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Research Article

Strong Convergence Theorems of Common Fixed Points for a Family of Quasi-ϕ-Nonexpansive Mappings

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We consider a modified Halpern type iterative algorithm for a family of quasi-ϕ-nonexpansive mappings in the framework of Banach spaces. Strong convergence theorems of the purposed iterative algorithms are established.

1. Introduction

Let E be a Banach space, C a nonempty closed and convex subset of E, and $T : C \to C$ a nonlinear mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,\ y \in C. \tag{1.1}$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping; see ([1, 2]). More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.2}$$

where $u \in C$ is a fixed element. Banach Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved the following well-known strong convergence theorem.

Theorem B. *Let C be a bounded closed convex subset of a Hilbert space H and T a nonexpansive mapping on C. Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1-t)Tz_t$ for any $t \in (0,1)$. Then $\{z_t\}$ converges strongly to an element of $F(T)$ nearest to u.*

Motivated by Theorem B, Halpern [3] considered the following explicit iteration:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.3}$$

and obtained the following theorem.

Theorem H. *Let C be a bounded closed convex subset of a Hilbert space H and T a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Then the sequence $\{x_n\}$ defined by* (1.3) *converges strongly to the element of $F(T)$ nearest to u.*

In [4], Lions improved the result of Halpern [3], still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T provided that the control sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}((\alpha_{n+1} - \alpha_n)/\alpha_{n+1}^2) = 0$.

It was observed that both the Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\{\alpha_n\} = 1/(n+1)$. This was overcome by Wittmann [5], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

In [6], Shioji and Takahashi extended Wittmann's results to the setting of Banach spaces under the assumptions (C1), (C2), and (C4) imposed on the control sequences $\{\alpha_n\}$. In [7], Xu remarked that the conditions (C1) and (C2) are necessary for the strong convergence of the iterative sequence defined in (1.3) for all nonexpansive self-mappings. It is well known that the iterative algorithm (1.3) is widely believed to have slow convergence because the restriction of condition (C2). Thus, to improve the rate of convergence of the iterative process (1.3), one cannot rely only on the process itself.

Recently, hybrid projection algorithms have been studied for the fixed point problems of nonlinear mappings by many authors; see, for example, [8–24]. In 2006, Martinez-Yanes and Xu [10] proposed the following modification of the Halpern iteration for a single nonexpansive mapping T in a Hilbert space. To be more precise, they proved the following theorem.

Theorem MYX. *Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0,1)$ is such that $\lim_{n\to\infty}\alpha_n = 0$. Then the sequence $\{x_n\}$ defined by*

$$\begin{gathered} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n) T x_n, \\ C_n = \left\{ z \in C : \left\| y_n - z \right\|^2 \leq \| x_n - z \|^2 + \alpha_n \left(\| x_0 \|^2 + 2\langle x_n - x_0, z \rangle \right) \right\}, \\ Q_n = \{ z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{gathered} \tag{1.4}$$

converges strongly to $P_{F(T)} x_0$.

Very recently, Qin and Su [17] improved the result of Martinez-Yanes and Xu [10] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n), \\ C_n = \{ v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n) \}, \\ Q_n = \{ v \in C : \langle J x_0 - J x_n, x_n - v \rangle \geq 0 \}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{gathered} \tag{1.5}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)} x_0$.

In this paper, motivated by Kimura and Takahashi [8], Martinez-Yanes and Xu [10], Qin and Su [17], and Qin et al. [19], we consider a hybrid projection algorithm to modify the iterative process (1.3) to have strong convergence under condition (C1) only for a family of closed quasi-ϕ-nonexpansive mappings.

2. Preliminaries

Let E be a Banach space with the dual space E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \left\|f^*\right\|^2 \right\}, \quad \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that, if E^* is strictly convex, then J is single-valued and, if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

We know that, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [25] recently introduced a generalized projection operator Π_C in a Banach space E, which is an analogue of the metric projection in Hilbert spaces.

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The space E is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n+y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the space E is said to be smooth if

$$\lim_{t\to 0} \frac{\left\|x + ty\right\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

In a smooth Banach space E, we consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \left\|y\right\|^2, \quad \forall x,\ y \in E. \tag{2.3}$$

Observe that, in a Hilbert space H, (2.3) reduces to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem:

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y, x). \tag{2.4}$$

The existence and uniqueness of the operator Π_C follows from some properties of the functional $\phi(x, y)$ and the strict monotonicity of the mapping J (see, e.g., [25–28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$\left(\left\|y\right\| - \|x\|\right)^2 \le \phi(y, x) \le \left(\left\|y\right\| + \|x\|\right)^2, \quad \forall x,\ y \in E. \tag{2.5}$$

Remark 2.1. If E is a reflexive, strictly convex, and smooth Banach space, then, for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. In fact, it is sufficient to show that, if $\phi(x, y) = 0$, then $x = y$.

From (2.5), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [27, 29] for more details).

Let C be a nonempty closed and convex subset of E and T a mapping from C into itself. A point $p \in C$ is said to be an asymptotic fixed point of T ([30]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive ([27, 31, 32]) if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied by some authors ([27, 31, 32]).

A mapping $T : C \to C$ is said to be ϕ-nonexpansive ([18, 19, 24]) if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$. The mapping T is said to be quasi-ϕ-nonexpansive ([18, 19, 24]) if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings, which requires the strong restriction: $F(T) = \widetilde{F(T)}$.

In order to prove our main results, we need the following lemmas.

Lemma 2.3 (see [28]). *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.4 (see [25, 28]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x \in C$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \ge 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.5 (see [25, 28]). *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C. \tag{2.7}$$

Lemma 2.6 (see [7, 18]). *Let E be a uniformly convex and smooth Banach space, C a nonempty, closed, and convex subset of E and T a closed quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed and convex subset of C.*

3. Main Results

From now on, we use I to denote an index set. Now, we are in a position to prove our main results.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_i\}_{i\in I} : C \to C$ a family of closed quasi-ϕ-nonexpansive mappings* *such that* $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$. *Define a sequence* $\{x_n\}$ *in* C *in the following manner:*

$$\begin{aligned}
x_0 &\in C \quad \textit{chosen arbitrarily}, \\
y_{(n,i)} &= J^{-1}[\alpha_n J x_0 + (1-\alpha_n) J T_i x_n], \\
C_{(n,i)} &= \{z \in C : \phi(z, y_{(n,i)}) \leq \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\
C_n &= \bigcap\nolimits_{i\in I} C_{(n,i)}, \\
Q_0 &= C, \\
Q_n &= \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.1}$$

then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to* $\Pi_F x_0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \geq 0$. From the definitions of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \geq 0$. We, therefore, only show that C_n is convex for each $n \geq 0$. Indeed, note that

$$\phi(z, y_{(n,i)}) \leq \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n) \tag{3.2}$$

is equivalent to

$$2\alpha_n \langle z, Jx_0 \rangle + 2(1-\alpha_n)\langle z, Jx_n \rangle - 2\langle z, Jy_{n,i} \rangle \leq \alpha_n \|x_0\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{(n,i)}\|^2. \tag{3.3}$$

This shows that $C_{(n,i)}$ is closed and convex for each $n \geq 0$ and $i \in I$. Therefore, we obtain that $C_n = \bigcap_{i\in I} C_{(n,i)}$ is convex for each $n \geq 0$.

Next, we show that $F \subset C_n$ for all $n \geq 0$. For each $w \in F$ and $i \in I$, we have

$$\begin{aligned}
\phi(w, y_{(n,i)}) &= \phi\left(w, J^{-1}[\alpha_n J x_0 + (1-\alpha_n) J T_i x_n]\right) \\
&= \|w\|^2 - 2\langle w, \alpha_n J x_0 + (1-\alpha_n) J T_i x_n \rangle + \|\alpha_n J x_0 + (1-\alpha_n) J T_i x_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n \langle w, Jx_0 \rangle + 2(1-\alpha_n)\langle w, JT_i x_n \rangle + \alpha_n \|x_0\|^2 + (1-\alpha_n)\|T_i x_n\|^2 \\
&\leq \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, T_i x_n) \\
&\leq \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, x_n),
\end{aligned} \tag{3.4}$$

which yields that $w \in C_{(n,i)}$ for all $n \geq 0$ and $i \in$ I. It follows that $w \in C_n = \bigcap_{i\in I} C_{(n,i)}$. This proves that $F \subset C_n$ for all $n \geq 0$.

Next, we prove that $F \subset Q_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_{n-1}$ for some $n \geq 1$. Next, we show that $F \subset Q_n$ for the same n. Since x_n is the projection of x_0 onto $C_{n-1} \cap Q_{n-1}$, we obtain that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_{n-1} \cap Q_{n-1}. \tag{3.5}$$

Since $F \subset C_{n-1} \cap Q_{n-1}$ by the induction assumption, (3.5) holds, in particular, for all $w \in F$. This together with the definition of Q_n implies that $F \subset Q_n$ for all $n \geq 0$. Noticing that $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$ and $x_n = \Pi_{Q_n} x_0$, one has

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{3.6}$$

We, therefore, obtain that $\{\phi(x_n, x_0)\}$ is nondecreasing. From Lemma 2.5, we see that

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(w, x_0) - \phi(w, x_n) \\ &\leq \phi(w, x_0), \quad \forall w \in F \subset C_n, \ \forall n \geq 0. \end{aligned} \tag{3.7}$$

This shows that $\{\phi(x_n, x_0)\}$ is bounded. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of Q_n, we see that $Q_m \subset Q_n$ and $x_m = \Pi_{Q_m} x_0 \in Q_n$ for any positive integer $m \geq n$. Notice that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.8}$$

Taking the limit as $m, n \to \infty$ in (3.8), we get that $\phi(x_m, x_n) \to 0$. From Lemma 2.3, one has $x_m - x_n \to 0$ as $m, n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in C. Since E is a Banach space and C is closed and convex, we can assume that $x_n \to q \in C$ as $n \to \infty$.

Finally, we show that $q = \Pi_F x_0$. To end this, we first show $q \in F$. By taking $m = n + 1$ in (3.8), we have

$$\phi(x_{n+1}, x_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.9}$$

From Lemma 2.3, we arrive at

$$x_{n+1} - x_n \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.10}$$

Noticing that $x_{n+1} \in C_n$, we obtain

$$\phi(x_{n+1}, y_{(n,i)}) \leq \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n) \phi(x_{n+1}, x_n). \tag{3.11}$$

It follows from the assumption on $\{\alpha_n\}$ and (3.9) that $\lim_{n\to\infty}\phi(x_{n+1}, y_{(n,i)}) = 0$ for each $i \in I$. From Lemma 2.3, we obtain

$$\lim_{n\to\infty}\left\|x_{n+1} - y_{(n,i)}\right\| = 0, \quad \forall i \in I. \tag{3.12}$$

On the other hand, we have $\|Jy_{(n,i)} - JT_ix_n\| = \alpha_n\|Jx_0 - JT_ix_n\|$. By the assumption on $\{\alpha_n\}$, we see that $\lim_{n\to\infty}\|Jy_{(n,i)} - JT_ix_n\| = 0$ for each $i \in I$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\lim_{n\to\infty}\left\|y_{(n,i)} - T_ix_n\right\| = 0. \tag{3.13}$$

On the other hand, we have

$$\|x_n - T_ix_n\| \le \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_{(n,i)}\right\| + \left\|y_{(n,i)} - T_ix_n\right\|. \tag{3.14}$$

From (3.10)–(3.13), we obtain $\lim_{n\to\infty}\|T_ix_n - x_n\| = 0$. From the closedness of T_i, we get $q \in F$.

Finally, we show that $q = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in F \subset C_n. \tag{3.15}$$

Taking the limit as $n \to \infty$ in (3.15), we obtain that

$$\left\langle q - w, Jx_0 - Jq\right\rangle \ge 0, \quad \forall w \in F, \tag{3.16}$$

and hence $q = \Pi_F x_0$ by Lemma 2.4. This completes the proof. □

Remark 3.2. Comparing the hybrid projection algorithm (3.1) in Theorem 3.1 with algorithm (1.5) in Theorem QS, we remark that the set Q_n is constructed based on the set Q_{n-1} instead of C for each $n \ge 1$. We obtain that the sequence generated by the algorithm (3.1) is a Cauchy sequence. The proof is, therefore, different from the one presented in Qin and Su [17].

As a corollary of Theorem 3.1, for a single quasi-ϕ-nonexpansive mapping, we have the following result immediately.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed quasi-ϕ-nonexpansive mappings with a fixed point.*

Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C in the following manner:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = J^{-1}[\alpha_n J x_0 + (1-\alpha_n) J T x_n], \\
C_n = \{z \in C : \phi(z, y_n) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\
Q_0 = C, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \ge 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{gathered} \tag{3.17}$$

then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.4. Corollary 3.3 mainly improves Theorem 2.2 of Qin and Su [17] from the class of relatively nonexpansive mappings to the class of quasi-ϕ-nonexpansive mappings, which relaxes the strong restriction: $\widetilde{F(T)} = F(T)$.

In the framework of Hilbert spaces, Theorem 3.1 is reduced to the following result.

Corollary 3.5. *Let C be a nonempty closed and convex subset of a Hilbert space H and $\{T_i\}_{i\in I} : C \to C$ a family of closed quasi-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_{(n,i)} = \alpha_n x_0 + (1-\alpha_n) T_i x_n, \\
C_{(n,i)} = \left\{z \in C : \left\|z - y_{(n,i)}\right\|^2 \le \alpha_n \|z - x_0\|^2 + (1-\alpha_n)\|z - x_n\|^2\right\}, \\
C_n = \bigcap\nolimits_{i\in I} C_{(n,i)}, \\
Q_0 = C, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0,
\end{gathered} \tag{3.18}$$

then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Remark 3.6. Corollary 3.5 includes the corresponding result of Martinez-Yanes and Xu [10] as a special case. To be more precise, Corollary 3.5 improves Theorem 3.1 of Martinez-Yanes and Xu [10] from a single mapping to a family of mappings and from nonexpansive mappings to quasi-nonexpansive mappings, respectively.

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