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Research Article

Robustness of Mann Type Algorithm with Perturbed Mapping for Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is to study the robustness of Mann type algorithm in the sense that approximately perturbed mapping does not alter the convergence of Mann type algorithm. It is proven that Mann type algorithm with perturbed mapping $x_{n+1} = \lambda_n x_n + (1-\lambda_n)(Tx_n + e_n) - \lambda_n \mu_n F(x_n)$ remains convergent in a Banach space setting where $\lambda_n, \mu_n \in [0,1]$, T a nonexpansive mapping, e_n, $n = 0, 1, \ldots$, errors and F a strongly accretive and strictly pseudocontractive mapping.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X, and $T : C \to C$ a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Throughout this paper it is assumed that $\mathrm{Fix}(T) \neq \emptyset$. Construction of fixed points of nonlinear mappings is an important and active research area. In particular, iterative methods for finding fixed points of nonexpansive mappings have received vast investigation since these methods find applications in a variety of applied areas of variational inequality problems, equilibrium problems, inverse problems, partial differential equations, image recovery, and signal processing (see, e.g., [1–17]).

In 1953, Mann [18] introduced an iterative algorithm which is now referred to as Mann's algorithm. Most of the literature deals with the special case of the general Mann's algorithm; that is, for an arbitrary initial guess $x_0 \in C$, the sequence $\{x_n\}$ is generated by the recursive manner

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)Tx_n, \quad \forall n \geq 0, \tag{1.1}$$

where C is a convex subset of a Banach space X, $T : C \to C$ is a mapping and $\{\lambda_n\}$ is a sequence in the interval $[0,1]$. It is well known that Mann's algorithm can be employed to approximate fixed points of nonexpansive mappings and zeros of (strongly) accretive mappings in Hilbert spaces and Banach spaces. Many convergence theorems have been announced and published by a large number of authors. A typical convergence result in connection with the Mann's algorithm is the following theorem proved by Ishikawa [19].

Theorem IS (see [19])

Let C be a nonempty subset of a Banach space X and let $T : C \to X$ be a nonexpansive mapping. Let $\{\lambda_n\}$ be a real sequence satisfying the following control conditions:

(a) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(b) $0 \leq \lambda_n \leq \lambda < 1$.

Let $\{x_n\}$ be defined by (1.1) such that $x_n \in C$ for all $n \geq 0$. If $\{x_n\}$ is bounded then $x_n - Tx_n \to 0$ as $n \to \infty$.

The interest and importance of Theorem IS lie in the fact that strong or weak convergence of the sequence $\{x_n\}$ can be achieved under certain appropriate assumptions imposed on the mapping T, the domain $D(T)$ or the space X. In an infinite-dimensional space X, Mann's algorithm has only weak convergence, in general. In fact, it is known that if the sequence $\{\lambda_n\}$ is such that $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$, then Mann's algorithm converges weakly to a fixed point of T provided that the underlying space X is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial's property. See, for example, [20, 21].

The study of the robustness of Mann's algorithm is initiated by Combettes [22] where he considered a parallel projection method algorithm in signal synthesis (design and recovery) problems in a real Hilbert space H as follows:

$$x_{n+1} = x_n + \lambda_n \left(\sum_{i=1}^{m} w_i (P_i x_n + c_{i,n}) - x_n \right), \tag{1.2}$$

where for each i, $P_i(x)$ is the (nearest point) projection of a signal $x \in H$ onto a closed convex subset S_i of H [23] (S_i is interpreted as the ith constraint set of the signals), $\{\lambda_n\}_{n\geq 0}$ is a sequence of relaxation parameters in $(0,2)$, $\{w_i\}_{i=1}^{m}$ are strictly positive weights such that $\sum_{i=1}^{m} w_i = 1$, and $c_{i,n}$ stands for the error made in computing the projection onto S_i at iteration n. Then he proved the following robustness result of algorithm (1.2).

Theorem 1.1 (see [22]). *Assume $G := \bigcap_{i=1}^{m} S_i \neq \emptyset$. Assume also*

(i) $\sum_{n=0}^{\infty} \lambda_n(2-\lambda_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \lambda_n \| \sum_{i=1}^{m} w_i c_{i,n} \| < \infty$.

Then the sequence $\{x_n\}$ generated by (1.2) *converges weakly to a point in G.*

Define a mapping $T : H \longrightarrow H$ *by*

$$Tx := 2\sum_{i=1}^{m} w_i P_i(x) - x, \quad \forall x \in H, \tag{1.3}$$

and put

$$e_n := 2\sum_{i=1}^{m} w_i c_{i,n}, \quad \alpha_n := \frac{\lambda_n}{2} \in (0,1), \quad \forall n \geq 0. \tag{1.4}$$

Since P_i *is a projection, the mapping* $V_i := 2P_i - I$ *is nonexpansive. Thus* $P_i = (I + V_i)/2$ *and algorithm* (1.2) *can be rewritten as*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(Tx_n + e_n), \tag{1.5}$$

where T *is given by* (1.3)*. Note that* T *can be written as* $T = \sum_{i=1}^{m} w_i V_i$ *and thus* T *is nonexpansive. Note also that* $\mathrm{Fix}(T) = \bigcap_{i=1}^{m} \mathrm{Fix}(V_i) = G$*. Furthermore, conditions (i) and (ii) in Theorem 1.1 can be stated as*

(i)′ $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$

(ii)″ $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Very early, some authors had considered Mann iterations in the setting of uniformly convex Banach spaces and established strong and weak convergence results for Mann iterations; see, e.g., [24, 25]. Recently, Kim and Xu [26] studied the robustness of Mann's algorithm for nonexpansive mappings in Banach spaces and extended Combettes' robustness result (Theorem 1.1 above) for projections from Hilbert spaces to the setting of uniformly convex Banach spaces.

Theorem 1.2 (see [26, Theorem 3.3]). *Assume that* X *is a uniformly convex Banach space. Assume, in addition, that either* X^* *has the KK- property or* X *satisfies Opial's property. Let* $T : X \longrightarrow X$ *be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$*. Given an initial guess* $x_0 \in X$*. Let* $\{x_n\}$ *be generated by the following perturbed Mann's algorithm:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(Tx_n + e_n), \quad \forall n \geq 0, \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{e_n\} \subset X$ *satisfy the following properties:*

(i) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Then the sequence $\{x_n\}$ *converges weakly to a fixed point of* T*.*

Further, Kim and Xu [26] also extended the robustness to nonexpansive mappings which are defined on subsets of a Hilbert space and to accretive operators.

Theorem 1.3 (see [26, Theorem 4.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ via one of the following algorithms* (1.7) *and (1.7):*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(Tx_n + e_n), \quad \forall n \geq 0,$$
$$x_{n+1} = P_C[(1-\alpha_n)x_n + \alpha_n(Tx_n + e_n)], \quad \forall n \geq 0, \tag{1.7}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{e_n\} \subset X$ are such that

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Theorem 1.4 (see [26, Theorem 5.1]). *Let X be a uniformly convex Banach space. Assume in addition that either X^* has the KK- property or X satisfies Opial's property. Let A be an m-accretive operator in X such that $A^{-1}(0) \neq \emptyset$. Moreover, assume that $\{\alpha_n\} \subset (0,1)$, $\{c_n\} \subset (0,\infty)$, and $\{e_n\} \subset X$ satisfy the following properties:*

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$;

(iii) $0 < c_* < c_n < c^* < \infty$, *where c_* and c^* are two constants;*

(iv) $\sum_{n=0}^{\infty} |c_{n+1} - c_n| < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in X$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(J_{c_n}x_n + e_n), \quad \forall n \geq 0, \tag{1.8}$$

converges weakly to a point of $A^{-1}(0)$.

Let X be a real reflexive Banach space. Let $T : X \to X$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$ where $\delta, \lambda \in (0,1)$. In this paper, inspired by Combettes' robustness result (Theorem 1.1 above) and Kim and Xu's robustness result (Theorem 1.2 above) we will consider the robustness of Mann type algorithm with perturbed mapping, which generates, from an arbitrary initial guess $x_0 \in X$, a sequence $\{x_n\}$ by the recursive manner

$$y_n = \lambda_n x_n + (1-\lambda_n)(Tx_n + e_n),$$
$$x_{n+1} = y_n - \lambda_n \mu_n F(x_n), \quad \forall n \geq 0, \tag{1.9}$$

where $\{\lambda_n\}$, $\{\mu_n\}$, and $\{e_n\}$ are sequences in $[0,1]$ and in X, respectively, such that

(i) $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(1-\lambda_n)\|e_n\| < \infty$;

(iii) $\sum_{n=0}^{\infty} \lambda_n\mu_n < \infty$.

More precisely, we will prove under conditions (i)–(iii) the weak convergence of the algorithm (1.9) in a uniformly convex Banach space X which either has the KK-property or satisfies Opial's property. This theorem extends Kim and Xu's robustness result (Theorem 1.2 above) from Mann's algorithm (1.6) with errors to Mann type algorithm (1.9) with perturbed mapping F. On the other hand, we also extend Kim and Xu's robustness results (Theorems 1.3 and 1.4 above) for nonexpansive mappings which are defined on subsets of a Hilbert space and accretive operators in a uniformly convex Banach space from Mann's algorithm with errors to Mann type algorithm with perturbed mapping.

Throughout this paper, we use the following notations:

(i) $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence,

(ii) $\omega_w(\{x_n\}) = \{x : \exists x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let X be a real Banach space. Recall that the norm of X is said to be Fréchet differentiable if, for each $x \in S(X)$, the unit sphere of X, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists and is attained uniformly in $y \in S(X)$. In this case, we have

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + b(\|h\|) \tag{2.2}$$

for all $x, h \in X$, where J is the normalized duality map from X to X^* defined by

$$j(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, \tag{2.3}$$

$\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^*, and b is a function defined on $[0, \infty)$ such that $\lim_{t \downarrow 0} b(t)/t = 0$. Examples of Banach spaces which have a Fréchet differentiable norm include l^p and L^p for $1 < p < \infty$ (these spaces are actually uniformly smooth).

It is known that a Banach space X is Fréchet differentiable if and only if the duality map J is single-valued and norm-to-norm continuous.

We need the concept of the KK-property. A Banach space X is said to have the KK-property (the Kadec-Klee property) if, for any sequence $\{z_n\}$ in X, the conditions $z_n \rightharpoonup z$ and $\|z_n\| \to \|z\|$ imply that $z_n \to z$. It is known [27, Remark 3.2] that the dual space of a reflexive Banach space with a Fréchet differentiable norm has the KK-property.

Recall now that X satisfies Opial's property [28] provided that, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad y \in X, \ y \ne x. \tag{2.4}$$

It is known [28] that each l^p $(1 \le p < \infty)$ enjoys this property, while L^p does not unless $p = 2$. It is known [29] that any separable Banach space can be equivalently renormed so that it satisfies Opial's property.

Recall that a Banach space X is said to be uniformly convex if, for each $0 < \varepsilon \le 2$, the modulus of convexity $\delta_X(\varepsilon)$ of X defined by

$$\delta_X(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in X,\ \|x\| \le 1,\ \|y\| \le 1,\ \text{and}\ \|x-y\| \ge \varepsilon\right\} \tag{2.5}$$

is positive.

We need an inequality characterization of uniform convexity.

Lemma 2.1 (see [30]). *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|) \tag{2.6}$$

for all $\lambda \in [0,1]$ and $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

A mapping F with domain $D(F)$ and range $R(F)$ in X is called δ-strongly accretive if for each $x, y \in D(F)$,

$$\langle Fx - Fy, j(x-y)\rangle \ge \delta\|x-y\|^2, \quad \forall j(x-y) \in J(x-y) \tag{2.7}$$

for some $\delta \in (0,1)$. F is called λ-strictly pseudocontractive if for each $x, y \in D(F)$,

$$\langle Fx - Fy, j(x-y)\rangle \ge \|x-y\|^2 - \lambda\|x - y - (Fx - Fy)\|^2, \quad \forall j(x-y) \in J(x-y) \tag{2.8}$$

for some $\lambda \in (0,1)$. It is easy to see that (2.8) can be rewritten as

$$\langle (I-F)x - (I-F)y, j(x-y)\rangle \ge \lambda\|(I-F)x - (I-F)y\|^2. \tag{2.9}$$

The following proposition will be used frequently throughout this paper. For the sake of completeness, we include its proof.

Proposition 2.2. *Let X be a real Banach space and $F : D(F) \to X$ a mapping.*

(i) *If F is a λ-strictly pseudocontractive, then F is Lipschitz continuous with constant* $(1 + 1/\lambda)$.

(ii) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \ge 1$, then for each fixed $\mu \in [0,1]$, the mapping $I - \mu F$ has the following property:*

$$\|(I-\mu F)x - (I-\mu F)y\| \le \left(1 - \mu\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|, \quad \forall x, y \in D(F). \tag{2.10}$$

Proof. (i) From (2.9), we derive

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \langle (I-F)x-(I-F)y, j(x-y)\rangle \\ &\le \|(I-F)x-(I-F)y\|\|x-y\|, \quad \forall j(x-y)\in J(x-y)\end{aligned} \tag{2.11}$$

which implies that

$$\|(I-F)x-(I-F)y\| \le \frac{1}{\lambda}\|x-y\|. \tag{2.12}$$

Thus

$$\|Fx-Fy\| \le \|(I-F)x-(I-F)y\| + \|x-y\| \le \left(1+\frac{1}{\lambda}\right)\|x-y\|, \tag{2.13}$$

and so F is Lipschitz continuous with constant $(1+1/\lambda)$.

(ii) From (2.8) and (2.9), we obtain

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \langle (I-F)x-(I-F)y, j(x-y)\rangle \\ &= \|x-y\|^2 - \langle Fx-Fy, j(x-y)\rangle \\ &\le (1-\delta)\|x-y\|^2.\end{aligned} \tag{2.14}$$

Since $\delta+\lambda\ge 1 \Leftrightarrow \sqrt{(1-\delta)/\lambda}\in(0,1]$, we have

$$\|(I-F)x-(I-F)y\| \le \left(\sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\|, \quad \forall x,y\in D(F). \tag{2.15}$$

Consequently, for each fixed $\mu\in[0,1]$, we have

$$\begin{aligned}\|x-y-\mu(F(x)-F(y))\| &= \|(1-\mu)(x-y)+\mu[(I-F)x-(I-F)y]\| \\ &\le (1-\mu)\|x-y\|+\mu\|(I-F)x-(I-F)y\| \\ &\le (1-\mu)\|x-y\|+\mu\left(\sqrt{\frac{1-\delta}{\lambda}}\right)\|x-y\| \\ &= \left(1-\mu\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|, \quad \forall x,y\in D(F).\end{aligned} \tag{2.16}$$

This shows that inequality (2.10) holds. □

Proposition 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X.*

(i) *Reference [31] (demiclosedness principle). If $T : C \to C$ is a nonexpansive mapping and if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$.*

(ii) *Reference [32]. If C is also bounded, then there exists a continuous, strictly increasing, and convex function $\gamma : [0, \infty) \to [0, \infty)$ (depending only on the diameter of C) with $\gamma(0) = 0$ and such that*

$$\gamma\left(\left\|T(\lambda x + (1 - \lambda)y) - (\lambda Tx + (1 - \lambda)Ty)\right\|\right) \leq \|x - y\| - \|Tx - Ty\| \tag{2.17}$$

for all $x, y \in C$, $\lambda \in [0, 1]$, and nonexpansive mappings $T : C \to X$.

We also use the following elementary lemma.

Lemma 2.4 (see [33]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.*

3. Robustness of Mann Type Algorithm with Perturbed Mapping

Let X be a real reflexive Banach space. Let $T : X \to X$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. We now discuss the robustness of Mann type algorithm with perturbed mapping, which generates, from an initial guess $x_0 \in X$, a sequence $\{x_n\}$ as follows:

$$\begin{aligned} y_n &= \lambda_n x_n + (1 - \lambda_n)(Tx_n + e_n), \\ x_{n+1} &= y_n - \lambda_n \mu_n F(x_n), \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

where $\{\lambda_n\}, \{\mu_n\}$, and $\{e_n\}$ are sequences in $[0, 1]$ and in X, respectively, such that

(i) $\sum_{n=0}^{\infty} \lambda_n(1 - \lambda_n) = \infty$;

(ii) $\sum_{\mathrm{n}=0}^{\infty}(1 - \lambda_n)\|e_n\| < \infty$;

(iii) $\sum_{n=0}^{\infty} \lambda_n \mu_n < \infty$.

We remark that Mann type algorithm with perturbed mapping is based on Mann iteration method and steepest-descent method. Indeed, in algorithm (3.1), one iteration step

"$y_n = \lambda_n x_n + (1-\lambda_n)(Tx_n + e_n)$" is taken from Mann iteration method, and another iteration step "$x_{n+1} = y_n - \lambda_n\mu_n F(x_n)$" is taken from steepest-descent method.

We first discuss some properties of algorithm (3.1).

Lemma 3.1. *Let $\{x_n\}$ be generated by algorithm* (3.1) *and let $p \in \mathrm{Fix}(T)$. Then $\lim_{n\to\infty}\|x_n - p\|$ exists.*

Proof. We have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad= \|y_n - \lambda_n\mu_n F(x_n) - p\| \\
&\quad= \|\lambda_n x_n + (1-\lambda_n)(Tx_n+e_n) - \lambda_n\mu_n F(x_n) - p\| \\
&\quad\le \|\lambda_n[(I-\mu_n F)x_n - p] + (1-\lambda_n)(Tx_n - p)\| + (1-\lambda_n)\|e_n\| \\
&\quad\le (1-\lambda_n)\|Tx_n - p\| + \lambda_n\|(I-\mu_n F)x_n - p\| + (1-\lambda_n)\|e_n\| \\
&\quad\le (1-\lambda_n)\|x_n - p\| + \lambda_n[(I-\mu_n F)x_n - (I-\mu_n F)p \\
&\qquad\qquad + \|(I-\mu_n F)p - p\|] + (1-\lambda_n)\|e_n\| \\
&\quad\le (1-\lambda_n)\|x_n-p\| + \lambda_n\left[1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|x_n-p\| \\
&\qquad + \lambda_n\mu_n\|F(p)\| + (1-\lambda_n)\|e_n\| \\
&\quad= \left(1-\lambda_n\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-p\| + \lambda_n\mu_n\|F(p)\| + (1-\lambda_n)\|e_n\| \\
&\quad\le \|x_n-p\| + \lambda_n\mu_n\|F(p)\| + (1-\lambda_n)\|e_n\|.
\end{aligned} \tag{3.2}$$

The conclusion of the lemma is a consequence of Lemma 2.4. □

Proposition 3.2. *Let X be a uniformly convex Banach space.*

(i) *For all $p, q \in \mathrm{Fix}(T)$ and $0 \le t \le 1$, $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists.*

(ii) *If, in addition, the dual space X^* of X has the KK-property, then the weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a singleton.*

Proof. (i) For integers $n, m \geq 1$, define the mappings T_n and $S_{n,m}$ as follows:

$$T_n x := \lambda_n x + (1-\lambda_n)Tx + (1-\lambda_n)e_n - \lambda_n \mu_n F(x), \quad \forall x \in X, \tag{3.3}$$

and $S_{n,m} := T_{n+m-1}T_{n+m-2}\cdots T_n$. It is easy to see that $x_{n+m} = S_{n,m}x_n$. First, let us show that T_n and $S_{n,m}$ are nonexpansive. Indeed, for all $x, y \in X$, using Proposition 2.2 no. (ii) we have

$$\begin{aligned}
\|T_n x - T_n y\| &= \big\| \left[\lambda_n x + (1-\lambda_n)Tx + (1-\lambda_n)e_n - \lambda_n \mu_n F(x)\right] \\
&\qquad - \left[\lambda_n y + (1-\lambda_n)Ty + (1-\lambda_n)e_n - \lambda_n \mu_n F(y)\right] \big\| \\
&= \big\| \left[\lambda_n (I - \mu_n F)x + (1-\lambda_n)Tx\right] - \left[\lambda_n (I - \mu_n F)y + (1-\lambda_n)Ty\right] \big\| \\
&\leq \lambda_n \|(I-\mu_n F)x - (I-\mu_n F)y\| + (1-\lambda_n)\|Tx - Ty\| \\
&\leq \lambda_n \left[1 - \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right] \|x-y\| + (1-\lambda_n)\|x-y\| \\
&= \left[1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right] \|x-y\|.
\end{aligned} \tag{3.4}$$

Thus $T_n : X \to X$ is nonexpansive (due to $\lambda_n \mu_n \in [0,1]$) and so is $S_{n,m}$.

Second, let us show that for each $v \in \mathrm{Fix}(T)$,

$$\|S_{n,m}v - v\| \leq \sum_{j=n}^{n+m-1} \left[(1-\lambda_j)\|e_j\| + \lambda_j \mu_j \|F(v)\|\right]. \tag{3.5}$$

Indeed, whenever $m = 1$, we have

$$\begin{aligned}
\|S_{n,1}v - v\| &= \|T_n v - v\| \\
&= \|\lambda_n v + (1-\lambda_n)Tv + (1-\lambda_n)e_n - \lambda_n \mu_n F(v) - v\| \\
&\leq (1-\lambda_n)\|e_n\| + \lambda_n \mu_n \|F(v)\| \\
&= \sum_{j=n}^{n+1-1} \left[(1-\lambda_j)\|e_j\| + \lambda_j \mu_j \|F(v)\|\right].
\end{aligned} \tag{3.6}$$

This implies that inequality (3.5) holds for $m = 1$. Assume that inequality (3.5) holds for some $m \geq 1$. Consider the case of $m+1$. Observe that

$$\begin{aligned}
&\|S_{n,m+1}v - v\| = \|T_{n+m}S_{n,m}v - v\| \\
&= \left\|\lambda_{n+m}S_{n,m}v + (1-\lambda_{n+m})TS_{n,m}v + (1-\lambda_{n+m})e_{n+m} - \lambda_{n+m}\mu_{n+m}F(S_{n,m}v) - v\right\| \\
&= \left\|\lambda_{n+m}\left[(I-\mu_{n+m}F)S_{n,m}v - v\right] + (1-\lambda_{n+m})(TS_{n,m}v - v)\right\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&\leq (1-\lambda_{n+m})\|TS_{n,m}v - v\| + \lambda_{n+m}\left\|(I-\mu_{n+m}F)S_{n,m}v - v\right\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&\leq (1-\lambda_{n+m})\|S_{n,m}v - v\| + \lambda_{n+m}\left[\left\|(I-\mu_{n+m}F)S_{n,m}v - (I-\mu_{n+m}F)v\right\|\right. \\
&\quad \left. + \left\|(I-\mu_{n+m}F)v - v\right\|\right] + (1-\lambda_{n+m})\|e_{n+m}\| \\
&\leq (1-\lambda_{n+m})\|S_{n,m}v - v\| + \lambda_{n+m}\left[1-\mu_{n+m}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|S_{n,m}v - v\| \\
&\quad + \lambda_{n+m}\mu_{n+m}\|F(v)\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&= \left[1-\lambda_{n+m}\mu_{n+m}\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right]\|S_{n,m}v - v\| + \lambda_{n+m}\mu_{n+m}\|F(v)\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&\leq \|S_{n,m}v - v\| + \lambda_{n+m}\mu_{n+m}\|F(v)\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&\leq \sum_{j=n}^{n+m-1}\left[(1-\lambda_j)\left\|e_j\right\| + \lambda_j\mu_j\|F(v)\|\right] + \lambda_{n+m}\mu_{n+m}\|F(v)\| + (1-\lambda_{n+m})\|e_{n+m}\| \\
&= \sum_{j=n}^{n+m+1-1}\left[(1-\lambda_j)\left\|e_j\right\| + \lambda_j\mu_j\|F(v)\|\right].
\end{aligned} \tag{3.7}$$

This shows that inequality (3.5) holds for the case of $m+1$. Thus, by induction, we know that inequality (3.5) holds for all $m \geq 1$.

Now set

$$\begin{aligned}
a_n &= \left\|tx_n + (1-\mathrm{t})p - q\right\|, \\
b_{n,m} &= \left\|S_{n,m}(tx_n + (1-t)p) - (tx_{n+m} + (1-t)p)\right\|.
\end{aligned} \tag{3.8}$$

By Proposition 2.3 no. (ii) and noticing inequality (3.5) we deduce that

$$\begin{aligned}
b_{n,m} &\leq \left\|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)S_{n,m}p)\right\| + (1-t)\left\|S_{n,m}p - p\right\| \\
&\leq \gamma^{-1}\left(\left\|x_n - p\right\| - \left\|x_{n+m} - S_{n,m}p\right\|\right) + \sum_{j=n}^{n+m-1}\left[(1-\lambda_j)\left\|e_j\right\| + \lambda_j\mu_j\left\|F(p)\right\|\right] \\
&\leq \gamma^{-1}\left(\left\|x_n - p\right\| - \left\|x_{n+m} - p\right\| + \left\|p - S_{n,m}p\right\|\right) + \sum_{j=n}^{n+m-1}\left[(1-\lambda_j)\left\|e_j\right\| + \lambda_j\mu_j\left\|F(p)\right\|\right].
\end{aligned} \tag{3.9}$$

Therefore,

$$b_{n,m} \leq \gamma^{-1}\left(\|x_n - p\| - \|x_{n+m} - p\| + \sum_{j=n}^{n+m-1} \left[(1-\lambda_j)\|e_j\| + \lambda_j\mu_j\|F(p)\|\right] \right) + \sum_{j=n}^{n+m-1} \left[(1-\lambda_j)\|e_j\| + \lambda_j\mu_j\|F(p)\|\right]. \tag{3.10}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists and $\sum_{n=0}^{\infty}(1-\lambda_n)\|e_n\|$ and $\sum_{n=0}^{\infty}\lambda_n\mu_n$ are convergent, we conclude from (3.10) that

$$\lim_{n,m\to\infty} b_{n,m} = 0. \tag{3.11}$$

Also, since, for all $n, m \geq 1$,

$$\begin{aligned} a_{n+m} &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)p) - S_{n,m}q\| + \|S_{n,m}q - q\| \\ &\leq a_n + b_{n,m} + \sum_{j=n}^{n+m-1}\left[(1-\lambda_j)\|e_j\| + \lambda_j\mu_j\|F(q)\|\right], \end{aligned} \tag{3.12}$$

it follows from (3.11) and (3.12) that $\lim_{n\to\infty} a_n$ exists.

(ii) This is Lemma 3.2 of [27].

□

Now we can state and prove the main result of this section.

Theorem 3.3. *Assume that X is a uniformly convex Banach space. Assume, in addition, that either X^* has the KK-property or X satisfies Opial's property. Let $T : X \to X$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$ and $F : X \to X$ δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Given an initial guess $x_0 \in X$. Let $\{x_n\}$ be generated by the following Mann type algorithm with perturbed mapping F :*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)(Tx_n + e_n) - \lambda_n\mu_n F(x_n), \quad \forall n \geq 0, \tag{3.13}$$

where $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$, and $\{e_n\}_{n=0}^{\infty}$ satisfy the following properties:

(i) $\sum_{n=0}^{\infty}\lambda_n(1-\lambda_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(1-\lambda_n)\|e_n\| < \infty$;

(iii) $\sum_{n=0}^{\infty}\lambda_n\mu_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Fix $p \in \mathrm{Fix}(T)$ and select a number $r > 0$ large enough so that $\|x_n - p\| \le r$ for all $n \ge 0$. Let $M > 0$ satisfy $M > 2r + (1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|$ for all $n \ge 0$. By Lemma 2.1, we have

$$\begin{aligned}
\left\|x_{n+1} - p\right\|^2 &= \left\|\lambda_n(x_n - p) + (1-\lambda_n)(Tx_n - p) + (1-\lambda_n)e_n - \lambda_n\mu_n F(x_n)\right\|^2 \\
&\le \left\|\lambda_n(x_n - p) + (1-\lambda_n)(Tx_n - p)\right\|^2 \\
&\quad + 2\left\|(1-\lambda_n)e_n - \lambda_n\mu_n F(x_n)\right\|\left\|\lambda_n(x_n - p) + (1-\lambda_n)(Tx_n - p)\right\| \\
&\quad + \left\|(1-\lambda_n)e_n - \lambda_n\mu_n F(x_n)\right\|^2 \\
&\le \lambda_n\left\|x_n - p\right\|^2 + (1-\lambda_n)\left\|Tx_n - p\right\|^2 - \lambda_n(1-\lambda_n)\phi(\|x_n - Tx_n\|) \\
&\quad + M\left[(1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|\right] \\
&\le \left\|x_n - p\right\|^2 - \lambda_n(1-\lambda_n)\phi(\|x_n - Tx_n\|) + M\left[(1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|\right].
\end{aligned} \tag{3.14}$$

It follows that

$$\lambda_n(1-\lambda_n)\phi(\|x_n - Tx_n\|) \le \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 + M\left[(1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|\right]. \tag{3.15}$$

This implies that

$$\sum_{n=0}^{\infty}\lambda_n(1-\lambda_n)\phi(\|x_n - Tx_n\|) < \infty. \tag{3.16}$$

In particular, $\lim_{n\to\infty}\lambda_n(1-\lambda_n)\phi(\|x_n - Tx_n\|) = 0$. Due to condition (i), we must have that $\liminf_{n\to\infty}\phi(\|x_n - \mathrm{T}x_n\|) = 0$. Hence

$$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.17}$$

However, since

$$Tx_{n+1} - x_{n+1} = (Tx_{n+1} - Tx_n) + \lambda_n(Tx_n - x_n) - (1-\lambda_n)e_n + \lambda_n\mu_n F(x_n), \tag{3.18}$$

we have

$$\begin{aligned}
\|Tx_{n+1} - x_{n+1}\| &\le \|Tx_{n+1} - Tx_n\| + \lambda_n\|Tx_n - x_n\| + (1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\| \\
&\le \|x_{n+1} - x_n\| + \lambda_n\|Tx_n - x_n\| + (1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\| \\
&\le \|Tx_n - x_n\| + 2(1-\lambda_n)\|e_n\| + 2\lambda_n\mu_n\|F(x_n)\|,
\end{aligned} \tag{3.19}$$

and, by Lemma 2.4, $\lim_{n\to\infty}\|Tx_n - x_n\|$ exists and hence, by (3.17),

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.20}$$

Notice that, by the demiclosedness principle of $I - T$, we obtain

$$\omega_w(x_n) \subset \operatorname{Fix}(T). \tag{3.21}$$

Hence to prove that $\{x_n\}$ converges weakly to a fixed point of T, it suffices to show that $\omega_w(x_n)$ is a singleton. We distinguish two cases. First assume that X^* has the KK-property. Then that $\omega_w(x_n)$ is a singleton is guaranteed by Proposition 3.2 no. (ii).

Next assume that X satisfies Opial's property. Take $p_1, p_2 \in \omega_w(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. If $p_1 \neq p_2$, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty}\left\|x_n - p_1\right\| &= \lim_{i\to\infty}\left\|x_{n_i} - p_1\right\| \\
&< \lim_{i\to\infty}\left\|x_{n_i} - p_2\right\| = \lim_{j\to\infty}\left\|x_{m_j} - p_2\right\| \\
&< \lim_{j\to\infty}\left\|x_{m_j} - p_1\right\| \\
&= \lim_{n\to\infty}\left\|x_n - p_1\right\|.
\end{aligned} \tag{3.22}$$

This shows that $\omega_w(x_n)$ is a singleton. The proof is therefore complete. □

4. The Case Where Mappings Are Defined on Subsets

We observe that if the domain $D(T)$ is a proper closed convex subset C of X, then the vectors $Tx_n + e_n$ and $(I - \mu_n F)x_n$ may not belong to C. In this case the next iterate x_{n+1} may not be well defined by (3.13). In order to consider this situation, we will use the nearest projection P_C and for the projection to be nonexpansive, we have to restrict our spaces to be Hilbert spaces.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Given a closed convex subset C of H. Recall that the (nearest point) projection P_C from H onto C assigns each point $x \in H$ with its (unique) nearest point in C which is denoted by $P_C x$. Namely, $P_C x \in C$ is the unique point in C with the property

$$\|x - P_C x\| = \inf\{\left\|x - y\right\| : y \in C\}. \tag{4.1}$$

Note that P_C is nonexpansive.

Let $T : C \to C$ be a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$ and $F : C \to H$ δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Starting with $x_0 \in C$ and after x_n in C is defined, we have two ways to define the next iterate x_{n+1}: either applying the projection P_C to the vectors $(I - \mu_n F)x_n$ and $Tx_n + e_n$ and defining x_{n+1} as the convex combination of

$P_C((I-\mu_n F)x_n)$ and $P_C(Tx_n+e_n)$, or projecting a convex combination of $(I-\mu_n F)x_n$ and (Tx_n+e_n) onto C to define x_{n+1}. More precisely, we define x_{n+1} as follows:

$$x_{n+1} = \lambda_n P_C\big((I-\mu_n F)x_n\big) + (1-\lambda_n)P_C(Tx_n+e_n), \quad \forall n \geq 0, \tag{4.2}$$

or

$$x_{n+1} = P_C\big[\lambda_n(I-\mu_n F)x_n + (1-\lambda_n)(Tx_n+e_n)\big], \quad \forall n \geq 0. \tag{4.3}$$

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $F : C \to H$ δ-strongly monotone and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Let $\{x_n\}$ be generated by either (4.2) or (4.3) where the sequences $\{\lambda_n\}$, $\{\mu_n\}$ and $\{e_n\}$ are such that*

(i) $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$;

(ii) $\sum_{n=0}^{\infty}(1-\lambda_n)\|e_n\| < \infty$;

(iii) $\sum_{n=0}^{\infty} \lambda_n\mu_n < \infty$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Given $p \in \mathrm{Fix}(T)$. Assume that $\{x_n\}$ is generated by (4.2). Then

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|\lambda_n P_C\big((I-\mu_n F)x_n\big) + (1-\lambda_n)P_C(Tx_n+e_n) - p\big\| \\
&\leq \lambda_n\big\|P_C\big((I-\mu_n F)x_n\big) - p\big\| + (1-\lambda_n)\big\|P_C(Tx_n+e_n)-p\big\| \\
&\leq \lambda_n\big\|(I-\mu_n F)x_n - p\big\| + (1-\lambda_n)\big\|(Tx_n+e_n)-p\big\| \\
&\leq \lambda_n\big[\big\|(I-\mu_n F)x_n - (I-\mu_n F)p\big\| + \big\|(I-\mu_n F)p - p\big\|\big] \\
&\quad + (1-\lambda_n)\big\|(Tx_n+e_n)-p\big\| \\
&\leq \lambda_n\left[\left(1-\mu_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-p\| + \big\|(I-\mu_n F)p-p\big\|\right] \\
&\quad + (1-\lambda_n)\|x_n-p\| + (1-\lambda_n)\|e_n\| \\
&\leq \lambda_n\|x_n-p\| + \lambda_n\mu_n\|F(p)\| + (1-\lambda_n)\|x_n-p\| + (1-\lambda_n)\|e_n\| \\
&= \|x_n-p\| + (1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(p)\|.
\end{aligned} \tag{4.4}$$

Hence $\lim_{n\to\infty}\|x_n-p\|$ exists; in particular, $\{x_n\}$ is bounded. Let $M > 0$ be a constant such that $M > 2\|x_n-p\| + (1-\lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|$ for all $n \geq 0$.

We compute

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad = \left\|\lambda_n P_C\left((I-\mu_n F)x_n\right)+(1-\lambda_n)P_C(Tx_n+e_n)-p\right\|^2 \\
&\quad = \big\|\lambda_n(x_n-p)+(1-\lambda_n)(Tx_n-p)+\lambda_n\left[P_C\left((I-\mu_n F)x_n\right)-x_n\right] \\
&\qquad +(1-\lambda_n)[P_C(Tx_n+e_n)-Tx_n]\big\|^2 \\
&\quad \le \left\|\lambda_n(x_n-p)+(1-\lambda_n)(Tx_n-p)\right\|^2 \\
&\qquad + \left\|\lambda_n\left[P_C\left((I-\mu_n F)x_n\right)-x_n\right]+(1-\lambda_n)[P_C(Tx_n+e_n)-Tx_n]\right\|^2 \\
&\qquad + 2\left\|\lambda_n(x_n-p)+(1-\lambda_n)(Tx_n-p)\right\|\big\|\lambda_n\left[P_C\left((I-\mu_n F)x_n\right)-x_n\right] \\
&\qquad\qquad \times +(1-\lambda_n)[P_C(Tx_n+e_n)-Tx_n]\big\| \\
&\quad \le \lambda_n\|x_n-p\|^2+(1-\lambda_n)\|Tx_n-p\|^2-\lambda_n(1-\lambda_n)\|x_n-Tx_n\|^2 \\
&\qquad + M\left[(1-\lambda_n)\|e_n\|+\lambda_n\mu_n\|F(x_n)\|\right] \\
&\quad \le \|x_n-p\|^2-\lambda_n(1-\lambda_n)\|x_n-Tx_n\|^2+M\left[(1-\lambda_n)\|e_n\|+\lambda_n\mu_n\|F(x_n)\|\right].
\end{aligned} \tag{4.5}$$

That is,

$$\lambda_n(1-\lambda_n)\|x_n-Tx_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+M\left[(1-\lambda_n)\|e_n\|+\lambda_n\mu_n\|F(x_n)\|\right]. \tag{4.6}$$

This implies that

$$\sum_{n=0}^{\infty}\lambda_n(1-\lambda_n)\|x_n-Tx_n\|^2<\infty. \tag{4.7}$$

In particular (noticing assumption (i)),

$$\liminf_{n\to\infty}\|x_n-Tx_n\|=0. \tag{4.8}$$

We also have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|\lambda_n P_C\left((I-\mu_n F)x_n\right)+(1-\lambda_n)P_C(Tx_n+e_n)-x_n\right\| \\
&\le \lambda_n\left\|P_C\left((I-\mu_n F)x_n\right)-x_n\right\|+(1-\lambda_n)\|P_C(Tx_n+e_n)-x_n\| \\
&\le \lambda_n\mu_n\|F(x_n)\|+(1-\lambda_n)(\|Tx_n-x_n\|+\|e_n\|).
\end{aligned} \tag{4.9}$$

Moreover, noticing

$$Tx_{n+1}-x_{n+1}=(Tx_{n+1}-Tx_n)+\lambda_n\left[Tx_n-P_C\left((I-\mu_n F)x_n\right)\right]+(1-\lambda_n)[Tx_n-P_C(Tx_n+e_n)], \tag{4.10}$$

we have

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\le \|Tx_{n+1} - Tx_n\| + \lambda_n \big\|Tx_n - P_C\big((I-\mu_n F)x_n\big)\big\| \\ &\quad + (1-\lambda_n)\|Tx_n - P_C(Tx_n + e_n)\| \\ &\le \|x_{n+1} - x_n\| + \lambda_n\big[\|Tx_n - x_n\| + \mu_n\|F(x_n)\|\big] + (1-\lambda_n)\|e_n\| \\ &\le \lambda_n\mu_n\|F(x_n)\| + (1-\lambda_n)(\|Tx_n - x_n\| + \|e_n\|) \\ &\quad + \lambda_n\big[\|Tx_n - x_n\| + \mu_n\|F(x_n)\|\big] + (1-\lambda_n)\|e_n\| \\ &= \|Tx_n - x_n\| + 2(1-\lambda_n)\|e_n\| + 2\lambda_n\mu_n\|F(x_n)\|.\end{aligned} \tag{4.11}$$

Similarly, if $\{x_n\}$ is generated by algorithm (4.3), then relations (4.4)–(4.11) still hold.

It is now readily seen that (4.11) together with Lemma 2.4 implies that $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists, which together with (4.8) further implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{4.12}$$

Equation (4.12) implies that $\omega_w(x_n) \subset \mathrm{Fix}(T)$, due to the demiclosedness principle. Finally, repeating the last part of the proof of Theorem 3.3 in the case of Opial's property, we see that $\{x_n\}$ converges weakly to a fixed point of T. The proof is therefore complete. □

Finally in this section, we consider the case of accretive operators. Recall that a multivalued operator A with domain $D(A)$ and range $R(A)$ in a Banach space X is said to be accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \ge 0, \tag{4.13}$$

where J is the duality map from X to the dual space X^*. An accretive operator A is m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$.

Denote by Ω the zero set of A; that is,

$$\Omega := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}. \tag{4.14}$$

Throughout the rest of this paper it is always assumed that A is m-accretive and Ω is nonempty.

Denote by J_r the resolvent of A for $r > 0$:

$$J_r = (I + rA)^{-1}. \tag{4.15}$$

It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex (this is so if X is uniformly convex). It is also known that $\mathrm{Fix}(J_r) = \Omega$ for $r > 0$.

Now consider the problem of finding a zero of an m-accretive operator A in a Banach space X,

$$0 \in Ax. \tag{4.16}$$

We will study the convergence of the following algorithm:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)(J_{c_n} x_n + e_n) - \lambda_n \mu_n F(x_n), \tag{4.17}$$

where $F : X \to X$ is a perturbed mapping, the initial guess $x_0 \in X$ is arbitrary, $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences in $[0,1]$, $\{c_n\}$ is a sequence of positive numbers, and $\{e_n\}$ is an error sequence in X.

Theorem 4.2. *Let X be a uniformly convex Banach space. Assume in addition that either X^* has the KK-property or X satisfies Opial's property. Let A be an m-accretive operator in X such that $\Omega \neq \emptyset$ and let $F : X \to X$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geq 1$. Moreover, assume that $\{\lambda_n\}$, $\{\mu_n\}$, $\{c_n\}$, and $\{e_n\}$ satisfy the following properties:*

(i) $\sum_{n=0}^{\infty} \lambda_n (1 - \lambda_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} (1 - \lambda_n) \|e_n\| < \infty$;

(iii) $\sum_{n=0}^{\infty} \lambda_n \mu_n < \infty$;

(iv) $0 < c_* < c_n < c^* < \infty$, *where c_* and c^* are two constants;*

(v) $\sum_{n=0}^{\infty} |c_{n+1} - c_n| < \infty$.

Then the sequence $\{x_n\}$ generated by algorithm (4.17) *converges weakly to a point of* Ω.

Proof. The proof is a refinement of that of Theorem 3.3 given in Section 3 and [34, Theorem 3.3] together with Proposition 3.2. So we only sketch it.

Let $p \in \Omega$. By (4.17), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\lambda_n x_n + (1 - \lambda_n)(J_{c_n} x_n + e_n) - \lambda_n \mu_n F(x_n) - p\| \\
&\leq \lambda_n \|(I - \mu_n F)x_n - p\| + (1 - \lambda_n)\|J_{c_n} x_n - p\| + (1 - \lambda_n)\|e_n\| \\
&\leq \lambda_n \left[\|(I - \mu_n F)x_n - (I - \mu_n F)p\| + \|(I - \mu_n F)p - p\|\right] \\
&\quad + (1 - \lambda_n)\|x_n - p\| + (1 - \lambda_n)\|e_n\| \\
&\leq \lambda_n \left[1 - \mu_n \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right] \|x_n - p\| + \lambda_n \mu_n \|F(p)\| \\
&\quad + (1 - \lambda_n)\|x_n - p\| + (1 - \lambda_n)\|e_n\| \\
&\leq \lambda_n \|x_n - p\| + \lambda_n \mu_n \|F(p)\| + (1 - \lambda_n)\|x_n - p\| + (1 - \lambda_n)\|e_n\| \\
&= \|x_n - p\| + \lambda_n \mu_n \|F(p)\| + (1 - \lambda_n)\|e_n\|.
\end{aligned} \tag{4.18}$$

By Lemma 2.4, we see that $\lim_{n\to\infty}\|x_n - p\|$ exists.

With slight modifications of the proof of Theorem 3.3 (replacing Tx_n by $J_{c_n} x_n$), we can obtain that

$$\liminf_{n\to\infty} \|x_n - J_{c_n} x_n\| = 0. \tag{4.19}$$

Now noticing

$$J_{c_{n+1}}x_{n+1} - x_{n+1} = J_{c_{n+1}}x_{n+1} - J_{c_n}x_n + \lambda_n\left[J_{c_n}x_n - (I - \mu_n F)x_n\right] - (1 - \lambda_n)e_n, \tag{4.20}$$

and letting $s_n := \|J_{c_n}x_n - x_n\|$ for all $n \geq 0$, we deduce that

$$\begin{aligned} s_{n+1} &\leq \lambda_n\left\|J_{c_n}x_n - (I - \mu_n F)x_n\right\| + \|J_{c_{n+1}}x_{n+1} - J_{c_n}x_n\| + (1 - \lambda_n)\|e_n\| \\ &\leq \lambda_n s_n + \|J_{c_{n+1}}x_{n+1} - J_{c_n}x_n\| + (1 - \lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|. \end{aligned} \tag{4.21}$$

By mimicking the proof of Theorem 3.3 in [34], we can show that, in the case of $c_n \leq c_{n+1}$,

$$\frac{s_{n+1}}{c_{n+1}} \leq \frac{s_n}{c_n} + \frac{2}{c_n}\left[(1 - \lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|\right], \tag{4.22}$$

and in the case of $c_n > c_{n+1}$,

$$\frac{s_{n+1}}{c_{n+1}} \leq \frac{s_n}{c_n} + \frac{2s^*}{c_*^2}|c_n - c_{n+1}| + \frac{2}{c_*}\left[(1 - \lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|\right], \tag{4.23}$$

where s^* is such that $s_n \leq s^*$ for all $n \geq 0$. In either case we conclude from (4.22) and (4.23) that $\{s_n\}$ satisfies

$$\frac{s_{n+1}}{c_{n+1}} \leq \frac{s_n}{c_n} + \sigma_n, \quad \forall n \geq 0, \tag{4.24}$$

where $\sigma_n := 2s^*|c_{n+1} - c_n|/c_*^2 + 2[(1 - \lambda_n)\|e_n\| + \lambda_n\mu_n\|F(x_n)\|]/c_*$ fulfills $\sum_{n=0}^{\infty}\sigma_n < \infty$. By Lemma 2.4, (4.24) implies that $\lim_{n\to\infty}(s_n/c_n)$ exists. This together with the assumption (iv) and (4.19) implies that $\limsup_{n\to\infty}\|x_n - J_{c_n}x_n\| = 0$. So, by Lemma 3.3 in [34], we have

$$\|x_n - J_{c_*}x_n\| \leq 2\|x_n - J_{c_n}x_n\| \longrightarrow 0. \tag{4.25}$$

By the demiclosedness principle, (4.25) ensures that $\omega_w(x_n) \subset \mathrm{Fix}(J_{c_*}) = \Omega$. Repeating the last part of the proof of Theorem 3.3, we conclude that $\{x_n\}$ converges weakly to a point of Ω. □

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