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Research Article

Browder's Fixed Point Theorem and Some Interesting Results in Intuitionistic Fuzzy Normed Spaces

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We define and study Browder's fixed point theorem and relation between an intuitionistic fuzzy convex normed space and a strong intuitionistic fuzzy uniformly convex normed space. Also, we give an example to show that uniformly convex normed space does not imply strongly intuitionistic fuzzy uniformly convex.

1. Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising the field of science and engineering. Now a large number of research papers appear by using the concept of fuzzy set/numbers, and fuzzification of many classical theories has also been made. It has also very useful applications in various fields, for example, nonlinear operator [1], stability problem [2, 3], and so forth. The fuzzy topology [4–8] proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. One of the most important problems in fuzzy topology is to obtain an appropriate concept of an intuitionistic fuzzy metric space and an intuitionistic fuzzy normed space. These problems have been investigated by Park [9] and Saadati and Park [10], respectively, and they introduced and studied a notion of an intuitionistic fuzzy normed space. The topic of fuzzy topology has important applications as quantum particle physics. On the other hand, these problems are also important in modified fuzzy spaces [11–14].

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm [10, 15–17] seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness through the intuitionistic fuzzy norm.

Schauder [18] introduced the fixed point theorem, and since then several generalizations of this concept have been investigated by various authors, namely, Kirk [19], Baillon [20], Browder [21, 22] and many others. Recently, fuzzy version of various fixed point theorems was discussed in [18, 23–28] and also its relations were investigated in [7, 29].

Quite recently the concepts of l-intuitionistic fuzzy compact set and strongly intuitionistic fuzzy uniformly convex normed space are studied, and Schauder fixed point theorem in intuitionistic fuzzy normed space is proved in [30]. As a consequence of Theorem 4.1 [30] and Browder's theorems [26, 27] in crisp normed linear space we have Browder's theorems in intuitionistic fuzzy normed space. Also we give relation between an intuitionistic fuzzy uniformly convex normed space and a strongly intuitionistic fuzzy uniformly convex normed space. Furthermore, we construct an example to show that intuitionistic fuzzy uniformly convex normed space does not imply strongly intuitionistic fuzzy uniformly convex.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative; (b) $*$ is continuous; (c) $a * 1 = a$ for all $a \in [0,1]$; (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2. A binary operation $\Diamond : [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a′) $\Diamond$ is associative and commutative; (b′) $\Diamond$ is continuous; (c′) $a \Diamond 0 = a$ for all $a \in [0,1]$; (d′) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$. Using the notions of continuous t-norm and t-conorm, Saadati and Park [10] have recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 1.3. The five-tuple $(X, \mu, \upsilon, *, \Diamond)$ is said to be *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and μ, υ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$, (i) $\mu(x,t)+\upsilon(x,t) \leq 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, (t/|\alpha|))$ for each $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$, (vi) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, (viii) $\upsilon(x,t) < 1$, (ix) $\upsilon(x,t) = 0$ if and only if $x = 0$, (x) $\upsilon(\alpha x, t) = \upsilon(x, (t/|\alpha|))$ for each $\alpha \neq 0$, (xi) $\upsilon(x,t) \Diamond \upsilon(y,s) \geq \upsilon(x+y, t+s)$, (xii) $\upsilon(x,\cdot) : (0,\infty) \to [0,1]$ is continuous and (xiii) $\lim_{t\to\infty} \upsilon(x,t) = 0$ and $\lim_{t\to 0} \upsilon(x,t) = 1$. In this case (μ, υ) is called an intuitionistic fuzzy norm.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [10].

Definition 1.4. Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNS. Then, a sequence $x = (x_k)$ is said to be intuitionistic fuzzy convergent to $L \in X$ if $\lim \mu(x_k - L, t) = 1$ and $\lim \upsilon(x_k - L, t) = 0$ for all $t > 0$. In this case we write $(\mu, \upsilon) - \lim x_n = x$ or $x_k \xrightarrow{\text{IF}} L$ as $k \to \infty$.

Definition 1.5. Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNS. Then, $x = (x_k)$ is said to be intuitionistic fuzzy Cauchy sequence if $\lim\ \mu(x_{k+p} - x_k, t) = 1$ and $\lim \upsilon(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \ldots$.

Definition 1.6. Let $(X, \mu, \upsilon, *, \diamondsuit)$ be an IFNS. Then $(X, \mu, \upsilon, *, \diamondsuit)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \upsilon, *, \diamondsuit)$ is intuitionistic fuzzy convergent in $(X, \mu, \upsilon, *, \diamondsuit)$.

Definition 1.7 (see [2]). Let $(X, \mu, \upsilon, *, \diamondsuit)$ be an IFNS with the condition

$$\mu(x,t) > 0, \quad \upsilon(x,t) < 1 \quad \text{implies that } x = 0,\ \forall t \in R. \tag{1.1}$$

Let $\|x\|_\alpha = \inf\{t > 0 : \mu(x,t) \geq \alpha \text{ and } \upsilon(x,t) \leq 1-\alpha\}$, for all $\alpha \in (0,1)$. Then $\{\|\cdot\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. These norms are called α-norms on X corresponding to intuitionistic fuzzy norm (μ, υ).

It is easy to see the following.

Proposition 1.8. *Let* $(X, \mu, \upsilon, *, \diamondsuit)$ *be an IFNS satisfying* (1.1), *and let* (x_k) *be a sequence in* X. *Then* $\lim \mu(x_k - x, t) = 1$ *and* $\lim \upsilon(x_k - x, t) = 0$ *if and only if* $\lim_{n\to\infty}\|x_k - x\|_\alpha = 0$ *for all* $\alpha \in (0,1)$.

Proposition 1.9 (see [25]). *Let* $(X, \mu, \upsilon, *, \diamondsuit)$ *be an IFNS satisfying* (1.1). *Then a subset* M *of* X *is l-intuitionistic fuzzy bounded if and only if* M *is bounded with respect to* $\|\cdot\|_\alpha$, *for all* $\alpha \in (0,1)$, *where* $\|\cdot\|_\alpha$ *denotes the* α-*norm of* (μ, υ).

Recently, [1, 29] introduced the concept of strong and weak intuitionistic fuzzy continuity as well as strong and weak intuitionistic fuzzy convergent.

Some notations and results which will be used in this paper [25] are given below.

(i) The linear space $V(= R \text{ and } C)$ is an intuitionistic fuzzy normed space with respect to the intuitionistic fuzzy norm (μ_2, υ_2) defined as

$$\mu_2(x,t) = \begin{cases} 1 & \text{if } t > |x|, \\ 0 & \text{if } t \leq |x|, \end{cases} \qquad \upsilon_2(x,t) = \begin{cases} 0 & \text{if } t > |x|, \\ 1 & \text{if } t \leq |x|, \end{cases} \tag{1.2}$$

satisfying (1.1) condition and $\|x\|_\alpha = |x|$, for all $\alpha \in (0,1)$.

(ii) If $(X, \mu_1, \upsilon_1, *, \diamondsuit)$ is a intuitionistic fuzzy normed space, then U_α^* denotes the set of all linear functions from X to Y which are bounded as linear operators from $(X, \|\cdot\|_\alpha^1)$ to $(Y, \|\cdot\|_\alpha^2)$, where $\|\cdot\|_\alpha^1$ and $\|\cdot\|_\alpha^2$ denote α-norms of (μ_1, υ_1) and (μ_2, υ_2), respectively.

(iii) U_α^* is clearly the first dual space of $(X, \|\cdot\|_\alpha^1)$ for $\alpha \in (0,1)$.

Definition 1.10. Let $(X, \mu_1, \upsilon_1, *, \diamondsuit)$ be an intuitionistic fuzzy normed space satisfying (1.1), and let $(Y, \mu_2, \upsilon_2, *, \diamondsuit)$ be the intuitionistic fuzzy normed space defined in the above remark. A sequence $(x_n) \in X$ is said to be l-intuitionistic fuzzy weakly convergent and converges to x_0 if for all $\alpha \in (0,1)$ and for all $f \in U_\alpha^*$ (dual space with respect to $\|\cdot\|_\alpha$), $f(x_n) \underset{n\to\infty}{\rightarrow} f(x_0)$, that is

$$\lim_{n\to\infty} \mu_2(f(x_n) - f(x_0), t) = 1, \qquad \lim_{n\to\infty} \upsilon_2(f(x_n) - f(x_0), t) = 0, \tag{1.3}$$

for all $f \in U_\alpha^*$.

Definition 1.11 (see [25]). Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNS. A mapping $T : (X, \mu, \upsilon, *, \Diamond) \to (X, \mu, \upsilon, *, \Diamond)$ is said to be intuitionistic nonexpansive if

$$\mu(T(x) - T(y), t) \geq \mu(x - y, t), \qquad \upsilon(T(x) - T(y), t) \leq \mu(x - y, t) \tag{1.4}$$

for all $x, y \in X$, for all $t \in R$.

2. Browder's Theorems and Some Results in IFNS

In this section, we discuss the idea of fuzzy type of some Browder's fixed point theorems in intuitionistic fuzzy normed space by [26, 27]. As a consequence of Theorem 4.1 [30] and Browder's theorems [26, 27] in crisp normed linear space we have Browder's theorems in intuitionistic fuzzy normed space.

Theorem 2.1. *Let K be a nonempty l-intuitionistic fuzzy weakly compact convex subset of a strong intuitionistic fuzzy uniformly convex normed space $(X, \mu, \upsilon, *, \Diamond)$ satisfying* (1.1)*. Then every intuitionistic fuzzy nonexpansive mapping $T : K \to K$ has a fixed point.*

Theorem 2.2. *Let $(X, \mu, \upsilon, *, \Diamond)$ be a strong intuitionistic fuzzy uniformly convex normed space satisfying* (1.1)*. Let K be an l-intuitionistic fuzzy bounded and l-intuitionistic fuzzy closed convex subset of X, and let T be an intuitionistic fuzzy nonexpansive mapping of K into X. Suppose that for $(x_n) \in K$, $(x_n - T(x_n)) \to 0$ as strongly intuitionistic fuzzy convergent while $(x_n - x_0) \to 0$ as l-intuitionistic fuzzy weakly convergent. Then $T(x_0) = x_0$.*

Now, we give relation between a intuitionistic fuzzy uniformly convex normed space and a strongly intuitionistic fuzzy uniformly convex normed space.

Proposition 2.3. *If $(X, \mu, \upsilon, *, \Diamond)$ is a strong intuitionistic fuzzy uniformly convex normed space, then it is a intuitionistic fuzzy uniformly convex normed space.*

Proof. Recall that $(X, \mu, \upsilon, *, \Diamond)$ is said to be strong intuitionistic fuzzy uniformly convex if for each $\varepsilon \in (0, 2)$, there exists $\delta \in (0, 1)$ such that $\phi \neq \mathfrak{D}^{\varepsilon} \subset \mathfrak{F}_{\delta}$. For all $x, y \in X$, where

$$\begin{aligned} \mathfrak{D}^{\varepsilon} &= \left[(x, y) : \mu(x - y, \varepsilon) < \mu(x, 1) * \mu(y, 1),\ \upsilon(x - y, \varepsilon) > \upsilon(x, 1) \Diamond \upsilon(y, 1)\right], \\ \mathfrak{F}_{\delta} &= \left[(x, y) : \mu\left(\frac{x + y}{2}, \delta\right) \geq \mu(x, 1) * \mu(y, 1),\ \upsilon\left(\frac{x + y}{2}, \delta\right) \leq \upsilon(x, 1) \Diamond \upsilon(y, 1)\right] \end{aligned} \tag{2.1}$$

and together with

$$\mu(x, t) = \lim_{s \to t^-} \mu(x, s), \qquad \upsilon(x, t) = \lim_{s \to t^+} \upsilon(x, s) \tag{2.2}$$

and also $(X, \mu, \upsilon, *, \Diamond)$ is said to be intuitionistic fuzzy uniformly convex by Definition 1.7 if for each $\varepsilon \in (0, 2)$, there exists $\delta \in (0, 1)$ such that

$$\left[k_x, k_y \leq 1; k_{x-y} > \varepsilon\right] \Longrightarrow k_{(x+y)/2} \leq \delta \tag{2.3}$$

for all $x, y \in X$, where

$$k_x = \min_{\alpha\in(0,1)} \left[\min_{t>0}\left[\mu(x,t) \geq \alpha,\ \upsilon(x,t) \leq 1-\alpha\right]\right]. \tag{2.4}$$

Suppose that $(X, \mu, \upsilon, *, \Diamond)$ is strong intuitionistic fuzzy uniformly convex normed space. Choose $\varepsilon \in (0,2)$ and $x, y \in X$ such that $[k_x, k_y \leq 1; k_{x-y} > \varepsilon]$. Now, $k_x \leq 1$ implies that

$$\begin{aligned} &\max_{\alpha\in(0,1)} \left[\min_{t>0}\left[\mu(x,t) \geq \alpha,\ \upsilon(x,t) \leq 1-\alpha\right]\right] \leq 1 \\ &\Longrightarrow \min_{t>0}\left[\mu(x,t) \geq \alpha,\ \upsilon(x,t) \leq 1-\alpha\right], \end{aligned} \tag{2.5}$$

for all $\alpha \in (0,1)$.

Case 2.4. If for some α_0(say)$\in (0,1)$,

$$\min_{t>0}\left[\mu(x,t) \geq \alpha_0,\ \upsilon(x,t) \leq 1-\alpha_0\right] < 1, \tag{2.6}$$

then

$$\mu\ (x,1) \geq \alpha_0, \qquad \upsilon(x,1) \leq 1-\alpha_0. \tag{2.7}$$

Case 2.5. If for some α_1(say)$\in (0,1)$, $\min_{t>0}[\mu(x,t) \geq \alpha_1,\ \upsilon(x,t) \leq 1-\alpha_1] = 1$ implies that there exists a sequence (t_n) with $t_n \downarrow 1$, $\mu(x,t_n) \geq \alpha_1$ and with $t_n \uparrow 1$, $\upsilon(x,t) \leq 1-\alpha_1$, for all n

$$\begin{aligned} &\Longrightarrow \lim_{t_n\downarrow 1}\mu(x,t_n) \geq \alpha_1, \qquad \lim_{t_n\uparrow 1}\upsilon(x,t_n) \leq 1-\alpha_1 \\ &\Longrightarrow \mu(x,1) \geq \alpha_1, \qquad \upsilon(x,1) \leq 1-\alpha_1. \end{aligned} \tag{2.8}$$

For (2.7), (2.8) we get that $k_x \leq 1$ implies $\mu(x,1) = 1$ and $\upsilon(x,1) = 1$. Similarly, $k_y \leq 1$ implies $\mu(y,1) = 1$ and $\upsilon(y,1) = 1$. Now, $k_{x-y} > \varepsilon$ implies

$$\max_{\alpha\in(0,1)} \left[\min_{t>0}\left[\mu(x-y,t) \geq \alpha,\ \upsilon(x-y,t) \leq 1-\alpha\right]\right] > \varepsilon \tag{2.9}$$

$\Rightarrow$ there exists α_2(say)$\in (0,1)$ such that

$$\min_{t>0}\left[\mu(x-y,t) \geq \alpha_2,\ \upsilon(x-y,t) \leq 1-\alpha_2\right] > \varepsilon \tag{2.10}$$

$\Rightarrow \mu(x-y,\varepsilon) < \alpha_2 < 1$ and $\upsilon(x-y,\varepsilon) > 1-\alpha_2 > 0$. We claim that

$$\mu(x-y,\varepsilon) < 1, \qquad \upsilon(x-y,\varepsilon) > 0. \tag{2.11}$$

If possible suppose that $\mu(x-y,\varepsilon)=1$ and $\upsilon(x-y,\varepsilon)=0$ implies that there exists a sequence (ε_n) with $\varepsilon_n \downarrow \varepsilon$ such that $\lim_{\varepsilon_n \downarrow \varepsilon}\mu(x-y,\varepsilon_n)=1$ and with $\varepsilon_n \uparrow \varepsilon$ such that $\lim_{\varepsilon_n \uparrow \varepsilon}\upsilon(x-y,\varepsilon_n)=0 \Rightarrow \mu(x-y,\varepsilon_n)=1$ and $\upsilon(x-y,\varepsilon_n)=0$, for all $n \Rightarrow \mu(x-y,\varepsilon_n)>\alpha$ and $\upsilon(x-y,\varepsilon_n)<1-\alpha$, for all n and $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x-y,t)\ge\alpha,\ \upsilon(x-y,t)\le 1-\alpha\right]\le\varepsilon_n, \tag{2.12}$$

for all n and $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x-y,t)\ge\alpha,\ \upsilon(x-y,t)\le 1-\alpha\right]\le\varepsilon, \tag{2.13}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\max_{\alpha\in(0,1)}\left[\min_{t>0}\left[\mu(x-y,t)\ge\alpha,\ \upsilon(x-y,t)\le 1-\alpha\right]\right]\le\varepsilon, \tag{2.14}$$

$\Rightarrow k_{x-y}\le\varepsilon$, a contradiction. Thus

$$k_{x-y}>\varepsilon \Longrightarrow \mu(x-y,\varepsilon)<1, \qquad \upsilon(x-y,\varepsilon)>0. \tag{2.15}$$

Now from above we get that, $[k_x,k_y\le 1; k_{x-y}>\varepsilon]$ implies

$$\mu(x-y,\varepsilon)<[\mu(x,1)*\mu(y,1)], \quad \upsilon(x-y,\varepsilon)>[\upsilon(x,1)\Diamond\upsilon(y,1)] \tag{2.16}$$

$\Rightarrow$

$$\mu\left(\frac{x+y}{2},\delta\right)\ge[\mu(x,1)*\mu(y,1)], \qquad \upsilon\left(\frac{x+y}{2},\delta\right)\le[\upsilon(x,1)\Diamond\upsilon(y,1)] \tag{2.17}$$

such that the idea of strong uniform convexity of $(X,\mu,\upsilon,*,\Diamond) \Rightarrow$

$$\mu\left(\frac{x+y}{2},\delta\right)\ge 1, \qquad \upsilon\left(\frac{x+y}{2},\delta\right)\le 0 \tag{2.18}$$

$\Rightarrow$

$$\mu\left(\frac{x+y}{2},\delta\right)\ge 1>\alpha, \qquad \upsilon\left(\frac{x+y}{2},\delta\right)\le 0<1-\alpha, \tag{2.19}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu\left(\frac{x+y}{2},t\right)\ge\alpha,\ \upsilon\left(\frac{x+y}{2},t\right)\le 1-\alpha\right]\le\delta, \tag{2.20}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\max_{\alpha\in(0,1)}\left[\min_{t>0}\left[\mu\left(\frac{x+y}{2},t\right)\geq\alpha,\ \upsilon\left(\frac{x+y}{2},t\right)\leq 1-\alpha\right]\right]\leq\delta. \tag{2.21}$$

Now for any positive number ε_1(say) we get

$$\mu(x,t+\varepsilon_1)\geq\mu(x,t),\qquad \upsilon(x,t+\varepsilon_1)\leq\upsilon(x,t). \tag{2.22}$$

Thus,

$$\mu(x,t)\geq\alpha,\qquad \upsilon(x,t)\leq 1-\alpha \tag{2.23}$$

implies that

$$\mu(x,t+\varepsilon_1)\geq\alpha,\qquad \upsilon(x,t+\varepsilon_1)\leq 1-\alpha, \tag{2.24}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x,t)\geq\alpha,\upsilon(x,t)\leq 1-\alpha\right]\geq\min_{t>0}\left[\mu(x,t+\varepsilon_1)\geq\alpha,\ \upsilon(x,t+\varepsilon_1)\leq 1-\alpha\right], \tag{2.25}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x,t)\geq\alpha,\ \upsilon(x,t)\leq 1-\alpha\right]\geq\min_{s-\varepsilon_1>0}\left[\mu(x,s)\geq\alpha,\ \upsilon(x,s)\leq 1-\alpha\right] \tag{2.26}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x,t)\geq\alpha,\ \upsilon(x,t)\leq 1-\alpha\right]\geq\min_{s>\varepsilon_1}\left[\mu(x,s)\geq\alpha,\ \upsilon(x,s)\leq 1-\alpha\right]-\varepsilon_1, \tag{2.27}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\min_{t>0}\left[\mu(x,t)\geq\alpha,\ \upsilon(x,t)\leq 1-\alpha\right]\geq\min_{s>0}\left[\mu(x,s)\geq\alpha,\ \upsilon(x,s)\leq 1-\alpha\right]-\varepsilon_1, \tag{2.28}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\max_{\alpha\in(0,1)}\left[\min_{t>0}\left[\mu(x,t)\geq\alpha,\ \upsilon(x,t)\leq 1-\alpha\right]\right]\geq\max_{\alpha\in(0,1)}\left[\min_{s>0}\left[\mu(x,s)\geq\alpha,\ \upsilon(x,s)\leq 1-\alpha\right]\right]-\varepsilon_1, \tag{2.29}$$

for all $\alpha \in (0,1) \Rightarrow$

$$\max_{\alpha\in(0,1)}\left[\min_{t>0}\left[\mu(x,t)\geq\alpha,\ \upsilon(x,t)\leq 1-\alpha\right]\right]\geq k_s-\varepsilon_1. \tag{2.30}$$

From (2.21), (2.30), we get that, $[k_x, k_y \leq 1; k_{x-y} > \varepsilon]$ implies $k_{(x+y)/2} - \varepsilon_1 \leq \delta$, for any $\varepsilon_1 > 0 \Rightarrow k_{(x+y)/2} - \delta \leq \varepsilon_1$, for any $\varepsilon_1 > 0 \Rightarrow k_{(x+y)/2} - \delta \leq 0$ that is $k_{(x+y)/2} \leq \delta$. Hence, we have $[k_x, k_y \leq 1; k_{x-y} > \varepsilon] \Rightarrow k_{(x+y)/2} \leq \delta$. So $(X, \mu, \upsilon, *, \Diamond)$ is intuitionistic fuzzy uniformly convex. □

Remark 2.6. However, the reverse of this Proposition 2.3 is untrue. This fact can be seen in the following example.

Let $X = R^2$ (R is the set of all real numbers). Suggest two norms on X as the following: $\|x\|' = (x_1^2 + x_2^2)^{1/2}$ and $\|x\|'' = \max[|x_1|, |x_2|]$, where $x = (x_1, x_2)$. Clearly $\|x\|' \geq \|x\|''$, for all $x \in X$, and it can be easily verified that X is uniformly convex with regard to $\|\cdot\|'$. Define a function $\mu : X \times R \to [0,1]$ and $\upsilon : X \times R \to [0,1]$ by

$$\mu(x,t) = \begin{cases} 1 & \text{if } t \geq \|x\|', \\ \dfrac{1}{2} & \text{if } \|x\|'' \leq t < \|x\|', \\ \dfrac{1}{3} & \text{if } 0 < t < \|x\|'', \\ 0 & \text{if } t \leq 0, \end{cases} \qquad \upsilon(x,t) = \begin{cases} 0 & \text{if } t \geq \|x\|', \\ \dfrac{1}{3} & \text{if } \|x\|'' \leq t < \|x\|', \\ \dfrac{1}{2} & \text{if } 0 < t < \|x\|'', \\ 1 & \text{if } t \leq 0. \end{cases} \tag{2.31}$$

It can be easily verified that (μ, υ) is a intuitionistic fuzzy norm on X. Now,

$$\left[k_x = \|x\|', k_y = \|y\|', k_{x-y} = \|x-y\|', k_{(x+y)/2} = \left\|\frac{x+y}{2}\right\|'\right]. \tag{2.32}$$

Thus $(X, \mu, \upsilon, *, \Diamond)$ is intuitionistic fuzzy uniformly convex normed space. So, since this is not uniformly convex with regard to $\|\cdot\|'$, therefore there exists $\varepsilon \in (0,2)$ such that for all $\delta \in (0,1)$ we have $x_\delta, y_\delta \in X$ such that,

$$\left[\|x_\delta\|'' \leq 1, \|y_\delta\|'' \leq 1, \|x_\delta - y_\delta\|'' > \varepsilon \text{ but } \left\|\frac{x+y}{2}\right\|'' > \delta\right]. \tag{2.33}$$

Now, $\|x_\delta\|'' \leq 1 \Rightarrow \mu(x_\delta, 1) \geq 1/2$ and $\upsilon(x_\delta, 1) \leq 1/2$. Reversely, $\mu(x_\delta, 1) \geq 1/2$ and $\upsilon(x_\delta, 1) \leq 1/2 \Rightarrow \lim_{\varepsilon_n \downarrow 1} \mu(x_\delta, \varepsilon_n) \geq 1/2$ and $\lim_{\varepsilon_n \uparrow 1} \upsilon(x_\delta, 1) \leq 1/2 \Rightarrow \|x_\delta\|'' \leq \varepsilon_n$, for $n \Rightarrow \|x_\delta\|'' \leq 1$. Hence

$$\|x_\delta\|'' \leq 1 \Longleftrightarrow \mu(x_\delta, 1) \geq \frac{1}{2}, \upsilon(x_\delta, 1) \leq \frac{1}{2}. \tag{2.34}$$

Again,

$$\|y_\delta\|'' \leq 1 \Longleftrightarrow \mu(y_\delta, 1) \geq \frac{1}{2}, \upsilon(y_\delta, 1) \leq \frac{1}{2}. \tag{2.35}$$

So, $\|x_\delta - y_\delta\|'' > \varepsilon > 0 \Rightarrow \mu(x_\delta - y_\delta, \varepsilon) = 1/3 \Rightarrow \mu(x_\delta - y_\delta, \varepsilon) \geq 1/3$ and $\upsilon(x_\delta - y_\delta, \varepsilon) \leq 1/3$. We suggest that $\mu(x_\delta - y_\delta, \varepsilon) = 1/3$ and $\upsilon(x_\delta - y_\delta, \varepsilon) = 1/3$. If $\mu(x_\delta - y_\delta, \varepsilon) > 1/3, \upsilon(x_\delta - y_\delta, \varepsilon) < 1/3 \Rightarrow \lim_{\varepsilon_n \downarrow \varepsilon} \mu(x_\delta - y_\delta, \varepsilon) > 1/3$, $\lim_{\varepsilon_n \uparrow \varepsilon} \upsilon(x_\delta - y_\delta, \varepsilon) < 1/3 \Rightarrow \mu(x_\delta - y_\delta, \varepsilon_n) > 1/3$, $\upsilon(x_\delta - y_\delta, \varepsilon_n) < 1/3$, for $n \Rightarrow \mu(x_\delta - y_\delta, \varepsilon_n) > 1/2$, $\upsilon(x_\delta - y_\delta, \varepsilon_n) < 1/2$, for $n \Rightarrow \|x_\delta - y_\delta\|'' \leq \varepsilon_n$, for $n \Rightarrow \|x_\delta - y_\delta\|'' \leq \varepsilon$; however this is a contradiction that is

$$\|x_\delta - y_\delta\|'' > \varepsilon \Longrightarrow \mu(x_\delta - y_\delta, \varepsilon) = \frac{1}{3}, \upsilon(x_\delta - y_\delta, \varepsilon) = \frac{1}{3}. \tag{2.36}$$

Otherwise $\mu(x_\delta - y_\delta, \varepsilon) = 1/3$, $\upsilon(x_\delta - y_\delta, \varepsilon) = 1/3 \Rightarrow \mu(x_\delta - y_\delta, \varepsilon) \leq 1/3$, $\upsilon(x_\delta - y_\delta, \varepsilon) \geq 1/3 \Rightarrow \|x_\delta - y_\delta\|'' > \varepsilon$, and thus

$$\mu(x_\delta - y_\delta, \varepsilon) = \frac{1}{3}, \upsilon(x_\delta - y_\delta, \varepsilon) = \frac{1}{3} \Longrightarrow \|x_\delta - y_\delta\|'' > \varepsilon. \tag{2.37}$$

By (2.36) and (2.37) we get

$$\|x_\delta - y_\delta\|'' > \varepsilon \Longleftrightarrow \mu(x_\delta - y_\delta, \varepsilon) = \frac{1}{3}, \upsilon(x_\delta - y_\delta, \varepsilon) = \frac{1}{3}. \tag{2.38}$$

Again

$$\left\|\frac{x_\delta - y_\delta}{2}\right\|'' > \delta \Longleftrightarrow \mu\left(\frac{x_\delta + y_\delta}{2}, \delta\right) = \frac{1}{3}, \quad \upsilon\left(\frac{x_\delta + y_\delta}{2}, \delta\right) = \frac{1}{3}. \tag{2.39}$$

Hence with (2.35), (2.36), (2.37), (2.38), and (2.39) we have

$$\mu(x_\delta - y_\delta, \varepsilon) < [\mu(x_\delta, 1) * \mu(y_\delta, 1)], \quad \upsilon(x_\delta - y_\delta, \varepsilon) > [\upsilon(x_\delta, 1) \Diamond \upsilon(y_\delta, 1)] \tag{2.40}$$

but

$$\mu\left(\frac{x_\delta + y_\delta}{2}, \delta\right) < [\mu(x_\delta, 1) * \mu(y_\delta, 1)], \quad \upsilon\left(\frac{x_\delta + y_\delta}{2}, \delta\right) > [\upsilon(x_\delta, 1) \Diamond \upsilon(y_\delta, 1)]. \tag{2.41}$$

Moreover $(x_\delta, y_\delta) \in \mathfrak{D}^\varepsilon$ but $(x_\delta, y_\delta) \notin \mathfrak{F}_\delta$ that is $\mathfrak{D}^\varepsilon \not\subset \mathfrak{F}_\delta$. However $(X, \mu, \upsilon, *, \Diamond)$ is not a uniformly convex intuitionistic fuzzy normed space.

3. Conclusion

We studied here the concept of intuitionistic fuzzy normed space as an extension of the fuzzy normed space, which provides a larger setting to deal with the uncertainly and vagueness in natural problems arising in many branches of science and engineering. In this new setup we established Browder's fixed point theorem and some interesting results in intuitionistic fuzzy normed space which could be very useful tools in the development of fuzzy set theory.

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