty$, then there are subsequences, denoted by $\{x_n\}$ and $\{y_n\}$, such that

$$\inf_n d(x_n, y_n) > 0. \tag{2.18}$$

Choose $\varepsilon \in (0,1]$ such that

$$d(x_n, y_n) \geq \varepsilon(r+1) > 0 \quad \forall n \in \mathbb{N}. \tag{2.19}$$

Since $0 < b(1-c) \leq 1/2$ and $0 < \eta(r,\varepsilon) \leq 1$, $0 < 2b(1-c)\eta(r,\varepsilon) \leq 1$. This implies $0 \leq 1 - 2b(1-c)\eta(r,\varepsilon) < 1$. Choose $R \in (r, r+1)$ such that

$$\big(1 - 2b(1-c)\eta(r,\varepsilon)\big)R < r. \tag{2.20}$$

Since

$$\limsup_n d(x_n, x) \leq r, \quad \limsup_n d(y_n, x) \leq r, \quad r < R, \tag{2.21}$$

there are further subsequences again denoted by $\{x_n\}$ and $\{y_n\}$, such that

$$d(x_n, x) \leq R, \quad d(y_n, x) \leq R, \quad d(x_n, y_n) \geq \varepsilon R \quad \forall n \in \mathbb{N}. \tag{2.22}$$

Then by Lemma 2.7 and (2.20),

$$\begin{aligned} d\big((1-t_n)x_n \oplus t_n y_n, x\big) &\leq \big(1 - 2t_n(1-t_n)\eta(R,\varepsilon)\big)R \\ &\leq \big(1 - 2b(1-c)\eta(r,\varepsilon)\big)R < r \end{aligned} \tag{2.23}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} d\big((1-t_n)x_n \oplus t_n y_n, x\big) < r, \tag{2.24}$$

which contradicts to the hypothesis. □

3. Main Results

In this section, we prove our main theorems.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to X$ be a nonexpansive mapping with $x^* \in F(T) := \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Then $\lim_{n\to\infty} d(x_n, x^*)$ exists.*

Proof. By Lemma 2.1(i) the nearest point projection $P : X \to K$ is nonexpansive. Then

$$\begin{aligned}
d(x_{n+1}, x^*) &= d\big(P\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big]\big), Px^*\big) \\
&\le d\big((1-\alpha_n)x_n \oplus \alpha_n TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], x^*\big) \\
&\le (1-\alpha_n)d(x_n, x^*) + \alpha_n d\big(TP\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], Tx^*\big) \\
&\le (1-\alpha_n)d(x_n, x^*) + \alpha_n d\big(P\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], x^*\big) \\
&\le (1-\alpha_n)d(x_n, x^*) + \alpha_n \big[(1-\beta_n)d(x_n, x^*) + \beta_n d(Tx_n, Tx^*)\big] \\
&\le (1-\alpha_n)d(x_n, x^*) + \alpha_n \big[(1-\beta_n)d(x_n, x^*) + \beta_n d(x_n, x^*)\big] \\
&= d(x_n, x^*).
\end{aligned} \tag{3.1}$$

Consequently, we have

$$d(x_n, x^*) \le d(x_1, x^*) \quad \forall n \ge 1. \tag{3.2}$$

This implies that $\{d(x_n, x^*)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_n d(x_n, x^*)$ exists. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1). *Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.3}$$

Proof. Let $x^* \in F(T)$. Then, by Theorem 3.1, $\lim_n d(x_n, x^*)$ exists. Let

$$\lim_{n\to\infty} d(x_n, x^*) = r. \tag{3.4}$$

If $r = 0$, then by the nonexpansiveness of T the conclusion follows. If $r > 0$, we let $y_n = P[(1-\beta_n)x_n \oplus \beta_n Tx_n]$. By Lemma 2.1(iv) we have

$$\begin{aligned}
d(y_n, x^*)^2 &= d\big(P\big[(1-\beta_n)x_n \oplus \beta_n Tx_n\big], Px^*\big)^2 \\
&\le d\big((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*\big)^2 \\
&\le (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\
&\le (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(x_n, x^*)^2 \\
&= d(x_n, x^*)^2.
\end{aligned} \tag{3.5}$$

Therefore

$$d(y_n, x^*) \le d\big((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*\big) \le d(x_n, x^*). \tag{3.6}$$

It follows from (3.6) and Lemma 2.1(iv) that

$$\begin{aligned} d(x_{n+1},x^*)^2 &= d\left(P\left[(1-\alpha_n)x_n \oplus \alpha_n T y_n\right], P x^*\right)^2 \\ &\le d\left((1-\alpha_n)x_n \oplus \alpha_n T y_n, x^*\right)^2 \\ &\le (1-\alpha_n)d(x_n,x^*)^2 + \alpha_n d\left(T y_n, x^*\right)^2 - \alpha_n(1-\alpha_n)d\left(x_n, T y_n\right)^2 \\ &\le (1-\alpha_n)d(x_n,x^*)^2 + \alpha_n d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d\left(x_n, T y_n\right)^2 \\ &= d(x_n,x^*)^2 - \alpha_n(1-\alpha_n)d\left(x_n, T y_n\right)^2. \end{aligned} \tag{3.7}$$

Therefore

$$d(x_{n+1},x^*)^2 \le d(x_n,x^*)^2 - W(\alpha_n)d\left(x_n, T y_n\right)^2, \tag{3.8}$$

where $W(\alpha) = \alpha(1-\alpha)$. Since $\alpha_n \in [\varepsilon, 1-\varepsilon], W(\alpha_n) \ge \varepsilon^2$.

By (3.8), we have

$$\varepsilon^2 \sum_{n=1}^{\infty} d\left(x_n, T y_n\right)^2 \le d(x_1,x^*)^2 < \infty. \tag{3.9}$$

This implies $\lim_{n\to\infty} d(x_n, T y_n) = 0$.

Since T is nonexpansive, we get that $d(x_n,x^*) \le d(x_n,Ty_n) + d(y_n,x^*)$, and hence

$$r \le \liminf_{n\to\infty} d\left(y_n, x^*\right). \tag{3.10}$$

On the other hand, we can get from (3.6) that

$$\limsup_{n\to\infty} d\left(y_n, x^*\right) \le r. \tag{3.11}$$

Thus $\lim_n d(y_n,x^*) = r$. This fact and (3.6) imply

$$\lim_{n\to\infty} d\left((1-\beta_n)x_n \oplus \beta_n T x_n, x^*\right) = r. \tag{3.12}$$

Since T is nonexpansive,

$$\limsup_{n\to\infty} d(T x_n, x^*) \le r. \tag{3.13}$$

It follows from (3.4), (3.12), (3.13), and Lemma 2.9 that

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0. \tag{3.14}$$

This completes the proof. □

The following theorem is an analog of [23, Theorem 3.5] and extends [22, Theorem 3.3] to nonself mappings.

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1)*. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. By Theorem 3.2, $\lim_n d(x_n, Tx_n) = 0$. It follows from (3.2) that $\{d(x_n, v)\}$ is bounded and decreasing for each $v \in F(T)$, and so it is convergent. By Lemma 2.4, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that the sequence $\{x_n\}$ Δ-converges to an element of $F(T)$. □

We now state two strong convergence theorems. Recall that a mapping $T : K \to X$ is said to satisfy *Condition I* ([26]) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \forall x \in K. \tag{3.15}$$

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1)*. Suppose that T satisfies condition I. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 3.5. *Let K be a nonempty compact convex subset of a complete CAT(0) space X and let $T : K \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.1)*. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Another result in [23] is that the author obtains a common fixed point theorem of two nonexpansive self-mappings. The proof is metric in nature and carries over to the present setting. Therefore, we can state the following result.

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S, T : K \to K$ be two nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n S\left[(1-\beta_n)x_n \oplus \beta_n T x_n\right]. \tag{3.16}$$

Then $\{x_n\}$ Δ-converges to a common fixed point of S and T.

Acknowledgments

The authors are grateful to Professor Sompong Dhompongsa for his suggestions and advices during the preparation of the article. The second author was supported by the Commission on Higher Education and Thailand Research Fund under Grant MRG5280025. This work is dedicated to Professor Wataru Takahashi.

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