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Research Article

An Iterative Algorithm for Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities

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We present an iterative algorithm for finding a common element x^* of the set of solutions of a mixed equilibrium problem and the set of a variational inclusion in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithms strongly converge to x^* which solves some variational inequality.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a nonlinear mapping, let $\varphi : C \to R$ be a function, and let Θ be a bifunction of $C \times C$ into R. Now we consider the following mixed equilibrium problem:

$$\text{Find } u \in C \text{ such that } \Theta(u, y) + \varphi(y) - \varphi(u) + \langle Fu, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solution of problem (1.1) is denoted by EP.

If $F = 0$, then the mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem:

$$\text{Find } u \in C \text{ such that } \Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was considered by Ceng and Yao [1]. If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{Find } u \in C \text{ such that } \Theta(u, y) + \langle Fu, y - u\rangle \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by S. Takahashi and W. Takahashi [2]. If $\varphi = 0$ and $F = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{Find } u \in C \text{ such that } \Theta(u, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\Theta(x, y) = 0$ for all $x, y \in C$, the mixed equilibrium problem (1.1) becomes the following variational inequality problem:

$$\text{Find } u \in C \text{ such that } \varphi(y) - \varphi(u) + \langle Fu, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [3–8]. Some methods have been proposed to solve the mixed equilibrium problem and the equilibrium problem. In 1997, Flaim and Antipen [4] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, S. Takahashi and W. Takahashi [9] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of fixed point points of a nonexpansive mapping. Furthermore, Yao et al. [10] introduced some new iterative schemes for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [1] considered a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Peng and Yao [11] developed a CQ method. They obtained some strong convergence results for finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of the variational inequality and the set of fixed points of a nonexpansive mapping. Their results extend and improve the corresponding results in [1, 9, 12].

Recall that a mapping $B : C \to C$ is said to be β-inverse strongly monotone if there exists a constant $\beta > 0$ such that $\langle Bx - By, x - y\rangle \geq \beta \|Bx - By\|^2$, for all $x, y \in C$. A mapping A is strongly positive on H if there exists a constant $\mu > 0$ such that $\langle Ax, x\rangle \geq \mu\|x\|^2$ for all $x \in H$.

Let $B : H \to H$ be a single-valued nonlinear mapping and let $R : H \to 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \tag{1.6}$$

where θ is the zero vector in H. The set of solutions of problem (1.6) is denoted by $I(B, R)$. If $H = R^m$, then problem (1.6) becomes the generalized equation introduced by Robinson [13]. If $B = 0$, then problem (1.6) becomes the inclusion problem introduced by Rockafellar [14]. It is known that (1.6) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory. Also various types of variational inclusions problems have been extended and generalized. Recently, Zhang et al. [15] introduced a new iterative scheme for finding a common element of the set of solutions to the problem (1.6) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [16] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping. For some related works, please see [1, 2, 9–11, 13–34] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we present an iterative algorithm for finding a common element x^* of the set of solutions of a mixed equilibrium problem and the set of a variational inclusion in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithms strongly converge to x^* which solves some variational inequality.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Longleftrightarrow \langle x - x^*, x^* - y \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let the set-valued mapping $R : H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}(x), \quad x \in H, \tag{2.3}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single-valued, nonexpansive, and 1-inverse strongly monotone and that a solution of problem (1.6) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see, for instance, [25].

Throughout this paper, we assume that a bifunction $\Theta : C \times C \to \mathbf{R}$ and a convex function $\varphi : C \to R$ satisfy the following conditions:

(H1) $\Theta(x,x) = 0$ for all $x \in C$;

(H2) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x, y \in C$;

(H3) for each $y \in C$, $x \mapsto \Theta(x,y)$ is weakly upper semicontinuous;

(H4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;

(H5) for each $x \in C$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{2.4}$$

Lemma 2.1 (see [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to R$ be a bifunction and let $\varphi : C \to R$ be a proper lower semicontinuous and convex function. For $r > 0$ and $x \in C$, define a mapping $S_r : C \to C$ as follows:*

$$S_r(x) = \left\{ z \in C : \Theta(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.5}$$

for all $x \in C$. Assume that the conditions (H1)–(H5) hold. Then one has the following results:

(1) *for each $x \in C$, $S_r(x) \neq \emptyset$ and S_r is single-valued;*

(2) *S_r is firmly nonexpansive, that is, for any $x, y \in C$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle; \tag{2.6}$$

(3) $\mathrm{Fix}(S_r) = \mathrm{EP}$;

(4) EP *is closed and convex.*

Lemma 2.2 (see [24]). *Let $R : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \to 2^H$ is maximal monotone.*

Lemma 2.3 (see [34]). *Assume taht $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$ where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we will prove our main result. First, we give some assumptions on the operators and the parameters. Subsequently, we introduce our iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of a variational inclusion. Finally, we will show that the proposed algorithm has strong convergence.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to R$ be a lower semicontinuous and convex function and let $\Theta : H \times C \to R$ be a bifunction satisfying conditions (H1)–(H5). Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and let $R : H \to 2^H$ be a maximal monotone mapping. Let the mappings $F, B : C \to C$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $r > 0$ and $\lambda > 0$ be two constants such that $r < 2\alpha$ and $\lambda < 2\beta$.

Now we introduce the following iteration algorithm.

Algorithm 3.1. For given $x_0 \in C$ arbitrarily, compute the sequences $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{gathered}\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C[(I - \alpha_n A) J_{R,\lambda}(I - \lambda B)u_n],\end{gathered} \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

Now we study the strong convergence of the algorithm (3.1).

Theorem 3.2. *Suppose that $\Omega := \mathrm{EP} \cap I(B, R) \neq \emptyset$. Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $x^* \in \Omega$ which solves the following variational inequality:*

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.2}$$

Proof. Take $x^* \in \Omega$. It is clear that

$$S_r(x^* - rFx^*) = J_{R,\lambda}(x^* - \lambda Bx^*) = x^*, \quad n \geq 0. \tag{3.3}$$

We divide our proofs into the following five steps:

(1) the sequences $\{x_n\}$ and $\{u_n\}$ are bounded;

(2) $\|x_{n+1} - x_n\| \to 0$;

(3) $\|Fx_n - Fx^*\| \to 0$ and $\|Bu_n - Bx^*\| \to 0$;

(4) $\limsup_{n\to\infty}\langle Ax^*, x_n - x^*\rangle \geq 0$;

(5) $x_n \to x^*$.

Proof of (1). Since F is α-inverse strongly monotone and B is β-inverse strongly monotone, we have

$$\left\|(I - rF)x - (I - rF)y\right\|^2 \leq \left\|x - y\right\|^2 + r(r - 2\alpha)\left\|Fx - Fy\right\|^2, \tag{3.4}$$

$$\left\|(I - \lambda B)x - (I - \lambda B)y\right\|^2 \leq \left\|x - y\right\|^2 + \lambda(\lambda - 2\beta)\left\|Bx - By\right\|^2. \tag{3.5}$$

It is clear that if $0 \le r \le 2\alpha$ and $0 \le \lambda \le 2\beta$, then $(I - rF)$ and $(I - \lambda B)$ are all nonexpansive. Set $y_n = J_{R,\lambda}(u_n - \lambda B u_n)$, $n \ge 0$. It follows that

$$\left\| y_n - x^* \right\| = \| J_{R,\lambda}(u_n - \lambda B u_n) - J_{R,\lambda}(x^* - \lambda B x^*) \| \le \| (u_n - \lambda B u_n) - (x^* - \lambda B x^*) \| \le \| u_n - x^* \|. \tag{3.6}$$

By Lemma 2.1, we have $u_n = S_r(x_n - rFx_n)$ for all $n \ge 0$. Then, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\ &\le \|x_n - rFx_n - (x^* - rFx^*)\|^2 \\ &\le \|x_n - x^*\|^2 + r(r - 2\alpha)\|Fx_n - Fx^*\|^2 \\ &\le \|x_n - x^*\|^2. \end{aligned} \tag{3.7}$$

Hence, we have

$$\left\| y_n - x^* \right\| \le \| x_n - x^* \|. \tag{3.8}$$

Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{ |\langle Au, u \rangle| : u \in H,\ \|u\| = 1 \}. \tag{3.9}$$

Observe that

$$\langle (I - \alpha_n A)u, u \rangle = 1 - \alpha_n \langle Au, u \rangle \ge 1 - \alpha_n \|A\| \ge 0, \tag{3.10}$$

that is to say $I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(I - \alpha_n A)\| &= \sup\{ \langle (I - \alpha_n A)u, u \rangle : u \in H,\ \|u\| = 1 \} \\ &= \sup\{ 1 - \alpha_n \langle Au, u \rangle : u \in H,\ \|u\| = 1 \} \\ &\le 1 - \alpha_n \mu. \end{aligned} \tag{3.11}$$

From (3.1), we deduce that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\| P_C\left[(I - \alpha_n A) y_n \right] - x^* \right\| \\ &\le \left\| \left[(I - \alpha_n A)(y_n - x^*) \right] - \alpha_n A x^* \right\| \\ &\le (1 - \alpha_n \mu) \left\| y_n - x^* \right\| + \alpha_n \|Ax^*\| \\ &\le (1 - \alpha_n \mu) \|x_n - x^*\| + \alpha_n \|Ax^*\| \\ &\le \max\left\{ \|x_0 - x^*\|, \frac{\|Ax^*\|}{\mu} \right\}. \end{aligned} \tag{3.12}$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{u_n\}$, $\{y_n\}$, and $\{Ay_n\}$ are all bounded. □

Proof of (2). From (3.1), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|P_C[(I-\alpha_n A)y_n]-P_C[(I-\alpha_{n-1}A)y_{n-1}]\big\| \\ &\le \big\|[(I-\alpha_n A)y_n]-[(I-\alpha_{n-1}A)y_{n-1}]\big\| \\ &= \big\|(I-\alpha_n A)(y_n-y_{n-1})+(\alpha_{n-1}-\alpha_n)Ay_{n-1}\big\| \\ &\le \big\|(I-\alpha_n A)(y_n-y_{n-1})\big\|+\big\|(\alpha_{n-1}-\alpha_n)Ay_{n-1}\big\| \\ &\le (1-\alpha_n\mu)\big\|y_n-y_{n-1}\big\|+|\alpha_n-\alpha_{n-1}|\big\|Ay_{n-1}\big\|.\end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned}\big\|y_n-y_{n-1}\big\| &= \|J_{R,\lambda}(u_n-\lambda Bu_n)-J_{R,\lambda}(u_{n-1}-\lambda Bu_{n-1})\| \\ &\le \|(u_n-\lambda Bu_n)-(u_{n-1}-\lambda Bu_{n-1})\| \\ &\le \|u_n-u_{n-1}\| \\ &= \|S_r(x_n-rFx_n)-S_r(x_{n-1}-rFx_{n-1})\| \\ &\le \|(x_n-rFx_n)-(x_{n-1}-rFx_{n-1})\| \\ &\le \|x_n-x_{n-1}\|.\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we get

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\alpha_n\mu)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\big\|Ay_{n-1}\big\| \\ &= (1-\alpha_n\mu)\|x_n-x_{n-1}\|+\alpha_n\mu\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|\frac{1}{\mu}\big\|Ay_{n-1}\big\|.\end{aligned} \tag{3.15}$$

Notice that $\lim_{n\to\infty}|1-\alpha_{n-1}/\alpha_n| = 0$. This together with the last inequality and Lemma 2.3 implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.16}$$

□

Proof of (3). From (3.5) and (3.7), we get

$$\begin{aligned}\big\|y_n-x^*\big\|^2 &= \|J_{R,\lambda}(u_n-\lambda Bu_n)-J_{R,\lambda}(x^*-\lambda Bx^*)\|^2 \\ &\le \|(u_n-\lambda Bu_n)-(x^*-\lambda Bx^*)\|^2 \\ &\le \|u_n-x^*\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2 \\ &\le \|x_n-x^*\|^2+r(r-2\alpha)\|Fx_n-Fx^*\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2.\end{aligned} \tag{3.17}$$

By (3.1), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|P_C\left[(I-\alpha_n A)y_n\right]-x^*\right\|^2 \\
&\le \left\|(I-\alpha_n A)y_n-x^*\right\|^2 \\
&= \left\|y_n-x^*-\alpha_n Ay_n\right\|^2 \\
&= \left\|y_n-x^*\right\|^2-2\alpha_n\langle y_n-x^*, Ay_n\rangle+\alpha_n^2\left\|Ay_n\right\|^2 \\
&\le \left\|y_n-x^*\right\|^2+\alpha_n\left(2\left\|y_n-x^*\right\|\left\|Ay_n\right\|+\left\|Ay_n\right\|^2\right) \\
&\le \left\|y_n-x^*\right\|^2+\alpha_n M,
\end{aligned} \tag{3.18}$$

where $M>0$ is some constant satisfying $\sup_n\{2\|y_n-x^*\|\|Ay_n\|+\|Ay_n\|^2\}\le M$. From (3.17) and (3.18), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2+r(r-2\alpha)\|Fx_n-Fx^*\|^2 \\
+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2+\alpha_n M.
\end{aligned} \tag{3.19}$$

Thus,

$$\begin{aligned}
&r(2\alpha-r)\|Fx_n-Fx^*\|^2+\lambda(2\beta-\lambda)\|Bu_n-Bx^*\|^2 \\
&\quad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n M \\
&\quad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|+\alpha_n M,
\end{aligned} \tag{3.20}$$

which imply that

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0, \qquad \lim_{n\to\infty}\|Bu_n-Bx^*\|=0. \tag{3.21}$$

□

Proof of (4). Since S_r is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\
&\le \langle x_n - rFx_n - (x^* - rFx^*), u_n - x^* \rangle \\
&= \frac{1}{2}\Big(\|x_n - rFx_n - (x^* - rFx^*)\|^2 + \|u_n - x^*\|^2 \\
&\quad - \|x_n - rFx_n - (x^* - rFx^*) - (u_n - x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n - r(Fx_n - Fx^*)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r\langle Fx_n - Fx^*, x_n - u_n \rangle - r^2\|Fx_n - Fx^*\|^2\Big).
\end{aligned} \tag{3.22}$$

Hence, we have

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\|. \tag{3.23}$$

Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\le \langle u_n - \lambda Bu_n - (x^* - \lambda Bx^*), y_n - x^* \rangle \\
&= \frac{1}{2}\Big(\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*)\|^2 + \|y_n - x^*\|^2 \\
&\quad - \|u_n - \lambda Bu_n - (x^* - \lambda Bx^*) - (y_n - x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|u_n - y_n - \lambda(Bu_n - Bx^*)\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|u_n - y_n\|^2 \\
&\quad + 2\lambda\langle Bu_n - Bx^*, u_n - y_n \rangle - \lambda^2\|Bu_n - Bx^*\|^2\Big),
\end{aligned} \tag{3.24}$$

which implies that

$$\|y_n - x^*\|^2 \le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda\|Bu_n - Bx^*\|\|u_n - y_n\|. \tag{3.25}$$

Thus, by (3.23) and (3.25), we obtain

$$\begin{aligned}\left\|y_n - x^*\right\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ - \left\|u_n - y_n\right\|^2 + 2\lambda\|Bu_n - Bx^*\|\left\|u_n - y_n\right\|.\end{aligned} \tag{3.26}$$

Substitute (3.26) into (3.18) to get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ - \left\|u_n - y_n\right\|^2 + 2\lambda\|Bu_n - Bx^*\|\left\|u_n - y_n\right\| + \alpha_n M.\end{aligned} \tag{3.27}$$

Then we derive

$$\begin{aligned}&\|x_n - u_n\|^2 + \left\|u_n - y_n\right\|^2 \\ &\quad \le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| \\ &\qquad + 2\lambda\|Bu_n - Bx^*\|\left\|u_n - y_n\right\| + \alpha_n M.\end{aligned} \tag{3.28}$$

So, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0, \qquad \lim_{n\to\infty}\left\|u_n - y_n\right\| = 0. \tag{3.29}$$

□

Proof of (5). We note that $P_\Omega(I - A)$ is a contraction. As a matter of fact,

$$\begin{aligned}\left\|P_\Omega(I - A)x - P_\Omega(I - A)y\right\| &\le \left\|(I - A)x - P_\Omega(I - A)y\right\| \\ &\le \|I - A\|\left\|x - y\right\| \\ &\le (1 - \mu)\left\|x - y\right\|\end{aligned} \tag{3.30}$$

for all $x, y \in H$. Hence $P_\Omega(I - A)$ has a unique fixed point, say $x^* \in \Omega$. That is, $x^* = P_\Omega(I - A)(x^*)$. This implies that $\langle Ax^*, y - x^*\rangle \ge 0$ for all $y \in \Omega$. Next, we prove that

$$\limsup_{n\to\infty}\langle Ax^*, x_n - x^*\rangle \ge 0. \tag{3.31}$$

First, we note that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Ax^*, x_n - x^*\rangle = \lim_{j\to\infty}\left\langle Ax^*, x_{n_j} - x^*\right\rangle. \tag{3.32}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{ji}}\}$ of $\{x_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$.

We next show that $w \in EP$. By $u_n = S_r(x_n - rFx_n)$, we know that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \quad \forall y \in C. \tag{3.33}$$

It follows from (H2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq \Theta(y, u_n), \quad \forall y \in C. \tag{3.34}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\right\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C. \tag{3.35}$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (3.35) we have

$$\begin{aligned}
\langle y_t - u_{n_i}, Fy_t\rangle &\geq \langle y_t - u_{n_i}, Fy_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rFx_{n_i})}{r}\right\rangle + \Theta(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, Fy_t - Fu_{n_i}\rangle + \langle y_t - u_{n_i}, Fu_{n_i} - Fx_{n_i}\rangle - \varphi(y_t) \\
&\quad + \varphi(u_{n_i}) - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\right\rangle + \Theta(y_t, u_{n_i}).
\end{aligned} \tag{3.36}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Fu_{n_i} - Fx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of F, we have $\langle y_t - u_{n_i}, Fy_t - Fu_{n_i}\rangle \geq 0$. So, from (H4) and the weakly lower semicontinuity of φ, $(u_{n_i} - x_{n_i})/r \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Fy_t\rangle \geq -\varphi(y_t) + \varphi(w) + \Theta(y_t, w). \tag{3.37}$$

From (H1), (H4), and (3.37), we also have

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\
&= t\left[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)\left[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)\right] \\
&\leq t\left[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)\langle y_t - w, Fy_t\rangle \\
&= t\left[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)t\langle y - w, Fy_t\rangle,
\end{aligned} \tag{3.38}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t\rangle. \tag{3.39}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw \rangle \geq 0. \tag{3.40}$$

This implies that $w \in \text{EP}$.

Next, we show that $w \in I(B, R)$. In fact, since B is β-inverse strongly monotone, B is Lipschitz continuous monotone mapping. It follows from Lemma 2.2 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$, that is, $g - Bv \in R(v)$. Again since $y_{n_i} = J_{R,\lambda}(u_{n_i} - \lambda B u_{n-i})$, we have $u_{n_i} - \lambda u_{n_i} \in (I + \lambda R)(y_{n_i})$, that is, $(1/\lambda)(u_{n_i} - y_{n_i} - \lambda B u_{n_i}) \in R(y_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i}) \right\rangle \geq 0, \tag{3.41}$$

and so

$$\begin{aligned} \langle v - y_{n_i}, g \rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i}) \right\rangle \\ &= \left\langle v - y_{n_i}, Bv - By_{n_i} + By_{n_i} - Bu_{n_i} + \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \right\rangle \\ &\geq \langle v - y_{n_i}, By_{n_i} - Bu_{n_i} \rangle + \left\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \right\rangle. \end{aligned} \tag{3.42}$$

It follows from $\|u_n - y_n\| \to 0$, $\|Bu_n - By_n\| \to 0$ and $y_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty} \langle v - y_{n_i}, g \rangle = \langle v - w, g \rangle \geq 0. \tag{3.43}$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is, $w \in I(B, R)$. Therefore, $w \in \Omega$. It follows that

$$\begin{aligned} \limsup_{n \to \infty} \langle Ax^*, x_n - x^* \rangle &= \lim_{j \to \infty} \left\langle Ax^*, x_{n_j} - x^* \right\rangle \\ &= \langle Ax^*, w - x^* \rangle \\ &\geq 0. \end{aligned} \tag{3.44}$$

□

Proof of (6). First, we note that $x_{n+1} = P_C[(I - \alpha_n A) y_n]$; then for all $x \in C$, we have $\langle x_{n+1} - (I - \alpha_n A) y_n, x_{n+1} - x \rangle \leq 0$.

From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, x_{n+1}-x^*\rangle \\
&= \left\langle x_{n+1}-(I-\alpha_n A)y_n+(I-\alpha_n A)y_n-x^*, x_{n+1}-x^*\right\rangle \\
&= \left\langle x_{n+1}-(I-\alpha_n A)y_n, x_{n+1}-x^*\right\rangle \\
&\quad + \langle (I-\alpha_n A)y_n-x^*, x_{n+1}-x^*\rangle \\
&\le \left\langle (I-\alpha_n A)(y_n-x^*)-\alpha_n Ax^*, x_{n+1}-x^*\right\rangle \\
&= \left\langle (I-\alpha_n A)(y_n-x^*), x_{n+1}-x^*\right\rangle \\
&\quad + 2\alpha_n\langle -Ax^*, x_{n+1}-x^*\rangle \\
&\le \left\|(I-\alpha_n A)(y_n-x^*)\right\|\|x_{n+1}-x^*\| \\
&\quad + 2\alpha_n\langle -Ax^*, x_{n+1}-x^*\rangle \\
&\le \frac{(1-\alpha_n\mu)}{2}\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right) \\
&\quad + 2\alpha_n\langle -Ax^*, x_{n+1}-x^*\rangle,
\end{aligned} \tag{3.45}$$

that is,

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n\mu)\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n\mu}\langle -Ax^*, x_{n+1}-x^*\rangle \\
&= (1-\delta_n)\|x_n-x^*\|^2+\delta_n\sigma_n,
\end{aligned} \tag{3.46}$$

where $\delta_n = \alpha_n\mu$ and $\sigma_n = (2/(1+\alpha_n\mu)\mu)\langle -Ax^*, x_{n+1}-x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, by Lemma 2.3, we conclude that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

□

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