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Research Article

On Mappings with Contractive Iterate at a Point in Generalized Metric Spaces

Ljiljana Gajić and Zagorka Lozanov-Crvenković

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

Correspondence should be addressed to Ljiljana Gajić, gajic@dmi.uns.ac.rs

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Using the setting of generalized metric space, the so-called G-metric space, fixed point theorems for mappings with a contractive and a generalized contractive iterate at a point are proved. These results generalize some comparable results in the literature. A common fixed point result is also proved.

1. Introduction

Sehgal in [1] proved fixed point theorem for mappings with a contractive iterate at a point and therefore generalized a well-known Banach theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous mapping with property that for every $x \in X$ there exists $n(x) \in \mathbb{N}$ so that for every $y \in X$*

$$d\left(T^{n(x)}x, T^{n(x)}y\right) \leq q \cdot d(x, y), \quad \text{where } q \in [0, 1). \tag{1.1}$$

Then T has a unique fixed point u in X and $\lim_k T^k(x_0) = u$, for each $x_0 \in X$.

Guseman [2] extended Sehgal's result by removing the condition of continuity of T and weakening (1.1) to hold on some subset B of X such that $T(B) \subseteq B$, where, for some $x_0 \in B$, B contains the closure of the iterates of x_0. Further extensions appear in [3, 4]. Our aim in this study is to show that these results are valid in more general class of spaces.

In 1963, S. Gähler introduced the notion of 2-metric spaces but different authors proved that there is no relation between these two function and there is no easy relationship between results obtained in the two settings. Because of that, Dhage [5] introduced a new concept of the measure of nearness between three or more object. But topological structure of so called D-metric spaces was incorrect. Finally, Mustafa and Sims [6] introduced correct definition of generalized metric space as follows.

Definition 1.2 (see [6]). Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$;

(G2) $0 < G(x,x,y)$; for all $x,y \in X$, with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$, with $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, (symmetry in all three variables);

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Clearly these properties are satisfied when $G(x,y,z)$ is the perimeter of triangle with vertices at x, y, and $z \in \mathbb{R}^2$, moreover taking a in the interior of the triangle shows that $(G5)$ is the best possible.

Example 1.3. Let (X,d) be an ordinary metric apace, then (X,d) can define G-metrics on X by

(E_s) $G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z)$,

(E_m) $G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$.

Example 1.4 (see [6]). Let $x = \{a,b\}$. Define G on $X \times X \times X$ by

$$G(a,a,a) = G(b,b,b) = 0, \quad G(a,a,b) = 1, \quad G(a,b,b) = 2, \tag{1.2}$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear the (X,G) is a G-metric space.

Definition 1.5 (see [6]). Let (X,G) be a G-metric space, and let $\{x_n\}$ be sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x

Thus, if $x_n \to x$ in a G-metric space (X,G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$.

Definition 1.6 (see [6]). Let (X,G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq N$; that is, if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

A G-metric space (X,G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Proposition 1.7 (see [6]). *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.8 (see [6]). A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$, for all $x, y \in X$.

Proposition 1.9 (see [6]). *Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1.3}$$

Note that if (X, G) is a symmetric G-metric space, then

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{1.4}$$

However, if (X, G) is nonsymmetric, then by the G-metric properties it follows that

$$\frac{3}{2}G(x, y, y) \le d_G(x, y) \le 3G(x, y, y), \quad \forall x, y \in X, \tag{1.5}$$

and that in general these inequalities cannot be improved.

Proposition 1.10 (see [6]). *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

In recent years a lot of interesting papers were published with fixed point results in G-metric spaces, see [7–18]. This paper is our contribution to the fixed point theory in G-metric spaces.

2. Fixed Point Results

Let (X, G) be a G-metric space, $f : X \to X$ a mapping, $B \subseteq X$ such that for some $q \in (0, 1)$ and each for $x \in B$ there exists a positive integer $n = n(x)$ such that

$$\begin{aligned} &G\left(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\right) \\ &\quad \le q \cdot \max\left\{G(z, x, x), G\left(z, f^{n(x)}(x), f^{n(x)}(x)\right), G\left(f^{n(x)}(z), x, x\right)\right\} \end{aligned} \tag{2.1}$$

for all $z \in B$. Then we write $f \in \langle 1 \rangle$. If

$$\begin{aligned} &G\left(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\right) \\ &\quad \le q \cdot \max\left\{G(z, x, x), \frac{1}{2}\left[G\left(z, f^{n(x)}(z), f^{n(x)}(z)\right) + G\left(x, f^{n(x)}(x), f^{n(x)}(x)\right)\right],\right. \\ &\qquad\qquad \left.\frac{1}{2}\left[G\left(z, f^{n(x)}(x), f^{n(x)}(x)\right) + G\left(f^{n(x)}(z), x, x\right)\right]\right\} \end{aligned} \tag{2.2}$$

for all $z \in B$, we write $f \in \langle 2 \rangle$.

Theorem 2.1. *Let $f \in \langle 1 \rangle$ or $f \in \langle 2 \rangle$. Let $B \subseteq X$, with $f(B) \subseteq B$. If there exists $u \in B$ such that for $n = n(u)$, $f^n(u) = u$, then u is the unique fixed point of f in B. Moreover, $f^k(y_0) \to u$, $k \to \infty$, for any $y_0 \in B$ for $f \in \langle 1 \rangle$, and for $f \in \langle 2 \rangle$ if $q < 2/3$.*

Proof. If (X, G) is a symmetric space than $d_G(z,x) = 2G(z,x,x)$ and (2.1) becomes

$$d_G\left(f^{n(x)}(z), f^{n(x)}(x)\right) \leq q \max\left\{d_G(z,x), d_G\left(z, f^{n(x)}(x)\right), d_G\left(x, f^{n(x)}(z)\right)\right\}, \tag{2.3}$$

and (2.2) becomes

$$\begin{aligned} d_G\left(f^{n(x)}(z), f^{n(x)}(x)\right) \leq q \max\Bigg\{ & d_G(z,x), \frac{1}{2}\left[d_G\left(z, f^{n(x)}(z)\right) + d_G\left(x, f^{n(x)}(x)\right)\right], \\ & \frac{1}{2}\left[d_G\left(z, f^{n(x)}(x)\right) + d_G\left(x, f^{n(x)}(z)\right)\right]\Bigg\}, \end{aligned} \tag{2.4}$$

thus the result follows from Theorem 12 in [3] and it is valid for any $q < 1$. Suppose now that (X, G) is nonsymmetric space. Then by inequality (1.5) we have that (2.1) becomes

$$\begin{aligned} & G\left(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\right) \\ & \quad \leq 2q \cdot \max\left\{G(z,x,x), G\left(z, f^{n(x)}(x), f^{n(x)}(x)\right), G\left(f^{n(x)}(z), x, x\right)\right\}, \end{aligned} \tag{2.5}$$

and (2.2) becomes

$$\begin{aligned} & G\left(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\right) \\ & \quad \leq 2q \cdot \max\Bigg\{G(z,x,x), \frac{1}{2}\left[G\left(z, f^{n(x)}(z), f^{n(x)}(z)\right) + G\left(x, f^{n(x)}(x), f^{n(x)}(x)\right)\right], \\ & \qquad \frac{1}{2}\left[G\left(z, f^{n(x)}(x), f^{n(x)}(x)\right) + G\left(f^{n(x)}(z), x, x\right)\right]\Bigg\}. \end{aligned} \tag{2.6}$$

Since $2q$ need not be less then 1 we can use metric fixed point results only for $q < 1/2$. On the other side, using the concept of G-metric space, we are going to prove the result, if the first case for any $0 < q < 1$, and in the second one for $0 < q < 2/3$. This means that our results are real generalization in the case of nonsymmetric G-metric spaces.

Let $f \in \langle 1 \rangle$. Uniqueness follows from (2.1), since for $f^n(z) = z$, it follows that $G(z,u,u) = G(f^n(z), f^n(u), f^n(u)) \leq qG(z,u,u)$. Now $f^n(f(u)) = f(u)$ implies that $f(u) = u$.

Let $y_0 \in B$, and assume $f^m(y_0) \neq u$ for each m. For m sufficiently large write $m = kn+r$, $k \geq 1$, and $1 \leq r < n$. Then

$$\begin{aligned}
G(f^m(y_0),u,u) &= G\Big(f^{kn+r}(y_0), f^n(u), f^n(u)\Big) \\
&\leq q\max\Big\{G\Big(f^{(k-1)n+r}(y_0),u,u\Big), G\Big(f^{(k-1)n+r}(y_0), f^n(u), f^n(u)\Big), \\
&\qquad G(f^m(y_0),u,u,)\Big\} \\
&= qG\Big(f^{(k-1)n+r}(y_0),u,u\Big) \leq \cdots \leq q^k G(f^r(y_0),u,u) \\
&\leq q^k \max\{G(f^p(y_0),u,u) : 1 \leq p < n\},
\end{aligned} \tag{2.7}$$

so $G(f^m(y_0),u,u) \to 0$, $m \to \infty$.

If $f \in \langle 2\rangle$, uniqueness follows from (2.2) since for $f^n(z) = z$, it follows that $G(z,u,u) = G(f^n(z), f^n(u), f^n(u)) \leq q\max\{G(z,u,u),0\}$ and further $f(u) = u$. Now for any $y_0 \in B$

$$G(f^m(y_0),u,u) = G\Big(f^{kn+r}(y_0), f^n(u), f^n(u)\Big) \leq qM(y_0,m,u), \tag{2.8}$$

where

$$M(y_0,m,u) = \begin{cases} G(f^{(k-1)n+r}(y_0),u,u), \\ \dfrac{1}{2}[G(f^{(k-1)n+r}(y_0), f^m(y_0), f^m(y_0)) + 0], \\ \dfrac{1}{2}[G(f^{(k-1)n+r}(y_0),u,u) + G(f^m(y_0),u,u)]. \end{cases} \tag{2.9}$$

For $M(y_0,m,u) = (1/2)[G(f^{(k-1)n+r}(y_0),u,u) + G(f^m(y_0),u,u)]$ we have

$$\begin{aligned}
&\frac{1}{2}G\Big(f^{(k-1)n+r}(y_0),u,u\Big) < \frac{1}{2}G(f^m(y_0),u,u), \\
&\frac{1}{2}G(f^m(y_0),u,u) < \frac{1}{2}G\Big(f^{(k-1)n+r}(y_0),u,u\Big)
\end{aligned} \tag{2.10}$$

which is a contradiction, and therefore

$$M(y_0,m,u) = \max\Big\{G\Big(f^{(k-1)n+r}(y_0),u,u\Big), \frac{1}{2}G\Big(f^{(k-1)n+r}(y_0), f^m(y_0), f^m(y_0)\Big)\Big\}. \tag{2.11}$$

If $M(y_0, m, u) = (1/2)G(f^{(k-1)n+r}(y_0), f^m(y_0), f^m(y_0))$ then

$$G(f^m(y_0), u, u) \leq \frac{q}{2} G\left(f^{(k-1)n+r}(y_0), u, u\right) + qG(u, u, f^m(y_0)). \tag{2.12}$$

So

$$2(1-q)G(f^m(y_0), u, u) \leq qG\left(f^{(k-1)n+r}(y_0), u, u\right). \tag{2.13}$$

Therefore, $G(f^m(y_0), u, u) \leq hG(f^{(k-1)n+r}(y_0), u, u)$, where $h = \max\{q, q/2(1-q)\}$. For $h < 1$, $G(f^m(y_0), u, u) \to \infty$, $m \to \infty$. □

For $f : X \to X$ the set $\mathcal{O}(f; x_0) = \{f^n(x_0) : n \in \mathbb{N}\}$ is called the orbit for $x_0 \in X$.

Theorem 2.2. *Let (X, G) be a complete G-metric space and let $f : X \to X$ be a mapping. Suppose that for some $x_0 \in X$ the orbit $\overline{\mathcal{O}(f; x_0)}$ is complete, and that: for some $q \in [0, 1)$ and each $x \in \mathcal{O}(f; x_0)$ there is an integer $n(x) \geq 1$ such that*

$$G\left(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\right) \leq q \cdot G(z, x, x) \tag{2.14}$$

for all $z \in \mathcal{O}(f; x_0)$.

Then $x_k = f^{n(x_{k-1})}(x_{k-1})$, $k \in \mathbb{N}$, converges to some $u \in X$ and for all $m, k \in \mathbb{N}$, $m > k$

$$G(x_k, x_k, x_m) \leq \frac{q^k}{(1-q)^2} \max\{G(f^p(x_0), x_0, x_0) : \ 1 \leq p \leq n(x_0)\} \tag{2.15}$$

If inequality in (2.14) holds for all $x \in \overline{\mathcal{O}(f; x_0)}$, then $f^{n(u)}(u) = u$ and $f^k(x_0) \to u, k \to \infty$. Moreover, if $f(\overline{\mathcal{O}(f; x_0)}) \subseteq \overline{\mathcal{O}(f; x_0)}$, then u is the fixed point of f.

Proof. If (X, G) is a symmetric G-metric space the statement easily follows from Guseman fixed point result [2]. Let (X, G) be nonsymmetric G-metric space. Then by inequality (1.5)

$$d_G\left(f^{n(x)}(z), f^{n(x)}(x)\right) \leq 2qd_G(z, x). \tag{2.16}$$

Thus, one can use the fixed point result in metric space only for $q < 1/2$. But here, using the concept of G-metric, we prove the result for any $0 < q < 1$. At first let us show that

$$\sup_m G(f^m(x_0), x_0, x_0) = M < +\infty. \tag{2.17}$$

For any $m \in \mathbb{N}$, sufficiently large, there exist $k, r \in \mathbb{N}$, $1 \leq r \leq n(x_0) - 1$ such that $m = k \cdot n(x_0) + r$. Then

$$\begin{aligned}
G(f^m(x_0), x_0, x_0) &\leq G\left(f^{kn(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\leq qG\left(f^{(k-1)n(x_0)+r}(x_0), x_0, x_0\right) + G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\leq qG\left(f^{(k-1)n(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\right) + (1+q)G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\leq q^2G\left(f^{(k-2)n(x_0)+r}(x_0), x_0, x_0\right) + (1+q)G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \leq \cdots \\
&\leq q^kG(f^r(x_0), x_0, x_0) + \left(1 + q + \cdots + q^{k-1}\right)G\left(f^{n(x_0)}(x_0), x_0, x_0\right) \\
&\leq \frac{1}{1-q}\max\{G(f^p(x_0), x_0, x_0) : 1 \leq p \leq n(x_0)\} = M < +\infty.
\end{aligned} \tag{2.18}$$

Now, for each $k \in \mathbb{N}$

$$\begin{aligned}
G(x_k, x_k, x_{k+1}) &= G\left(f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_k)} f^{n(x_{k-1})}(x_{k-1})\right) \\
&\leq qG\left(x_{k-1}, x_{k-1}, f^{n(x_k)}(x_{k-1})\right) \leq \cdots \\
&\leq q^kG\left(x_0, x_0, f^{n(x_k)}(x_0)\right) \leq q^kM.
\end{aligned} \tag{2.19}$$

For all $m, k \in \mathbb{N}$, $m > k$, it follows that

$$G(x_k, x_k, x_m) \leq G(x_k, x_k, x_{k+1}) + G(x_{k+1}, x_{k+1}, x_{k+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \leq \frac{q^k}{1-q}M, \tag{2.20}$$

so $\{x_k\}$ is Cauchy sequence and there exists $u = \lim_k x_k$, for some $u \in X$, and inequality (2.15) is proved.

If we suppose that inequality in (2.14) is satisfied for all $x \in \overline{\mathcal{O}(f; x_0)}$, then, for all $k \in \mathbb{N}$,

$$G\left(f^{n(u)}(u), f^{n(u)}(u), f^{n(u)}(x_k)\right) \leq qG(u, u, x_k) \tag{2.21}$$

so $\lim_k f^{n(u)}(x_k) = f^{n(u)}(u)$.

On the other hand,

$$\begin{aligned}
G\left(f^{n(u)}(x_k), x_k, x_k\right) &= G\left(f^{n(u)} f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1})\right) \\
&\leq qG\left(f^{n(u)}(x_{k-1}), x_{k-1}, x_{k-1}\right) \leq \cdots \leq q^kG\left(f^{n(u)}(x_0), x_0, x_0\right)
\end{aligned} \tag{2.22}$$

implies that $\lim_k G(f^{n(u)}(x_k), x_k, x_k) = 0$.

Since G is continuous it means that

$$G\left(f^{n(u)}(u), u, u\right) = 0. \tag{2.23}$$

Hence $f^{n(u)}(u) = u$.

Now, let us suppose that $f(\overline{\mathcal{O}(f;x_0)}) \subseteq \overline{\mathcal{O}(f;x_0)}$. Since $f \in \langle 1 \rangle$ by Theorem 2.1 u is the fixed point of f in X and $\lim_k f^k(x_0) = u$. $\square$

For $n(x) = 1$, in inequality (2.14) independently on x, we are going to simplify the proof and to relax the condition in (2.14).

Corollary 2.3. *Let (X, G) be a complete G-metric apace and let $f : X \to X$. Suppose that there exist a point $x_0 \in X$ and $q \in [0,1)$ with $\overline{\mathcal{O}(f;x_0)}$ complete and*

$$G(f(z), f(x), f(x)) \leq qG(z, x, x) \tag{2.24}$$

for each $x, z = f(x) \in \mathcal{O}(f;x_0)$. Then $\{f^k(x_0)\}$ converges to some point $u \in X$ and for all $k, m \in \mathbb{N}$, $m > k$,

$$G(x_k, x_k, x_m) \leq \frac{q^k}{1-q} G(x_0, x_0, f(x_0)). \tag{2.25}$$

If (2.24) holds, for all $x \in \overline{\mathcal{O}(f;x_0)}$ or f is orbitally continuous at u, then u is a fixed point of f.

Proof. If (X, G) is a symmetric space than $d_G(x, z) = 2G(z, x, x)$ so (2.24) becomes

$$d_G(f(z), f(x)) \leq q d_G(z, x), \tag{2.26}$$

and result follows from Theorem 2 in [19].

Now, let (X, G) be a nonsymmetric G-metric space. Then since $x_k = f(x_{k-1})$, $k \in \mathbb{N}$,

$$G(x_k, x_k, x_{k+1}) \leq qG(x_{k-1}, x_{k-1}, x_k) \ \leq \cdots \leq q^k G(x_0, x_0, f(x_0)), \tag{2.27}$$

so for all $m, k \in \mathbb{N}$, $m > k$,

$$G(x_k, x_k, x_m) \leq \frac{q^k}{1-q} G(x_0, x_0, f(x_0)), \tag{2.28}$$

and there exists $u = \lim_k x_k$. If (2.24) holds for all $x \in \overline{\mathcal{O}(f;x_0)}$, then by Theorem 2.2, since $n(u) = 1$, it follows that $f(u) = u$.

The fact that f is orbitally continuous at $x = u$, and that $\lim_k f^k(x_0) = u$, implies that $\lim_k f^{k+1}(x_0) = f(u)$, and therefore $u = f(u)$. $\square$

Remark 2.4. Let us note that this result is very close to Theorem 2.1 in [8].

Remark 2.5. In the statements above f does not have to be continuous.

The next theorems are generalizations of Ćirić fixed point results in [4].

Theorem 2.6. *Let (X, G) be a complete metric space and $T : X \to X$ a mapping. Suppose that for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$\begin{aligned} &G(T^n x, T^n x, T^n y) \\ &\quad \le q \max\left\{G(x,x,y), G(x,x,Ty), \dots, G(x,x,T^n y), \frac{1}{2}[G(x,x,T^n x) + G(x,T^n x,T^n x)]\right\} \end{aligned} \tag{2.29}$$

holds for some $q < 2/3$ and all $y \in X$. Then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $\lim_m T^m(x) = u$.

Proof. If (X, G) is a symmetric space then $d_G(x,y) = 2G(x,x,y)$ and inequality (2.29) becomes

$$d_G(T^n x, T^n y) \le q \max\{d_G(x,y), d_G(x,Ty), \dots, d_G(x,T^n x)\}, \tag{2.30}$$

for all $y \in X$. Then the result follows from Theorem 2.1 in [4] and it is true for all $q < 1$.

Now suppose that (X, G) is nonsymmetric space. Then, by definition of the metric d_G and inequality (1.5) we have

$$d_G(T^n x, T^n y) \le 2q \max\{d_G(x,y), d_G(x,Ty), \dots, d_G(x,T^n y), d_G(x,T^n x)\}. \tag{2.31}$$

But $2q$ need not to be less than 1, so we will prove the statement by using G-metric.

First, let us prove prove that

$$G(x,x,T^m x) \le \frac{1}{1-q} b(x), \quad m = 1,2,\dots, \tag{2.32}$$

where

$$b(x) = \max\left\{G(x,x,Tx), G(x,x,T^2 x), \dots, G(x,x,T^n x), \frac{1}{2}[G(x,x,T^n x) + G(x,T^n x,T^n x)]\right\}. \tag{2.33}$$

Clearly (2.32) is true for $m = 1,2,\dots,n$. Suppose that $m > n$, and that (2.32) is true for $i \le m$ and let us prove it for $i = m+1$. Let $m + 1 - n = r$. Now

$$\begin{aligned} G\left(x,x,T^{m+1}x,\right) &\le G(x,x,T^n x) + G\left(T^n x, T^n x, T^{m+1}x\right), \\ G\left(T^n x, T^n x, T^{m+1}x\right) &\le q b(x), \end{aligned} \tag{2.34}$$

where

$$b(x) = \max\Big\{G(x,x,T^r x), G\Big(x,x,T^{r+1}x\Big),\dots,G(x,x,T^{r+n}x),$$
$$\frac{1}{2}[G(x,x,T^n x)+G(x,T^n x,T^n x)]\Big\}. \tag{2.35}$$

If $b(x) = G(x,x,T^{n+r})$, then (2.34) imply

$$G\Big(x,x,T^{m+1}x\Big) \le \frac{1}{1-q}G(x,x,T^n x) \le \frac{1}{1-q}b(x). \tag{2.36}$$

If $b(x) \ne G(x,x,T^{n+r})$, then (2.34) imply

$$G\Big(x,x,T^{m+1}x\Big) \le G(x,x,T^n x) + \frac{q}{1-q}b(x) \le \frac{1}{1-q}b(x). \tag{2.37}$$

Thus by induction we obtain (2.32).

Let us prove that $\{T^m x\}_m$ is a Cauchy sequence. Let $x_0 = x$, $n_0 = n(x_0)$, $x_1 = T^{n_0}x_0$, and we define inductively a sequence of integers and a sequence of points $\{x_k\}_k$ in X as follows: $n_k = n(x_k)$, and $x_{k+1} = T^{n_k}x_k$, $k = 0,1,\dots$. Evidently, $\{x_k\}_k$ is a subsequence of the orbit $\{T^m x_0\}_m$. Using this sequence we will prove that $\{T^m x_0\}_m$ is a Cauchy sequence.

Let x_k be any fixed member of $\{x_k\}_k$ and let $x_p = T^p x_0$ and $x_q = T^q x_0$ be any two members of the orbit which follow after x_k. Then $x_p = T^r x_k$ and $x_q = T^s x_k$ for some r and s, respectively. Now, using (2.29) we get

$$G(x_k,x_k,x_p) = G\Big(T^{n_{k-1}}x_{k-1},T^{n_{k-1}}x_{k-1},T^{n_k-1}T^r x_{k-1}\Big) \le \frac{3}{2}qG(x_{k-1},x_{k-1},T^{r_1}x_{k-1}), \tag{2.38}$$

where

$$G(x_{k-1},x_{k-1},T^{r_1}x_{k-1})$$
$$= \max\Big\{G(x_{k-1},x_{k-1},T^r x_{k-1}), G\Big(x_{k-1},x_{k-1},T^{r+1}x_{k-1}\Big),\dots,$$
$$G(x_{k-1},x_{k-1},T^{r+n_{k-1}}x_{k-1}), G(x_{k-1},x_{k-1},T^{n_{k-1}}x_{k-1})\Big\}. \tag{2.39}$$

Similarly, $G(x_{k-1},x_{k-1},T^{r_1}x_{k-1}) \le (3/2)qG(x_{k-2},x_{k-2},T^{r_2}x_{k-2})$, where

$$G(x_{k-2},x_{k-2},T^{r_2}x_{k-2}) = \max\{G(x_{k-2},x_{k-2},T^{r_1}x_{k-2}),\dots,G(x_{k-2},x_{k-2},T^{n_{k-2}}x_{k-2})\} \tag{2.40}$$

Repeating this argument k times we get

$$G(x_k,x_k,x_p) \le \left(\frac{3}{2}q\right)^k G(x_0,x_0,T^{r_k}x_0). \tag{2.41}$$

Hence $G(x_k, x_k, x_p) \leq ((3/2)q)^k b(x_0)$. Similarly $G(x_k, x_k, x_q) \leq ((3/2)q)^k b(x_0)$, so

$$G(x_p, x_p, x_q) \leq 2G(x_k, x_k, x_p) + G(x_k, x_k, x_q) \leq \left(\frac{3}{2}q\right)^k \cdot 3b(x_0). \tag{2.42}$$

Since $q < 2/3$, it follows that $\{T^m x_0\}_m$ is a Cauchy sequence. Let $\lim_m T^m(x_0) = u \in X$. We show that u is a fixed point of T. First, let us prove that $T^n u = u$, where $n = n(u)$. For $m \geq n = n(u)$, we now have

$$\begin{aligned} &G(T^n u, T^n u, T^n T^m x_0) \\ &\quad \leq q \max\Big\{ G(u, u, T^m x_0), G\left(u, u, T^{m+1} x_0\right), \dots, G(u, u, T^{m+n} x_0), \\ &\qquad\qquad \frac{1}{2}[G(u, u, T^n u) + G(u, T^n u, T^n u)] \Big\}, \end{aligned} \tag{2.43}$$

and on letting m tend to infinity it follows that

$$G(T^n u, T^n u, u) \leq q \max\left\{ 0, \frac{1}{2}[G(u, u, T^n u) + G(u, T^n u, T^n u)] \right\}. \tag{2.44}$$

For $q < 2/3$ we have $T^n u = u$.

To show that u is a fixed point of T, let us suppose that $Tu \neq u$ and let $G(u, u, T^k u) = \max\{G(u, u, T^r u) : 1 \leq r \leq n = n(u)\}$. Then

$$\begin{aligned} G\left(u, u, T^k u\right) &= G\left(T^n u, T^n u, T^n T^k u\right) \\ &\leq q \max\Big\{ G\left(u, u, T^k u\right), G\left(u, u, T^{k+1} u\right), \dots, G\left(u, u, T^{k+n} u\right), \dots, \\ &\qquad\qquad \frac{1}{2}[G(u, u, T^n u) + G(u, T^n u, T^n u)] \Big\} \\ &\leq \frac{3}{2} q G\left(u, u, T^k u\right). \end{aligned} \tag{2.45}$$

Since $q < 2/3$, it follows that $G(u, u, T^k u) = 0$, which implies that u is a fixed point of T.

Let us suppose that for some $z \in X$, $Tz = z$. Then,

$$G(u, u, z) = G(T^n u, T^n u, T^n z) \leq q \ \max\{G(u, u, z), 0\} \tag{2.46}$$

implies that $z = u$ and thus u is the unique fixed point in X. □

If we suppose that T is continuous, then we may prove the following theorem.

Theorem 2.7. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a continuous mapping which satisfies the condition: for each $x \in X$ there is a positive integer $n = n(x)$ such that*

$$\begin{aligned} &G(T^n x, T^n x, T^n y) \\ &\quad \leq \max\{G(x, x, y), G(x, x, Ty), \ldots, G(x, x, T^n y), G(x, x, Tx), \ldots, G(x, x, T^n x)\} \end{aligned} \tag{2.47}$$

for some $q < 1$ and all $y \in X$. Then T has a unique fixed point $u \in X$ and $\lim_m T^m(x) = u \in X$, for every $x \in X$.

Proof. Let x be an arbitrary point in X. Then, as in the proof of Theorem 2.6, the orbit $\{T^m x\}_m$ is bounded and is a Cauchy sequence in the complete G-metric space X and so it has a limit u in X. Since by the hypothesis T is continuous,

$$T^{n(u)} u = T^{n(u)} \lim_m T^m x = \lim_m T^{m+n(u)} = u. \tag{2.48}$$

Therefore, u is a fixed point of $T^{n(u)}$. By the same argument as in the proof of Theorem 2.6, it follows that u is a unique fixed point of T. □

Remark 2.8. The condition that T is a continuous mapping can be relaxed by the following condition: $T^{n(x)}$ is continuous at a point $x \in X$.

3. A Common Fixed Point Result

Now, we are going to prove Hadžić [20] fixed point theorem in 2-metric space, in a manner of G-metric spaces.

Theorem 3.1. *Let (X, G) be a complete G-metric space, S and $T : X \to X$ one to one continuous mappings, $A : X \to SX \cap TX$ continuous mapping commutative with S and T. Suppose that there exists a point $x_0 \in X$ such that $\overline{\mathcal{O}(A; x_0)}$ is complete and that the following conditions are satisfied:*

(i) *For every $x \in \overline{\mathcal{O}(A; x_0)}$ there exists $n(x) \in \mathbb{N}$ so that for all $z \in X$ and some $q \in [0, 1)$*

$$\begin{aligned} &G\left(A^{n(x)} z, A^{n(x)} x, A^{n(x)} x\right) \\ &\quad \leq q \min\{G(Tx, Tx, Sz), G(Tx, Sx, Tz), G(Tx, Sx, Sz), G(Sx, Sx, Tz)\}. \end{aligned} \tag{3.1}$$

(ii) *There exists $M > 0$ such that for all $z \in \mathcal{O}(A; x_0)$*

$$G(Sx_0, Sx_0, z) \leq M < +\infty. \tag{3.2}$$

Then there exists one and only one element $u \in X$ such that

$$Au = Su = Tu = u. \tag{3.3}$$

(e.g., there exists a unique common fixed point for A, S, and T)

Proof. Since $AX \subseteq SX \cap TX$ starting with x_0 we can define the sequence $\{x_n\} \subseteq X$ such that

$$\begin{aligned} Tx_{2k-1} &= A^{n(x_{2k-2})}x_{2k-2}, \\ Sx_{2k} &= A^{n(x_{2k-1})}x_{2k-1}. \end{aligned} \tag{3.4}$$

Let

$$y_n = \begin{cases} Tx_{2k-1}, & n = 2k-1, \\ Sx_{2k}, & n = 2k, \end{cases} \quad k \in \mathbb{N}. \tag{3.5}$$

We are going to prove that $\{y_n\}$ is Cauchy sequence

$$\begin{aligned} G(y_{2k-1}, y_{2k-1}, y_{2k}) &= G\Big(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-1})}x_{2k-1}\Big) \\ &= G\Big(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-1})}T^{-1}A^{n(x_{2k-2})}x_{2k-2}\Big) \\ &\leq qG\Big(Sx_{2k-2}, Sx_{2k-2}, A^{n(x_{2k-1})}x_{2k-2}\Big) \\ &= qG\Big(A^{n(x_{2k-3})}x_{2k-3}, A^{n(x_{2k-3})}x_{2k-3}, A^{n(x_{2k-1})}S^{-1}A^{n(x_{2k-3})}x_{2k-3}\Big) \\ &\leq \cdots \leq q^{2k-2}G\Big(Tx_1, Tx_1, A^{n(x_{2k-1})}x_1\Big) \\ &= q^{2k-2}G\Big(A^{n(x_0)}x_0, A^{n(x_0)}x_0, A^{n(x_{2k-1})}T^{-1}A^{n(x_0)}x_0\Big) \\ &\leq q^{2k-1}G\Big(Sx_0, Sx_0, A^{n(x_{2k-1})}x_0\Big) \leq q^{2k-1}M. \end{aligned} \tag{3.6}$$

Similarly one can prove that $G(y_{2k}, y_{2k}, y_{2k+1}) \leq q^{2k}M$, $k \in \mathbb{N}$, for all $m, k \in \mathbb{N}$, $m > k$,

$$\begin{aligned} G(y_k, y_k, y_m) &\leq G(y_k, y_k, y_{k+1}) + G(y_{k+1}, y_{k+1}, y_m) \leq \cdots \\ &\leq \sum_{j=k}^{m-1} G(y_j, y_j, y_{j+1}) \leq \frac{q^k}{1-q}M. \end{aligned} \tag{3.7}$$

Thus we proved that $\{y_n\}$ is a Cauchy sequence, so there exists $u \in X$ such that

$$\lim_n y_n = u. \tag{3.8}$$

It obvious that $\lim_k Tx_{2k-1} = \lim_k Sx_{2k} = u$.

At first we will prove that $Au = u$

$$\begin{aligned}
G(y_{2k}, y_{2k}, Ay_{2k+1}) &= G(Sx_{2k}, Sx_{2k}, ATx_{2k+1}) \\
&= G\Big(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, AA^{n(x_{2k})}x_{2k}\Big) \\
&= G\Big(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, AA^{n(x_{2k})}S^{-1}A^{n(x_{2k-1})}x_{2k-1}\Big) \\
&\le qG\Big(Tx_{2k-1}, Tx_{2k-1}, AA^{n(x_{2k})}x_{2k-1}\Big) \\
&= qG\Big(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}x_{2k-2}, AA^{n(x_{2k})}T^{-1}A^{n(x_{2k-2})}x_{2k-2}\Big) \\
&\le q^2G\Big(Sx_{2k-2}, Sx_{2k-2}, AA^{n(x_{2k})}x_{2k-2}\Big) \le \cdots \\
&\le q^{2k}G\Big(Sx_0, Sx_0, AA^{n(x_{2k})}x_0\Big) \le q^{2k} \cdot M,
\end{aligned} \tag{3.9}$$

so $\lim_k G(y_{2k}, y_{2k}, Ay_{2k+1}) = 0$.

Now, since that G and A are continuous we have that $G(u, u, Au) = 0$ so $Au = u$.

Further, let us prove that $Tu = u$.

$$\begin{aligned}
G(y_{2k}, y_{2k}, Ty_{2k}) &= G\Big(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, TA^{n(x_{2k-1})}x_{2k-1}\Big) \\
&\le qG(Tx_{2k-1}, Tx_{2k-1}, STx_{2k-1}) \\
&= qG\Big(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}x_{2k-2}, SA^{n(x_{2k-2})}x_{2k-2}\Big) \\
&\le q^2G(Sx_{2k-2}, Sx_{2k-2}, TSx_{2k-2}) \\
&= q^2G\Big(A^{n(x_{2k-3})}x_{2k-3}, A^{n(x_{2k-3})}x_{2k-3}, TA^{n(x_{2k-3})}x_{2k-3}\Big) \le \cdots \\
&\le q^{2k}G(Sx_0, Sx_0, TSx_0)
\end{aligned} \tag{3.10}$$

implies that

$$\lim_k G(y_{2k}, y_{2k}, Ty_{2k}) = G(u, u, Tu) = 0, \tag{3.11}$$

and $Tu = u$. Similarly one can see that $Su = u$, so we prove that

$$Au = Su = Tu = u. \tag{3.12}$$

If we suppose that ω is some other common fixed point for A, S, and T then we have that

$$\begin{aligned} G(u,u,\omega) &= G\left(A^{n(u)}u, A^{n(u)}u, A^{n(u)}\omega\right) \\ &\le qG(Su,Su,T\omega) = qG(u,u,\omega) < G(u,u,\omega) \end{aligned} \tag{3.13}$$

which is contradiction!

So, the common fixed point for A, S, and T is unique, and proof is completed. □

Remark 3.2. For $S = T = Id_X$ condition (2.14) is satisfied but the Theorem 2.2 is not just a consequence of Theorem 3.1 since in Theorem 2.2 we do not suppose that f is continuous.

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