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Research Article

Fixed Point Theorems for Set-Valued Contraction Type Maps in Metric Spaces

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We first give some fixed point results for set-valued self-map contractions in complete metric spaces. Then we derive a fixed point theorem for nonself set-valued contractions which are metrically inward. Our results generalize many well-known results in the literature.

1. Introduction and Preliminaries

Let (X, d) be a metric space and let $\mathrm{CB}(X)$ denote the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\left\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\right\} \tag{1.1}$$

for every $A, B \in \mathrm{CB}(X)$, where $d(u, B) = \inf\{d(u, y) : y \in B\}$. In 1969, Nadler [1] extended the Banach contraction principle [2] to set-valued mappings.

Theorem 1.1 (Nadler [1]). *Let (X, d) be a complete metric space and let $T : X \to \mathrm{CB}(X)$ be a set-valued map. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \le r d(x, y) \tag{1.2}$$

for all $x, y \in X$. Then T has a fixed point.

Mizoguchi and Takahashi [3] proved the following generalization of Theorem 1.1.

Corollary 1.2 (Mizoguchi and Takahashi [3]). *Let* (X, d) *be a complete metric space and let* $T : X \to CB(X)$ *be a set-valued map satisfying*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \quad \text{for each } x, y \in X, \tag{1.3}$$

where $\alpha : [0, \infty) \to [0, 1)$ *satisfies* $\limsup_{s \to t^+} \alpha(s) < 1$ *for each* $t \in [0, \infty)$. *Then* T *has a fixed point.*

Also, Reich [4] has proved that if for each $x \in X$, Tx is nonempty and compact, then the above result holds under the weaker condition $\limsup_{s \to t^+} \alpha(s) < 1$ for each $t > 0$. To set up our results in the next section, we introduce some definitions and facts.

Definition 1.3. Throughout the paper, let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) $\psi(s) = 0 \Leftrightarrow s = 0$;

(b) ψ is lower semicontinuous and nondecreasing;

(c) $\limsup_{s \to 0+}(s/\psi(s)) < \infty$.

Theorem 1.4 (Bae [5]). *Let* (M, ρ) *be a complete metric space,* $\phi : M \to [0, \infty)$ *a lower semicontinuous function, and* $\varphi : [0, \infty) \to [0, \infty)$ *a lower semicontinuous function such that* $\varphi(t) > 0$ *for* $t > 0$ *and*

$$\limsup_{s \to 0+} \frac{s}{\varphi(s)} < \infty. \tag{1.4}$$

Let $g : M \to M$ *be a map such that for any* $x \in M$, $\rho(x, gx) \leq \phi(x)$ *and*

$$\varphi(\rho(x, gx)) \leq \phi(x) - \phi(g(x)) \tag{1.5}$$

hold. Then g *has a fixed point in* M.

Definition 1.5. Let (X, d) be a complete metric space and D be a nonempty closed subset of X.

(i) Set

$$\begin{aligned} MI_D(x) = \{z \in X : z = x \text{ or there exits } y \in D \text{ satisfying } y \neq x, \\ d(x, z) = d(x, y) + d(y, z)\}. \end{aligned} \tag{1.6}$$

Then $MI_D(x)$ is called the metrically inward set of D at x (see [5]);

(ii) Let $T : D \to CB(X)$ be a set-valued map. T is said to be *metricaly inward*, if for each $x \in D$,

$$Tx \subseteq MI_D(x). \tag{1.7}$$

In Section 2 we generalize Corollary 1.2 and Theorem 1.4.

2. Extension of Mizoguchi-Takahashi's Theorem

In the first result of this section, we use the technique in [6] to extend Corollary 1.2.

Theorem 2.1. *Let* (X, d) *be a complete metric space and let* $T : X \to CB(X)$ *be a set-valued map satisfying*

$$\psi(H(Tx, Ty)) \leq \alpha(\psi(d(x, y)))\psi(d(x, y)), \quad \text{for each } x, y \in X, \tag{2.1}$$

where $\alpha : [0, \infty) \to [0, 1)$ *satisfies* $\limsup_{s \to t^+} \alpha(s) < 1$ *for each* $t \in [0, \infty)$ *and* $\psi \in \Psi$. *Then* T *has a fixed point.*

Proof. Define a function $\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = (\alpha(t) + 1)/2$. Then $\alpha(t) < \beta(t)$ and $\limsup_{s \to t^+} \beta(s) < 1$ for all $t \in [0, \infty)$. Since ψ is nondecreasing, then from (1.3), for each $x \neq y$, we have

$$\begin{aligned} &\max\left\{\sup_{u \in Tx} \psi(d(u, Ty)), \sup_{v \in Ty} \psi(d(v, Tx))\right\} \\ &\quad = \max\left\{\psi\left(\sup_{u \in Tx} d(u, Ty)\right), \psi\left(\sup_{v \in Ty} d(v, Tx)\right)\right\} \\ &\quad = \psi(H(Tx, Ty)) < \beta(\psi(d(x, y)))\psi(d(x, y)). \end{aligned} \tag{2.2}$$

Hence for each $x \in X$ and $y \in Tx$, there exists an element $z \in Ty$ such that $\psi(d(y, z)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$. Thus we can define a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in Tx_n, \quad \psi(d(x_{n+1}, x_{n+2})) \leq \beta(\psi(d(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})), \tag{2.3}$$

for each $n \in \mathbb{N}$. Let us show that $\{x_n\}$ is convergent. Since $\beta(t) < 1$ for each $t \in [0, \infty)$, then $\{\psi(d(x_n, x_{n+1}))\}$ is a nonincreasing sequence of non-negative numbers and so is convergent to a real number, say r_0. Since $\limsup_{s \to r_0^+} \beta(s) < 1$ and $\beta(r_0) < 1$, there exist $r \in [0, 1)$ and $\epsilon > 0$ such that $\beta(s) \leq r$ for all $s \in [r_0, r_0 + \epsilon]$. We can take $n_0 \in \mathbb{N}$ such that $r_0 \leq \psi(d(x_n, x_{n+1})) \leq r_0 + \epsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Since

$$\psi(d(x_{n+1}, x_{n+2})) \leq \beta(\psi(d(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})) \leq r\psi(d(x_n, x_{n+1})) \tag{2.4}$$

for all $n \geq n_0$, then we have $r_0 \leq rr_0$ and so $r_0 = 0$ (note that $r < 1$). If $d(x_m, x_{m+1}) = 0$ for some $m \in \mathbb{N}$, then $d(x_n, x_{n+1}) = 0$ for each $n \geq m$ (note that $\{\psi(d(x_n, x_{n+1}))\}$ is nonincreasing). Thus $\{x_n\}$ is eventually constant, so we have a fixed point of T (note that $x_{n+1} \in Tx_n$). Now, we assume that $d(x_n, x_{n+1}) \neq 0$ for each $n \in \mathbb{N}$. Since $\{\psi(d(x_n, x_{n+1}))\}$ is decreasing and ψ is nondecreasing, then the nonnegative sequence $d(x_n, x_{n+1})$ converges to some nonnegative real number τ. Since ψ is nondecreasing and $d(x_n, x_{n+1})$ is nonincreasing, then $\psi(\tau) \leq \psi(d(x_n, x_{n+1}))$ for each $n \in \mathbb{N}$. Thus

$$\psi(\tau) \leq \lim_{n \to \infty} \psi(d(x_n, x_{n+1})) = r_0 = 0. \tag{2.5}$$

Thus $\tau = 0$ (note that $\psi(\tau) = 0$ implies $\tau = 0$). Also we have (note $\psi(d(x_{n+1}, x_{n+2})) \leq r\psi(d(x_n, x_{n+1}))$ for $n \geq n_0$)

$$\sum_{1}^{\infty}\psi(d(x_n, x_{n+1})) \leq \sum_{1}^{n_0}\psi(d(x_n, x_{n+1})) + \sum_{1}^{\infty} r^n \psi(d(x_{n_0}, x_{n_0+1})) < \infty. \tag{2.6}$$

Since

$$\limsup_{n\to\infty}\frac{d(x_n, x_{n+1})}{\psi(d(x_n, x_{n+1}))} \leq \limsup_{s\to 0+}\frac{s}{\psi(s)} < \infty, \tag{2.7}$$

then $\sum_{1}^{\infty} d(x_n, x_{n+1}) < \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $x_0 \in X$. Since ψ is lower semicontinuous and nondecreasing (recall also from above that $\lim_{n\to\infty}\psi(d(x_n, x_{n+1})) = 0$), then

$$\begin{aligned}
\psi(d(x_0, Tx_0)) &\leq \liminf_{n\to\infty}\psi(d(x_{n+1}, Tx_0)) \leq \liminf_{n\to\infty}\psi(H(Tx_n, Tx_0)) \\
&\leq \liminf_{n\to\infty}\beta(\psi(d(x_n, x_0)))\psi(d(x_n, x_0)) \leq \liminf_{n\to\infty}\psi(d(x_n, x_0)) \\
&= \lim_{s\to 0^+}\psi(s) = \lim_{n\to\infty}\psi(d(x_n, x_{n+1})) = 0,
\end{aligned} \tag{2.8}$$

and this with Tx_0 closed and (a) of Definition 1.3 implies $x_0 \in Tx_0$. □

Corollary 2.2. *Let (X, d) be a complete metric space and let $T : X \to \mathrm{CB}(X)$ be a set-valued map satisfying*

$$\psi(H(Tx, Ty)) \leq \psi(d(x, y)) - \tilde{\varphi}(\psi(d(x, y))), \quad \textit{for each } x, y \in X, \tag{2.9}$$

where $\psi \in \Psi$ and $\tilde{\varphi} : [0, \infty) \to [0, \infty)$ satisfying $\liminf_{s\to t+}(\tilde{\varphi}(s)/\psi(s)) > 0$ for each $t \in [0, \infty)$. Then T has a fixed point.

Proof. Let $\alpha(s) = 1 - \tilde{\varphi}(s)/\psi(s)$ and apply Theorem 2.1. □

In the following, we present a fixed point theorem for nonself set-valued contraction type maps which are metrically inward.

Theorem 2.3. *Let D be a nonempty closed subset of a complete metric space (X, d) and $T : D \to \mathrm{CB}(X)$ be a set-valued map satisfying*

$$\psi(H(Tx, Ty)) \leq \psi(d(x, y)) - \tilde{\varphi}(\psi(d(x, y))), \quad \textit{for each } x, y \in X, \tag{2.10}$$

for which $\psi \in \Psi$ is continuous and

$$\psi(r - s) + \psi(s + t) \leq \psi(r) + \psi(t), \quad \textit{for each } 0 \leq s \leq r \leq s + t. \tag{2.11}$$

Assume that $\widetilde{\varphi} : [0,\infty) \to [0,\infty)$ *is a lower semicontinuous function satisfying* $\liminf_{s\to 0+}(\widetilde{\varphi}(s)/\psi(s)) > 0$ *and* $\widetilde{\varphi}(s) > 0$ *for* $s > 0$*. Suppose that* T *is metrically inward on* D*. Then* T *has a fixed point in* D*.*

Proof. We first show that $\limsup_{s\to 0+}(s/\widetilde{\varphi}(s)) < \infty$. On the contrary, we assume that there exists a sequence $s_n \to 0+$ for which

$$\limsup_{n\to\infty}\frac{s_n}{\widetilde{\varphi}(s_n)} = \limsup_{n\to\infty}\frac{s_n/\psi(s_n)}{\widetilde{\varphi}(s_n)/\psi(s_n)} = \infty. \tag{2.12}$$

Since $\liminf_{n\to\infty}(\widetilde{\varphi}(s_n)/\psi(s_n)) > 0$, then we get $\limsup_{n\to\infty}(s_n/\psi(s_n)) = \infty$, which contradicts our assumption on ψ. Let $M = \{(x,y) : x \in X,\ y \in Tx\}$ be the graph of T. Let $\rho : M \times M \to [0,\infty)$ be given by

$$\rho((x,z),(u,v)) = \max\{\psi(d(x,u)),\psi(d(z,v))\}. \tag{2.13}$$

We show that (M,ρ) is a complete metric space. First note that since $\psi(s) = 0 \Leftrightarrow s = 0$ then $\rho((x,z),(u,v)) = 0 \Leftrightarrow (x,z) = (u,v)$. Clearly, $\rho((x,z),(u,v)) = \rho((u,v),(x,z))$. Now we show the triangle inequality. From (2.11), we have $\psi(r+t) \le \psi(r) + \psi(t),\ \forall r,t \ge 0$. Hence,

$$\begin{aligned}
&\rho((x,z),(r,s)) + \rho((r,s),(u,v))\\
&\quad = \max\{\psi(d(x,r)),\psi(d(z,s))\} + \max\{\psi(d(r,u)),\psi(d(s,v))\}\\
&\quad \ge \max\{\psi(d(x,r)) + \psi(d(r,u)),\psi(d(z,s)) + \psi(d(s,v))\}\\
&\quad \ge \max\{\psi(d(x,r) + d(r,u)),\psi(d(z,s) + d(s,v))\}\\
&\quad \ge \max\{\psi(d(x,u)),\psi(d(z,v))\} = \rho((x,z),(u,v)).
\end{aligned} \tag{2.14}$$

To prove the completeness of ρ, we first need to show that T is Hausdorff continuous. To prove this, let (x_n) be a sequence in D such that $x_n \to x \in D$. Since ψ is continuous at 0, then $\lim_{n\to\infty}\psi(d(x_n,x)) = \psi(0) = 0$. Hence from (2.10), we get $\lim_{n\to\infty}\psi(H(Tx_n,Tx)) = 0$. We claim that $\lim_{n\to\infty}H(Tx_n,Tx) = 0$ (and then we are finished). On the contrary, assume that there exist $\epsilon > 0$ and a subsequence x_{n_k} such that $H(Tx_{n_k},Tx) \ge \epsilon$, k=1,2,3,... . Since ψ is nondecreasing, then $\psi(H(Tx_{n_k},Tx)) \ge \psi(\epsilon) > 0$, a contradiction. Now, let (x_n,z_n) be a Cauchy sequence in M with respect to ρ. Then $\{x_n\}$ and $\{z_n\}$ are Cauchy sequences in the complete metric space (X,d). Then there exist $x,z \in X$ such that $d(x_n,x) \to 0$ and $d(z_n,z) \to 0$. Since $z_n \in Tx_n$ and T is Hausdorff continuous, then $z \in Tx$. Thus $(x,z) \in M$ and $\rho((x_n,z_n),(x,z)) \to 0$. Therefore, (M,ρ) is a complete metric space. Suppose that T has no fixed point. Then for each $(x,z) \in M$, we have $x \ne z$. Since $z \in Tx \subseteq \mathrm{MI}_D(x)$, we can choose $u \in D$ such that $u \ne x$ and

$$d(x,z) = d(x,u) + d(u,z). \tag{2.15}$$

Since T satisfies (2.10) and ψ is continuous, then we can choose $v \in Tu$ such that

$$\psi(d(z,v)) \le \psi(d(x,u)) - \frac{1}{2}\tilde{\varphi}(\psi(d(x,u))). \tag{2.16}$$

Let $\varphi(t) = \tilde{\varphi}(t)/2$. Then by combining (2.15) and (2.16), we get

$$\begin{aligned}\varphi(\psi(d(x,u))) &\le \psi(d(x,u)) - \psi(d(z,v)) \\ &= \psi(d(x,z) - d(u,z)) - \psi(d(z,v)).\end{aligned} \tag{2.17}$$

From (2.11), we have (note that ψ is nondecreasing)

$$\begin{aligned}\psi(d(x,z) - d(u,z)) - \psi(d(z,v)) &\le \psi(d(x,z)) - \psi(d(z,v) + d(u,z)) \\ &\le \psi(d(x,z)) - \psi(d(u,v)).\end{aligned} \tag{2.18}$$

Thus (2.17) and (2.18) yield

$$\varphi(\psi(d(x,u))) \le \psi(d(x,z)) - \psi(d(u,v)). \tag{2.19}$$

Since $\rho((x,z),(u,v)) = \max\{\psi(d(x,u)), \psi(d(z,v))\} = \psi(d(x,u)) \le \psi(d(x,z)) \equiv \phi(x,z)$, by defining $g : M \to M$ by $g(x,z) = (u,v)$, from Theorem 1.4, g must have a fixed point, say (x_0, z_0). Then $(x_0, z_0) = g(x_0, z_0) = (u_0, v_0)$. Hence $x_0 = u_0$. This is a contradiction. Therefore, T has a fixed point. □

Remark 2.4. Note that Theorem 2.3 does not follow from Theorem 3.3 of Bae [5] by replacing the metric d by $\psi(d)$. In Theorem 2.3, we assume T is metrically inward with respect to d but to apply Theorem 3.3 of [5] with $\psi(d)$ rather than d, we need T to be metrically inward with respect to $\psi(d)$.

Letting $\psi(s) = s$ for each $s \in [0,\infty)$, we get the following corollary due to Bae [5].

Corollary 2.5. *Let D be a nonempty closed subset of a complete metric space (X,d) and $T : D \to \mathrm{CB}(X)$ be a set-valued map satisfying*

$$H(Tx,Ty) \le d(x,y) - \tilde{\varphi}(d(x,y)), \quad \text{for each } x,y \in X, \tag{2.20}$$

for which $\tilde{\varphi} : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function satisfying $\liminf_{s\to 0+}(\tilde{\varphi}(s)/s) > 0$. *Suppose that T is metrically inward on D. Then T has a fixed point in D.*

Examples 2.6. Let $\psi : [0,\infty) \to [0,\infty)$ be a differentiable function with $\psi(0) = 0$ such that ψ' is positive and decreasing in $(0,\infty)$ and $\lim_{s\to 0+}\psi'(s) = \infty$. Now we show that (ψ) satisfies all the conditions of Theorem 2.3. Obviously, ψ is continuous and increasing. Since $\lim_{s\to 0+}(1/\psi'(s)) = 0$, then by L'Hopital's rule $\lim_{s\to 0+}(s/\psi(s)) = 0$. Thus $\limsup_{s\to 0+}(s/\psi(s)) < \infty$. Now we prove that for each $0 \le t \le r$, $\psi(r+t) \le \psi(r) + \psi(t)$. To show this let $h(t) = \psi(r) + \psi(t) - \psi(r+t)$ *for* $0 \le t \le r$. Then $h'(t) = \psi'(t) - \psi'(r+t) > 0$.

Since $h(0) = 0$ and h is increasing, we get $h(t) \geq 0$ for each $0 \leq t \leq r$ and we are done. Finally, we show that for each $0 \leq s \leq r \leq s + t$, we have $\psi(r - s) + \psi(s + t) \leq \psi(r) + \psi(t)$. Let $k(s) = \psi(r)+\psi(t)-\psi(r-s)+\psi(s+t)$ for $0 \leq s \leq r$. Then $k'(s) = \psi'(r-s)-\psi'(s+t)$. If $r \leq t$, then $k'(s) > 0$. Since $k(0) = 0$, we obtain $k(s) \geq 0$ for each $0 \leq s \leq r$ and we are finished. In the case, $r > t$, $k'(s) = 0$ if and only if $s = (r-t)/2$. Since $k'(s) > 0$ for $0 < s < (r-t)/2$ and $k'(s) < 0$ for $(r - t)/2 < s \leq t$, then $\inf_{0 \leq s \leq r} k(s) = \min(k(0), k(r)) = \min(0, \psi(r) + \psi(t) - \psi(r + t)) = 0$, and we are finished (note that we proved above that $\psi(r) + \psi(t) - \psi(r + t) \geq 0$).

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