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Research Article

Moduli and Characteristics of Monotonicity in Some Banach Lattices

Paweł Foralewski,[1] Henryk Hudzik,[1] Radosław Kaczmarek,[1] and Miroslav Krbec[2]

[1] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland*
[2] *Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Prague 1, Czech Republic*

Correspondence should be addressed to Henryk Hudzik, hudzik@amu.edu.pl

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First the characteristic of monotonicity of any Banach lattice X is expressed in terms of the left limit of the modulus of monotonicity of X at the point 1. It is also shown that for Köthe spaces the classical characteristic of monotonicity is the same as the characteristic of monotonicity corresponding to another modulus of monotonicity $\widehat{\delta}_{m,E}$. The characteristic of monotonicity of Orlicz function spaces and Orlicz sequence spaces equipped with the Luxemburg norm are calculated. In the first case the characteristic is expressed in terms of the generating Orlicz function only, but in the sequence case the formula is not so direct. Three examples show why in the sequence case so direct formula is rather impossible. Some other auxiliary and complemented results are also presented. By the results of Betiuk-Pilarska and Prus (2008) which establish that Banach lattices X with $\varepsilon_{0,m}(X) < 1$ and weak orthogonality property have the weak fixed point property, our results are related to the fixed point theory (Kirk and Sims (2001)).

1. Introduction

Let us denote $S_+(X) = S(X) \cap X_+$, where $S(X)$ is the unit sphere of a Banach lattice X (for its definition, see [1–3]) and X_+ is the positive cone of X.

A Banach lattice X is said to be strictly monotone ($X \in$ (SM)) if for all $x, y \in X_+$ such that $y \le x$ and $y \ne x$ we have $\|y\| < \|x\|$. A Banach lattice X is said to be uniformly monotone ($X \in$ (UM)) if for any $\varepsilon \in (0,1)$ there is $\delta(\varepsilon) \in (0,1)$ such that $\|x - y\| \le 1 - \delta(\varepsilon)$ whenever $0 \le y \le x$, $\|x\| = 1$, and $\|y\| \ge \varepsilon$ (see [1]).

For a given Banach lattice X, the function $\delta_{m,X} : [0,1] \to [0,1]$ defined by

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \le y \le x, \|x\| = 1, \|y\| \ge \varepsilon\} \tag{1.1}$$

is said to be the lower modulus of monotonicity of X. It is easy to show that (see [4])

$$\begin{aligned}\delta_{m,X}(\varepsilon) &= \inf\{1-\|x-y\| : 0 \le y \le x,\ \|x\| = 1,\ \|y\| = \varepsilon\}\\ &= 1-\sup\{\|x-y\| : 0 \le y \le x,\ \|x\| = 1,\ \|y\| \ge \varepsilon\}\\ &= 1-\sup\{\|x-y\| : 0 \le y \le x,\ \|x\| = 1,\ \|y\| = \varepsilon\}.\end{aligned} \tag{1.2}$$

The lower modulus of monotonicity $\delta_{m,X}$ is a convex function on the interval $[0,1]$ (see [5]) (so $\delta_{m,X}$ is continuous on the interval $[0,1)$ and nondecreasing on $[0,1]$ as well). It is also clear that $\delta_{m,X}(\varepsilon) \le \varepsilon$ for any $\varepsilon \in [0,1]$. Obviously, X is uniformly monotone if and only if $\delta_{m,X}(\varepsilon) > 0$ for every $\varepsilon \in (0,1]$. It is easy to see that a Banach lattice X is strictly monotone if and only if $\delta_{m,X}(1) = 1$.

The number $\varepsilon_{0,m}(X) \in [0,1]$ defined by

$$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0,1] : \delta_{m,X}(\varepsilon) = 0\} = \inf\{\varepsilon \in [0,1] : \delta_{m,X}(\varepsilon) > 0\} \tag{1.3}$$

is said to be the characteristic of monotonicity of X. Obviously, a Banach lattice X is uniformly monotone if and only if $\varepsilon_{0,m}(X) = 0$.

We can also define another characteristic of monotonicity of X, namely,

$$\widetilde{\varepsilon}_{0,m}(X) = \sup\{\varepsilon \ge 0 : \eta_{m,X}(\varepsilon) = 0\} = \inf\{\varepsilon \ge 0 : \eta_{m,X}(\varepsilon) > 0\}, \tag{1.4}$$

where $\eta_{m,X}$ is the upper modulus of monotonicity defined for all $\varepsilon > 0$ by the formula

$$\begin{aligned}\eta_{m,X}(\varepsilon) &= \inf\{\|x+y\| - 1 : x, y \in X_+,\ \|x\| = 1,\ \|y\| \ge \varepsilon\}\\ &= \inf\{\|x+y\| - 1 : x, y \in X_+,\ \|x\| = 1,\ \|y\| = \varepsilon\}\end{aligned} \tag{1.5}$$

(see [6, 7]). It is clear by the triangle inequality for the norm that $\eta_{m,X}(\varepsilon) \le \varepsilon$ for all $\varepsilon > 0$. Obviously, a Banach lattice X is uniformly monotone if and only if $\eta_{m,X}(\varepsilon) > 0$ for all $\varepsilon > 0$ or equivalently if $\widetilde{\varepsilon}_{0,m}(X) = 0$.

Let us also recall relationships between two moduli of monotonicity $\delta_{m,X}$ and $\eta_{m,X}$ as well as relationships between the characteristic of monotonicity $\varepsilon_{0,m}(X)$ and $\widetilde{\varepsilon}_{0,m}(X)$.

For arbitrary $\varepsilon \in (0,1)$ the following inequalities hold true (see [6]):

$$\frac{\delta_{m,X}(\varepsilon/(1+\varepsilon))}{1-\delta_{m,X}(\varepsilon/(1+\varepsilon))} \le \eta_{m,X}(\varepsilon) \le \frac{\delta_{m,X}(\varepsilon)}{1-\delta_{m,X}(\varepsilon)}. \tag{1.6}$$

Notice that inequalities (1.6) are equivalent to the following ones:

$$\frac{\eta_{m,X}(\varepsilon)}{1+\eta_{m,X}(\varepsilon)} \le \delta_{m,X}(\varepsilon) \le \frac{\eta_{m,X}(\varepsilon/(1-\varepsilon))}{1+\eta_{m,X}(\varepsilon/(1-\varepsilon))} \tag{1.7}$$

for any $\varepsilon \in (0,1)$. In [4, Theorem 1], it has been shown that

$$\varepsilon_{0,m}(X) \le \widetilde{\varepsilon}_{0,m}(X) \le 2\varepsilon_{0,m}(X). \tag{1.8}$$

It is easy to show that the upper estimate of the characteristic of monotonicity $\tilde{\varepsilon}_{0,m}(X)$ of a Banach lattice X given above can be improved. Namely, since $\|x+y\| \geq \max(\|x\|,\|y\|)$ for any couple $x, y \geq 0$, we have $\eta_{m,X}(\varepsilon) > 0$ for all $\varepsilon > 1$, whence we get $\tilde{\varepsilon}_{0,m}(X) \leq 1$. Therefore

$$\varepsilon_{0,m}(X) \leq \tilde{\varepsilon}_{0,m}(X) \leq \min\{1, 2\varepsilon_{0,m}(X)\} \tag{1.9}$$

for any Banach lattice X.

For more information on the monotonicity properties and coefficient of monotonicity in some Köthe spaces, we refer to [4–14].

2. Some General Results

In this part of the paper we will give a few general results. First we will present a new formula for the characteristic of monotonicity $\varepsilon_{0,m}(X)$ and we will introduce another modulus of monotonicity and characteristic of monotonicity for Köthe spaces. Obtained results will be useful in the last part of the paper in order to calculate the characteristic of monotonicity in Orlicz spaces. Finally we will investigate $\tilde{\varepsilon}_{0,m}(X)$.

2.1. A New Formula for the Characteristic of Monotonicity $\varepsilon_{0,m}(X)$

Theorem 2.1. *For any normed lattice X the following equality is true:*

$$\varepsilon_{0,m}(X) = 1 - \delta_{m,X}(1^-), \tag{2.1}$$

where $\delta_{m,X}(1^-) = \lim_{\varepsilon \to 1^-} \delta_{m,X}(\varepsilon)$. Moreover,

$$\delta_{m,X}(1 - \delta_{m,X}(\varepsilon)) = 1 - \varepsilon \tag{2.2}$$

for arbitrary $\varepsilon \in (\varepsilon_{0,m}(X), 1]$ if $\varepsilon_{0,m}(X) < 1$ as well as in the case when $\varepsilon = \varepsilon_{0,m}(X) = 1$.

Proof. If $\varepsilon_{0,m}(X) = 1$, then by the definition of $\varepsilon_{0,m}(X)$, we have $\delta_{m,X}(\varepsilon) = 0$ for any $\varepsilon \in (0,1)$, whence we get $1 - \delta_{m,X}(1^-) = 1$.

Let now $\varepsilon_{0,m}(X) < 1$, $\varepsilon \in (\varepsilon_{0,m}(X), 1)$, and $\eta \in (0, 1 - \delta_{m,X}(\varepsilon))$. Then for any $x \in S(X)$ and $y \in X$ satisfying $0 \leq y \leq x$, $\|y\| = \varepsilon$, and $\|x - y\| \geq 1 - \delta_{m,X}(\varepsilon) - \eta$, we have

$$\varepsilon = \|y\| = \|x - (x - y)\| \leq 1 - \delta_{m,X}(\|x - y\|) \leq 1 - \delta_{m,X}(1 - \delta_{m,X}(\varepsilon) - \eta). \tag{2.3}$$

Since $\delta_{m,X}$ is a continuous function on the interval $[0,1)$, by $\delta_{m,X}(\varepsilon) > 0$ and arbitrariness of $\eta \in (0, 1 - \delta_{m,X}(\varepsilon))$, we get

$$\varepsilon \leq 1 - \delta_{m,X}(1 - \delta_{m,X}(\varepsilon)). \tag{2.4}$$

Letting $\varepsilon \to 1^-$, we have

$$1 \leq 1 - \delta_{m,X}(1 - \delta_{m,X}(1^-)), \tag{2.5}$$

that is, $\delta_{m,X}(1-\delta_{m,X}(1^-)) \le 0$, whence

$$\delta_{m,X}\left(1-\delta_{m,X}\left(1^-\right)\right) = 0. \tag{2.6}$$

Therefore, $\varepsilon_{0,m}(X) \ge 1-\delta_{m,X}(1^-)$. Letting $\varepsilon \searrow \varepsilon_{0,m}(X)$ in (2.4), we get the opposite inequality, which ends the proof of equality (2.1).

Now we will show that equality (2.2) holds true. Suppose first that $\varepsilon \in (\varepsilon_{0,m}(X), 1)$. Since $\delta_{m,X}$ is a nondecreasing function on the interval $[0,1]$, by inequality (2.4), defining $t = 1-\delta_{m,X}(\varepsilon)$, we get

$$1-\delta_{m,X}(\varepsilon) \ge 1-\delta_{m,X}(1-\delta_{m,X}(1-\delta_{m,X}(\varepsilon))) = 1-\delta_{m,X}(1-\delta_{m,X}(t)). \tag{2.7}$$

Simultaneously, since $\delta_{m,X}$ is strictly increasing on the interval $(\varepsilon_{0,m}(X), 1]$, by equality (2.1), we have

$$\varepsilon_{0,m}(X) = 1-\delta_{m,X}\left(1^-\right) < 1-\delta_{m,X}(\varepsilon) = t < 1 \tag{2.8}$$

for any $\varepsilon \in (\varepsilon_{0,m}(X), 1)$. In consequence, inequality (2.4) holds also for t in place of ε, which means that

$$1-\delta_{m,X}(1-\delta_{m,X}(t)) \ge t = 1-\delta_{m,X}(\varepsilon). \tag{2.9}$$

Combining inequalities (2.7) and (2.9), we get the equality

$$1-\delta_{m,X}(1-\delta_{m,X}(t)) = 1-\delta_{m,X}(\varepsilon). \tag{2.10}$$

Since $\varepsilon, t \in (\varepsilon_{0,m}(X), 1)$ and $\delta_{m,X}$ is strictly increasing on this interval, we get the equality $\delta_{m,X}(t) = 1-\varepsilon$, which is just equality (2.2) for $\varepsilon \in (\varepsilon_{0,m}(X), 1)$.

Let now $\varepsilon = 1$. Since $\delta_{m,X}(1^-) \le \delta_{m,X}(1)$, by inequality (2.1), we get $1-\delta_{m,X}(1) \le 1-\delta_{m,X}(1^-) = \varepsilon_{0,m}(X)$, whence $\delta_{m,X}(1-\delta_{m,X}(1)) = 0$. Indeed, if $\delta_{m,X}$ is continuous at 1, then $\delta_{m,X}(1^-) = \delta_{m,X}(1)$ and so $1-\delta_{m,X}(1) = 1-\delta_{m,X}(1^-) = \varepsilon_{0,m}(X)$, whence $\delta_{m,X}(1-\delta_{m,X}(1)) = \delta_{m,X}(\varepsilon_{0,m}(X)) = 0$. If $\delta_{m,X}$ is not continuous at 1, then $\delta_{m,X}(1^-) < \delta_{m,X}(1)$ and so $1-\delta_{m,X}(1) < 1-\delta_{m,X}(1^-) = \varepsilon_{0,m}(X)$, whence, by the definition of $\varepsilon_{0,m}(X)$, we have $\delta_{m,X}(1-\delta_{m,X}(1)) = 0$. Therefore equality (2.2) holds also in this case. □

Remark 2.2. In equality (2.1), $\delta_{m,X}(1^-)$ cannot be replaced by $\delta_{m,X}(1)$. In Examples 2.3 and 2.4 we will present Banach lattices X for which $\delta_{m,X}(\varepsilon) = 0$ for any $\varepsilon \in [0,1)$ and $\delta_{m,X}(1) = 1$.

Example 2.3. Let us first consider the space $L^p = L^p([0,1],\Sigma,m)$ with $1 \le p < \infty$ over the Lebesgue measure space $([0,1],\Sigma,m)$. If $x \in S_+(L^p)$ and $A \in \Sigma$ is such that $\|x\chi_A\|_p = \varepsilon \in [0,1]$, then we have

$$1 = \|x\|_p^p = \left\|x\chi_{[0,1]\setminus A}\right\|_p^p + \left\|x\chi_A\right\|_p^p. \tag{2.11}$$

Hence for $y = x\chi_A$, we get $\|y\|_p = \varepsilon$ and $\|x-y\|_p = \|x\chi_{[0,1]\setminus A}\|_p = (1-\varepsilon^p)^{1/p}$, whence $1 - \|x-y\|_p = 1-(1-\varepsilon^p)^{1/p}$. In consequence $\delta_{m,L^p}(\varepsilon) \le 1-(1-\varepsilon^p)^{1/p}$. In order to show the opposite inequality, let us take arbitrary $0 \le y \le x \in L^p$, $\|x\|_p = 1$, $\|y\|_p \ge \varepsilon$. Then

$$\begin{aligned}1 = \|x\|_p^p &= \int_0^1 x^p(t)d\mu(t) = \int_0^1 \left[(x-y)+y\right]^p(t)d\mu(t)\\ &\ge \int_0^1 (x-y)^p(t)d\mu(t) + \int_0^1 y^p(t)d\mu(t) = \left\|x-y\right\|_p^p + \left\|y\right\|_p^p,\end{aligned} \tag{2.12}$$

whence

$$\left\|x-y\right\|_p \le \left(1-\left\|y\right\|_p^p\right)^{1/p} \le (1-\varepsilon^p)^{1/p}. \tag{2.13}$$

This means that $1-\|x-y\|_p \ge 1-(1-\varepsilon^p)^{1/p}$, whence, by arbitrariness of x and y, we get $\delta_{m,L^p}(\varepsilon) \ge 1-(1-\varepsilon^p)^{1/p}$. Therefore we have $\delta_{m,L^p}(\varepsilon) = 1-(1-\varepsilon^p)^{1/p}$ for every $\varepsilon \in [0,1]$.

Let us define $X = \oplus L^{p_n}$, the ℓ^1-direct sum of the spaces L^{p_n}, where $p_n \ge 1$ for any $n \in \mathbb{N}$, and $p_n \nearrow \infty$ as $n \to \infty$, equipped with the norm $\|x\| = \sum_{n=1}^{\infty} \|x_n\|_{p_n}$ for any $x = (x_n)_{n=1}^{\infty} \in X$ with $x_n \in L^{p_n}$ for any $n \in \mathbb{N}$. Since any space L^{p_n} is order linearly isometrically embedded into X, where the embedding operator is defined by

$$L^{p_n} \ni x_n \longmapsto (0,0,\ldots,0,x_n,0,0,\ldots) \tag{2.14}$$

with x_n on the nth place, for any $\varepsilon \in [0,1)$, we have

$$0 \le \delta_{m,X}(\varepsilon) \le \delta_{m,L^{p_n}}(\varepsilon) = 1-(1-\varepsilon^{p_n})^{1p_n} \searrow 0 \tag{2.15}$$

as $n \to \infty$, and consequently, $\delta_{m,X}(\varepsilon) = 0$ for any $\varepsilon \in [0,1)$. Simultaneously, the space X is strictly monotone as the ℓ^1-direct sum of uniformly monotone spaces L^{p_n} with $1 \le p_n < \infty$ for any $n \in \mathbb{N}$. Therefore $\delta_{m,X}(1) = 1$ and $\delta_{m,X}(1^-) = 0$.

Example 2.4. Let now $L^0 = L^0([0,\infty))$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0,\infty)$. For any $x \in L^0$ we define its distribution function μ by

$$\mu_x(\lambda) = m\left\{t \in \left[0,\gamma\right) : |x(t)| > \lambda\right\} \tag{2.16}$$

(see [3, 15, 16]) and the nonincreasing rearrangement x^* of x as

$$x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\} \tag{2.17}$$

(under the convention $\inf \emptyset = \infty$).

Let $\omega : [0,\infty) \to R_+$ be a nonincreasing locally integrable function, called a weight function. We say that the weight function is regular if there exists $\eta > 0$ such that $\int_0^{2t} \omega(t)dt \geq (1+\eta)\int_0^t \omega(t)dt$ for any $t \in [0,\infty)$ (see [9, 17]).

For any weight function ω, we define the Lorentz space by the formula

$$\Lambda_\omega = \left\{ x \in L^0 : \|x\| = \int_0^\infty x^*(t)\omega(t)dt < \infty \right\}. \tag{2.18}$$

Now we will show that for any Lorentz space Λ_ω such that the weight function is not regular but $\int_0^\infty \omega(t)dt = \infty$ (e.g., $\omega(t) = \min(1, 1/t)$ for $t \in [0,\infty)$), we have $\delta_{m,\Lambda_\omega}(1^-) = 0 < 1 = \delta_{m,\Lambda_\omega}(1)$.

In fact, since Λ_ω is strictly monotone (see [18, Proposition 4.1]), we have the equality $\delta_{m,\Lambda_\omega}(1) = 1$. Simultaneously, since ω is not regular, there exists an increasing sequence $(t_n)_{n=1}^\infty$ in the interval $[0,\infty)$ such that

$$\int_0^{2t_n} \omega(t)dt \leq \left(1 + \frac{1}{n}\right) \int_0^{t_n} \omega(t)dt. \tag{2.19}$$

We can find a decreasing sequence of positive numbers $(u_n)_{n=1}^\infty$ such that

$$\int_0^{2t_n} u_n \omega(t)dt = 1 \tag{2.20}$$

for any $n \in \mathbb{N}$. For $x_n := u_n \chi_{[0,2t_n)}$ and $y_n := u_n \chi_{[0,t_n)}$ $(n \in \mathbb{N})$, we get $0 \leq y_n \leq x_n, \|x_n\| = 1$ and, by inequality (2.19), $n/(n+1) \leq \|y_n\| \leq 1$. Since $(x_n - y_n)^* = y_n^*$, we also have that $n/(n+1) \leq \|x_n - y_n\| \leq 1$ for any $n \in \mathbb{N}$. Therefore $\delta_{m,\Lambda_\omega}(\varepsilon) = 0$ for any $\varepsilon \in [0,1)$.

Problem 1. In the above examples it has been shown that there are Banach lattices for which $\delta_{m,X}(1^-) < \delta_{m,X}(1)$ and $\varepsilon_{0,m}(X) = 1$, that is, $\delta_{m,X}(1^-) = 0$. It is natural to ask whether there exist Banach lattices X, for which $0 < \delta_{m,X}(1^-) < \delta_{m,X}(1)$.

From Theorem 2.1 and the definition of the modulus $\delta_{m,X}$ (see the preliminaries), we have the following.

Corollary 2.5. *For arbitrary Banach lattice* X *the following formulas hold true:*

$$\begin{aligned} \varepsilon_{0,m}(X) &= \lim_{\varepsilon \to 1^-} \left(\sup\{ \|x - y\| : 0 \leq y \leq x,\ \|x\| = 1,\ \|y\| \geq \varepsilon \} \right) \\ &= \lim_{\varepsilon \to 1^-} \left(\sup\{ \|x - y\| : 0 \leq y \leq x,\ \|x\| = 1,\ \|y\| = \varepsilon \} \right). \end{aligned} \tag{2.21}$$

2.2. Modulus and Characteristic of Monotonicity in Köthe Spaces

Denote by (T, Σ, μ) a positive, complete, and σ-finite measure space and by $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) real-valued and Σ-measurable functions defined on T. For two functions $x, y \in L^0$ we write $x \leq y$ if $x(t) \leq y(t)$ μ—a.e. in T. By $E = (E, \leq, \|\cdot\|_E)$ we denote a Köthe space over the measure space (T, Σ, μ), that is, E is a Banach subspace of L^0 which satisfies the following conditions (see [2, 3]).

(i) If $|x| \leq |y|$, $y \in E$, and $x \in L^0$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.

(ii) There exists a function $x \in E$ which is strictly positive μ—a.e. in T.

In Köthe spaces the definition of the characteristic of monotonicity can be simplified by using another modulus. Using the new formula for the characteristic of monotonicity of Köthe spaces, it should be easier to calculate this coefficient in concrete classes of Köthe spaces. We will see this advantage of the new formula in the class of Orlicz sequence spaces endowed with the Luxemburg norm. Let us define for E the modulus $\widehat{\delta}_{m,E} : [0,1] \to [0,1]$ by the formula

$$\widehat{\delta}_{m,E}(\varepsilon) = \inf\{1 - \|x - x\chi_A\|_E : x \geq 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \|x\chi_A\|_E \geq \varepsilon\}. \tag{2.22}$$

Obviously, the modulus $\widehat{\delta}_{m,E}$ is nondecreasing with respect to $\varepsilon \in [0,1]$ and $\delta_{m,X}(\varepsilon) \leq \widehat{\delta}_{m,E}(\varepsilon) \leq \varepsilon$ for any $\varepsilon \in [0,1]$. It is also possible to prove similarly as for the modulus $\delta_{m,X}$ in [4] that

$$\begin{aligned} \widehat{\delta}_{m,E}(\varepsilon) &= \inf\{1 - \|x - x\chi_A\|_E : x \geq 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \|x\chi_A\|_E = \varepsilon\} \\ &= 1 - \sup\{\|x - x\chi_A\|_E : x \geq 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \|x\chi_A\|_E \geq \varepsilon\} \\ &= 1 - \sup\{\|x - x\chi_A\|_E : x \geq 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \|x\chi_A\|_E = \varepsilon\}. \end{aligned} \tag{2.23}$$

The characteristic of monotonicity $\widehat{\varepsilon}_{0,m}(E)$ corresponding to the modulus $\widehat{\delta}_{m,E}$ is defined by

$$\widehat{\varepsilon}_{0,m}(E) = \sup\left\{\varepsilon \in [0,1] : \widehat{\delta}_{m,E}(\varepsilon) = 0\right\} = \inf\left\{\varepsilon \in [0,1] : \widehat{\delta}_{m,E}(\varepsilon) > 0\right\}. \tag{2.24}$$

We have the following:

Proposition 2.6. *For arbitrary Köthe space E the following formula holds true:*

$$\widehat{\varepsilon}_{0,m}(E) = \sup\left\{\limsup_{n\to\infty} \|x_n\chi_{A_n'}\|_E : (x_n) \subset S_+(E), (A_n) \subset \Sigma,\ \|x_n\chi_{A_n}\|_E \longrightarrow 1\right\}. \tag{2.25}$$

Proof. Let us denote

$$\widetilde{\alpha}(E) = \sup\left\{\limsup_{n\to\infty} \|x_n\chi_{A_n'}\|_E : (x_n) \subset S_+(E), (A_n) \subset \Sigma,\ \|x_n\chi_{A_n}\|_E \longrightarrow 1\right\}. \tag{2.26}$$

First, we will show that $\widehat{\varepsilon}_{0,m}(E) \le \widetilde{\alpha}(E)$. In order to do it, assume that $\widehat{\varepsilon}_{0,m}(E) > 0$ and $\varepsilon \in [0, \widehat{\varepsilon}_{0,m}(E))$. Then $\widehat{\delta}_{m,E}(\varepsilon) = 0$ and so

$$\sup\left\{ \left\| x\chi_A \right\|_E : x \ge 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \left\| x\chi_{A'} \right\|_E = \varepsilon \right\} = 1. \tag{2.27}$$

Next there exist a sequence (x_n) in $S_+(E)$ and a sequence (A_n) in Σ such that $\|x_n\chi_{A'_n}\|_E = \varepsilon$ and $\|x_n\chi_{A_n}\|_E \to 1$. Therefore $\varepsilon \le \widetilde{\alpha}(E)$, whence $\widehat{\varepsilon}_{0,m}(E) \le \widetilde{\alpha}(E)$.

In order to prove the opposite inequality assume that $\widehat{\varepsilon}_{0,m}(E) < 1$ and $\varepsilon \in (\widehat{\varepsilon}_{0,m}(E), 1]$, that is,

$$\sup\left\{ \left\| x\chi_A \right\|_E : x \ge 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \left\| x\chi_{A'} \right\|_E \ge \varepsilon \right\} < 1 \tag{2.28}$$

because of $\widehat{\delta}_{m,E}(\varepsilon) > 0$. We will show that $\widetilde{\alpha}(E) \le \varepsilon$. Otherwise we would have $\widetilde{\alpha}(E) > \varepsilon$ and then there were a sequence (x_n) in $S_+(E)$ and a sequence of sets (A_n) in Σ such that $\|x_n\chi_{A_n}\|_E \to 1$ and $\|x_n\chi_{A'_n}\|_E > \varepsilon$ for n large enough. Hence we have

$$\sup\left\{ \left\| x\chi_A \right\|_E : x \ge 0,\ \|x\|_E = 1,\ A \in \Sigma,\ \left\| x\chi_{A'} \right\|_E \ge \varepsilon \right\} = 1, \tag{2.29}$$

which contradicts inequality (2.28). Therefore, $\widetilde{\alpha}(E) \le \varepsilon$ and in consequence, by the arbitrariness of $\varepsilon \in (\widehat{\varepsilon}_{0,m}(E), 1]$, we conclude that $\widetilde{\alpha}(E) \le \widehat{\varepsilon}_{0,m}(E)$. □

Now we will show that both characteristics of monotonicity $\varepsilon_{0,m}(E)$ and $\widehat{\varepsilon}_{0,m}(E)$ are equal in Köthe spaces. In order to prove this fact we will prove first a result that will be helpful to prove this equality.

Lemma 2.7. *If E is a Köthe space, then for any positive ε and δ satisfying the condition $\varepsilon + \delta < 1$ the inequality $\delta_{m,E}(\varepsilon + \delta) \ge \delta\widehat{\delta}_{m,E}(\varepsilon)$ holds true.*

Proof. Let $\varepsilon, \delta \in (0, 1)$ be such that $\varepsilon + \delta < 1$ and $\widehat{\delta}_{m,E}(\varepsilon) > 0$. Assume that $0 \le y \le x$, $\|x\|_E = 1$, and $\|y\|_E \ge \varepsilon + \delta$. Let us define

$$A = \left\{ t \in T : y(t) < \delta x(t) \right\}. \tag{2.30}$$

Then $\|y\chi_A\|_E \le \|\delta x\|_E = \delta$. Since $\varepsilon + \delta \le \|y\|_E \le \|y\chi_A\|_E + \|y\chi_{A'}\|_E$, we get that $\|y\chi_{A'}\|_E \ge \varepsilon$. Therefore

$$\begin{aligned} \left\| x - y \right\|_E &\le \left\| x - y\chi_{A'} \right\|_E \le \left\| x - \delta x\chi_{A'} \right\|_E = \left\| (1-\delta)x + \delta x - \delta x\chi_{A'} \right\|_E \\ &\le (1-\delta)\|x\|_E + \delta\left\| x - x\chi_{A'} \right\|_E \le (1-\delta) + \delta\left(1 - \widehat{\delta}_{m,E}(\varepsilon)\right) \\ &= 1 - \delta\widehat{\delta}_{m,E}(\varepsilon). \end{aligned} \tag{2.31}$$

Hence for all $0 \le y \le x$ such that $\|x\|_E = 1$, $\|y\|_E \ge \varepsilon + \delta$, we have that $1 - \|x - y\|_E \ge \delta\widehat{\delta}_{m,E}(\varepsilon)$, whence $\delta_{m,E}(\varepsilon + \delta) \ge \delta\widehat{\delta}_{m,E}(\varepsilon)$. □

Theorem 2.8. *For arbitrary Köthe space E one has the equality*

$$\varepsilon_{0,m}(E) = \widehat{\varepsilon}_{0,m}(E). \tag{2.32}$$

Proof. Since $\delta_{m,E}(\varepsilon) \le \widehat{\delta}_{m,E}(\varepsilon)$ for all $\varepsilon \in [0,1]$, we have

$$\widehat{\varepsilon}_{0,m}(E) \le \varepsilon_{0,m}(E). \tag{2.33}$$

In order to get the inequality $\widehat{\varepsilon}_{0,m}(E) \ge \varepsilon_{0,m}(E)$, we need to consider separately two cases, namely, the case when $\varepsilon_{0,m}(E) < 1$ and the case when $\varepsilon_{0,m}(E) = 1$.

Case 1. Assume that $\varepsilon_{0,m}(E) < 1$. By virtue of inequality (2.33), we have $\widehat{\varepsilon}_{0,m}(E) < 1$ and $\widehat{\delta}_{m,E}(\varepsilon) > 0$ for all $\varepsilon \in (\widehat{\varepsilon}_{0,m}(E), 1)$. By Lemma 2.7, we have

$$\delta_{m,E}(\varepsilon_1) \ge (\varepsilon_1 - \varepsilon)\widehat{\delta}_{m,E}(\varepsilon) > 0 \tag{2.34}$$

for all ε and ε_1 such that $\widehat{\varepsilon}_{0,m}(E) < \varepsilon < \varepsilon_1 < 1$. Therefore, we obtained that $\delta_{m,E}(\varepsilon_1) > 0$ for any $\varepsilon_1 \in (\widehat{\varepsilon}_{0,m}(E), 1)$. Hence

$$\varepsilon_{0,m}(E) := \inf\{\varepsilon_1 : \delta_{m,E}(\varepsilon_1) > 0\} \le \widehat{\varepsilon}_{0,m}(E). \tag{2.35}$$

Case 2. Assume now that $\varepsilon_{0,m}(E) = 1$. We will prove that $\widehat{\varepsilon}_{0,m}(E) = 1$. Assume for the contrary that $\widehat{\varepsilon}_{0,m}(E) < 1$. Then, similarly as in Case 1, we get that $\delta_{m,E}(\varepsilon_1) > 0$ for all $\varepsilon_1 \in (\widehat{\varepsilon}_{0,m}(E), 1)$, whence $\varepsilon_{0,m}(E) \le \widehat{\varepsilon}_{0,m}(E) < 1$, a contradiction. Therefore $\varepsilon_{0,m}(E) = 1$ implies that $\widehat{\varepsilon}_{0,m}(E) = 1$.

□

Now, we will prove the following:

Corollary 2.9. *For arbitrary Köthe space E the following formulas are true:*

$$\begin{aligned}\varepsilon_{0,m}(E) = \widehat{\varepsilon}_{0,m}(E) &= \lim_{\varepsilon\to 1^-} \sup\{\|x\chi_{A'}\|_E : x \in S_+(E), A \in \Sigma,\ \|x\chi_A\|_E \ge \varepsilon\}\\ &= \lim_{\varepsilon\to 1^-} \sup\{\|x\chi_{A'}\|_E : x \in S_+(E),\ A \in \Sigma,\ \|x\chi_A\|_E = \varepsilon\}.\end{aligned} \tag{2.36}$$

Proof. Note that, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}&\sup\left\{\limsup_{n\to\infty}\|x_n\chi_{A_n'}\|_E : (x_n) \subset S_+(E), (A_n) \subset \Sigma,\ \|x_n\chi_{A_n}\|_E \longrightarrow 1\right\}\\ &\quad\le \sup\{\|x\chi_{A'}\|_E : x \in S_+(E),\ A \in \Sigma,\ \|x\chi_A\|_E \ge \varepsilon\}.\end{aligned} \tag{2.37}$$

Hence, by Proposition 2.6 and the arbitrariness of $\varepsilon \in (0,1)$, we get

$$\widehat{\varepsilon}_{0,m}(E) \le \lim_{\varepsilon\to 1^-} \sup\{\|x\chi_{A'}\|_E : x \in S_+(E), A \in \Sigma,\ \|x\chi_A\|_E \ge \varepsilon\}. \tag{2.38}$$

Simultaneously, by Corollary 2.5 and Theorem 2.8,

$$\begin{aligned}&\lim_{\varepsilon\to 1^-}\sup\{\|x\chi_{A'}\|_E : x\in S_+(E),\ A\in\Sigma,\ \|x\chi_A\|_E\geq\varepsilon\}\\&\quad\leq\lim_{\varepsilon\to 1^-}(\sup\{\|x-y\| : 0\leq y\leq x,\ \|x\|=1,\ \|y\|\geq\varepsilon\})=\varepsilon_{0,m}(E)=\widehat{\varepsilon}_{0,m}(E).\end{aligned}\tag{2.39}$$

Combining (2.38) and (2.39), we get inequality (2.36). □

Problem 2. We have $\delta_{m,X}(\varepsilon)\leq\widehat{\delta}_{m,E}(\varepsilon)\leq\varepsilon$, $\varepsilon_{0,m}(E)=\widehat{\varepsilon}_{0,m}(E)$ (i.e., $\delta_{m,X}(\varepsilon)=\widehat{\delta}_{m,E}(\varepsilon)=0$ for any $\varepsilon\in[0,\varepsilon_{0,m}(E))$, and $\lim_{\varepsilon\to 1^-}\delta_{m,X}(\varepsilon)=\lim_{\varepsilon\to 1^-}\widehat{\delta}_{m,E}(\varepsilon)$. It follows from Example 2.3 that $\delta_{m,X}(\varepsilon)=\widehat{\delta}_{m,E}(\varepsilon)$ for any $\varepsilon\in[0,1]$ for the space $E=L^p([0,1],\Sigma,m)$. So, it is natural to ask whether these two moduli are equal in arbitrary Köthe spaces.

2.3. Characteristic of Monotonicity $\widetilde{\varepsilon}_{0,m}(X)$ of a Banach Lattice X

Analogously as for $\varepsilon_{0,m}(X)$ (see [4, Theorem 5]) we get the following:

Proposition 2.10. *For arbitrary Banach lattice X the following formula holds true:*

$$\widetilde{\varepsilon}_{0,m}(X)=\sup\left\{\limsup_{n\to\infty}\|z_n-x_n\| : 0\leq x_n\leq z_n,\ \|x_n\|=1,\ \|z_n\|\longrightarrow 1\right\}.\tag{2.40}$$

Proof. Let us denote

$$\alpha(X)=\sup\left\{\limsup_{n\to\infty}\|z_n-x_n\| : 0\leq x_n\leq z_n,\ \|x_n\|=1,\ \|z_n\|\longrightarrow 1\right\}.\tag{2.41}$$

First, we will show that $\widetilde{\varepsilon}_{0,m}(X)\leq\alpha(X)$. In order to do it, assume that $\varepsilon>0$ and let $\eta_{m,X}(\varepsilon)=0$, that is,

$$\inf\{\|z\| : 0\leq x\leq z,\ \|x\|=1,\ \|z-x\|=\varepsilon\}=1.\tag{2.42}$$

Therefore there exist sequences $(x_n)_{n=1}^{\infty}\subset S_+(X)$ and $(z_n)_{n=1}^{\infty}\subset X_+$ such that $0\leq x_n\leq z_n$ and $\|z_n-x_n\|=\varepsilon$ for any $n\in\mathbb{N}$ and $\|z_n\|\to 1$. Hence, for arbitrary $\varepsilon>0$ such that $\eta_{m,X}(\varepsilon)=0$, we have

$$\varepsilon\leq\sup\left\{\limsup_{n\to\infty}\|z_n-x_n\| : 0\leq x_n\leq z_n,\ \|x_n\|=1,\ \|z_n\|\longrightarrow 1\right\}=\alpha(X).\tag{2.43}$$

Therefore

$$\widetilde{\varepsilon}_{0,m}(X)\leq\alpha(X).\tag{2.44}$$

Now, we will show the opposite inequality. In order to do this, assume that $\varepsilon > 0$ and $\eta_{m,X}(\varepsilon) > 0$, that is,

$$\inf\{\|z\| : 0 \leq x \leq z, \|x\| = 1, \|z - x\| \geq \varepsilon\} > 1. \tag{2.45}$$

Then $\alpha(X) \leq \varepsilon$. Indeed, in the opposite case it would be $\alpha(X) > \varepsilon$, and then there would exist sequences $(x_n)_{n=1}^{\infty} \subset S_+(X)$ and $(z_n)_{n=1}^{\infty} \subset X_+$ such that $0 \leq x_n \leq z_n$ for all $n \in \mathbb{N}$, $\|z_n\| \to 1$ and $\|z_n - x_n\| > \varepsilon$ for $n \in \mathbb{N}$ large enough. Hence we get

$$\inf\{\|z\| : 0 \leq x \leq z, \|x\| = 1, \|z - x\| \geq \varepsilon\} = 1, \tag{2.46}$$

which contradicts inequality (2.45). Therefore, $\alpha(X) \leq \varepsilon$ whenever $\varepsilon > 0$ and $\eta_{m,X}(\varepsilon) > 0$. Consequently, $\alpha(X) \leq \tilde{\varepsilon}_{0,m}(X)$, which together with (2.44) ends the proof. □

3. Characteristics of Monotonicity in Orlicz Spaces

In the last part of our paper we will present formulas for the characteristic of monotonicity in Orlicz function spaces and Orlicz sequence spaces. Let us start with some basic notions.

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if Φ is a nonzero function that is convex, even, vanishing and continuous at zero and left continuous on $\mathbb{R}_+$, which means that $\lim_{u \to b(\Phi)^-} \Phi(u) = \Phi(b(\Phi))$ (for the definition of $b(\Phi)$, see below).

Given any Orlicz function Φ, we define on $L^0 = L^0(T, \Sigma, \mu)$, where μ is nonatomic, a convex modular by the formula

$$I_\Phi(x) = \int_T \Phi(x(t))\,d\mu \tag{3.1}$$

(see [19–24]). The Orlicz function space $L^\Phi = L^\Phi(T, \Sigma, \mu)$ generated by an Orlicz function Φ is defined as

$$L^\Phi = \left\{x \in L^0 : I_\Phi(\lambda x) < +\infty \quad \text{for some } \lambda > 0\right\}. \tag{3.2}$$

We equip this space with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \leq 1\right\}. \tag{3.3}$$

In the sequence case, that is, when $T = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$, and $\mu(A) = \operatorname{card}(A)$ for any $A \subset \mathbb{N}$, we define on $\ell^0 = \ell^0(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ a convex modular I_Φ by

$$I_\Phi(x) = \sum_{n=1}^{\infty} \Phi(x(n)). \tag{3.4}$$

We define the Orlicz sequence space ℓ^Φ analogously as L^Φ and also consider it with the Luxemburg norm.

We say that an Orlicz function Φ satisfies condition Δ_2 for all $u \in \mathbb{R}_+$(at infinity) [at zero] if there is $K > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$ (for all $u \in \mathbb{R}$ satisfying $|u| \geq u_0$ with some $u_0 > 0$ such that $\Phi(u_0) < \infty$) [for all $u \in \mathbb{R}$ satisfying $|u| \leq u_0$ with some $u_0 > 0$ such that $\Phi(u_0) > 0$]. We write then $\Phi \in \Delta_2(\mathbb{R}_+)$ $(\Phi \in \Delta_2(\infty))[\Phi \in \Delta_2(0)]$, respectively. Let us note that $\Phi \in \Delta_2(0)$ implies that Φ vanishes only at zero and $\Phi \in \Delta_2(\infty)$ implies that $\Phi(u) < \infty$ for all $u \in \mathbb{R}$.

We will use two well-known parameters for the Orlicz function Φ: $a(\Phi) := \sup\{u > 0 : \Phi(u) = 0\}$ and $b(\Phi) := \sup\{u > 0 : \Phi(u) < \infty\}$.

3.1. The Characteristic of Monotonicity $\varepsilon_{0,m}(L^\Phi)$ of Orlicz Function Spaces

We start with the following:

Lemma 3.1. *Assume that Φ is an Orlicz function with $a(\Phi) > 0$ and satisfying the condition $\Delta_2(\infty)$ and let $c \in (a(\Phi), +\infty)$. Then for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that if $x \in L^\Phi$, $|x(t)| \geq c$ for μ—a.e. $t \in T$, and $I_\Phi(x) \leq \delta(\varepsilon)$, then $\|x\|_\Phi \leq \varepsilon$.*

Proof. Since $\Phi \in \Delta_2(\infty)$, so there are $u_0 > a(\Phi)$ and $K \geq 2$ such that $\Phi(2u) \leq K\Phi(u)$ for any $u \geq u_0$. We can assume that $c < u_0$. Since the interval $[c, u_0]$ is compact and the function $\Phi(2u)/\Phi(u)$ is continuous on this interval, we have that $L := \sup\{(\Phi(2u)/\Phi(u)) : u \in [c, u_0]\} < \infty$. In consequence, $\Phi(2u) \leq \max(K, L)\Phi(u)$ for all $u \geq c$. Let us denote by φ the right-hand side derivative of Φ. Since for any $t \geq c$, $t\varphi(t) \leq \Phi(2t) \leq \gamma\Phi(t)$, where $\gamma := \max(K, L)$, we have

$$M := \sup_{t \geq c} \frac{t\varphi(t)}{\Phi(t)} < \infty. \tag{3.5}$$

Therefore, taking any $u \geq c$ and $\alpha \geq 1$, we have

$$\int_u^{\alpha u} \frac{\varphi(t)}{\Phi(t)} dt \leq \int_u^{\alpha u} \frac{M}{t} dt, \tag{3.6}$$

whence

$$\Phi(\alpha u) \leq \alpha^M \Phi(u). \tag{3.7}$$

In consequence, if $0 < \beta \leq 1$ and $u \geq 0$ are such that $\beta u \geq c$, then we have

$$\Phi(u) = \Phi\left(\frac{1}{\beta}(\beta u)\right) \leq \frac{1}{\beta^M}\Phi(\beta u), \tag{3.8}$$

whence

$$\Phi(\beta u) \geq \beta^M \Phi(u). \tag{3.9}$$

Therefore, if $x \in L^{\Phi}$ and ε are as in the formulation of the Lemma, then, assuming that $I_{\Phi}(x) \leq \varepsilon^{M} < 1$, we have

$$\varepsilon^{M} \geq I_{\Phi}(x) = I_{\Phi}\left(\|x\|_{\Phi}\frac{x}{\|x\|_{\Phi}}\right) \geq \|x\|_{\Phi}^{M} I_{\Phi}\left(\frac{x}{\|x\|_{\Phi}}\right) = \|x\|_{\Phi}^{M}, \tag{3.10}$$

whence $\|x\|_{\Phi} \leq \varepsilon$. In such a way we proved our lemma with $\delta(\varepsilon) := \varepsilon^{M}$. □

Lemma 3.2 (see [4, Lemma 4]). *Let $\mu(T) < \infty$ and $\Phi \in \Delta_2(\infty)$. Then for any $\varepsilon \in (0,1)$ there is $p(\varepsilon) \in (0,1)$ such that if $1 \geq \|x_n\|_{\Phi} \geq 1 - p(\varepsilon)$, then $I_{\Phi}(x) \geq 1 - \varepsilon$.*

Theorem 3.3 (see [4, Theorem 6]). *If $\mu(T) < \infty$, $\Phi \in \Delta_2(\infty)$, and $a(\Phi) > 0$, then*

$$\delta_{m,L^{\Phi}}(1) = 1 - \frac{a(\Phi)}{c(\Phi)}, \tag{3.11}$$

where $c(\Phi)$ is the nonnegative constant satisfying the equality $\Phi(c(\Phi))\mu(T) = 1$.

Theorem 3.4. *Let L^{Φ} be an Orlicz function space. If $\mu(T) < \infty$, then the following statements hold true.*

(i) *If $\Phi \in \Delta_2(\infty)$ and $a(\Phi) = 0$, then $\varepsilon_{0,m}(L^{\Phi}) = 0$.*

(ii) *If $\Phi \in \Delta_2(\infty)$ and $a(\Phi) > 0$, then $\varepsilon_{0,m}(L^{\Phi}) = a(\Phi)/c(\Phi)$, where $c(\Phi)$ is the nonnegative constant satisfying the equality $\Phi(c(\Phi))\mu(T) = 1$.*

(iii) *If $\Phi \notin \Delta_2(\infty)$, then $\varepsilon_{0,m}(L^{\Phi}) = 1$.*

Proof. (i) If $\Phi \in \Delta_2(\infty)$ and $a(\Phi) = 0$, then the Orlicz space L^{Φ} is uniformly monotone (see [6]), so $\varepsilon_{0,m}(L^{\Phi}) = 0$.

(ii) By Theorems 2.1 and 3.3, we have

$$\varepsilon_{0,m}\left(L^{\Phi}\right) \geq \frac{a(\Phi)}{c(\Phi)}. \tag{3.12}$$

Now, we will show that for any $\theta \in (0,1)$ there exists $\sigma(\theta) \in (0,1)$ (close enough to 1) such that if $0 \leq y \leq x \in S_{+}(L^{\Phi})$ and $\|y\|_{\Phi} \geq \sigma(\theta)$, then

$$\left\|x - y\right\|_{\Phi} \leq (1+\theta)\frac{a(\Phi)}{c(\Phi)} + \theta. \tag{3.13}$$

Then, by Corollary 2.5 and inequality (3.12), we will get $\varepsilon_{0,m}(L^{\Phi}) = a(\Phi)/c(\Phi)$.

For any fixed $\theta \in (0,1)$, by Lemma 3.1, we can find $\delta(\theta) \in (0,1)$ such that $\|z\|_{\Phi} \leq \theta$ for any z satisfying $I_{\Phi}(z) \leq \delta(\theta)$ and $|z(t)| \geq (1+\theta)a(\Phi)$ for μ—a.e. $t \in T$. Next, by Lemma 3.2, we can find that $p(\delta(\theta)) \in (0,1)$ such that $I_{\Phi}(z) \geq 1 - \delta(\theta)$ whenever $\|z\|_{\Phi} \geq 1 - p(\delta(\theta))$. Denote $\sigma(\theta) = 1 - p(\delta(\theta))$.

Now for any fixed x and y such that $0 \leq y \leq x \in S_{+}(L^{\Phi})$ and $\|y\|_{\Phi} \geq \sigma(\theta)$, we define the set

$$A_{x,y} = \left\{t \in T : x(t) - y(t) > (1+\theta)a(\Phi)\right\}. \tag{3.14}$$

Since Φ is superadditive on $\mathbb{R}_+$, we have

$$1 = I_\Phi(x) = I_\Phi((x-y)+y) \geq I_\Phi(x-y) + I_\Phi(y), \tag{3.15}$$

whence, by $\|y\|_\Phi \geq \sigma(\theta)$, we get

$$I_\Phi(x-y) \leq 1 - I_\Phi(y) \leq 1 - (1-\delta(\theta)) = \delta(\theta). \tag{3.16}$$

In consequence

$$I_\Phi\left((x-y)\chi_{A_{x,y}}\right) \leq \delta(\theta), \tag{3.17}$$

and, by virtue of Lemma 3.1,

$$\left\|(x-y)\chi_{A_{x,y}}\right\|_\Phi \leq \theta. \tag{3.18}$$

Simultaneously, $0 \leq (x-y)\chi_{A'_{x,y}} \leq (1+\theta)a(\Phi)\chi_{A'_{x,y}} \leq (1+\theta)a(\Phi)\chi_T$, whence

$$\left\|(x-y)\chi_{A'_{x,y}}\right\|_\Phi \leq (1+\theta)a(\Phi)\|\chi_T\|_\Phi = (1+\theta)\frac{a(\Phi)}{c(\Phi)}. \tag{3.19}$$

Combining (3.18) and (3.19), we get (3.13), and the proof is finished.

(iii) Recall also that if $\Phi \notin \Delta_2(\infty)$, then the Orlicz space L^Φ contains an order isomorphically isometric copy of l^∞ (see [25, 26]), whence $\delta_{m,L^\Phi}(1) = 0$ and consequently $\varepsilon_{0,m}(L^\Phi) = 1$. □

Proceeding analogously as in proof of Theorem 3.4(i) and (iii), we get the following:

Theorem 3.5. *Let L^Φ be an Orlicz function space. If $\mu(T) = \infty$, then $\varepsilon_{0,m}(L^\Phi) = 0$ whenever $\Phi \in \Delta_2(\mathbb{R})$ and $\varepsilon_{0,m}(L^\Phi) = 1$ otherwise.*

3.2. Characteristic of Monotonicity of Orlicz Sequence Spaces

We start with a result that will be important for proving the main result of this section.

Theorem 3.6. *If the Orlicz function Φ satisfies the condition $\Delta_2(0)$ and $\Phi(b(\Phi)) \in (1/2, 1)$, then*

$$\delta_{m,\ell^\Phi}(1) = 1 - \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}. \tag{3.20}$$

Proof. Let us take arbitrary x such that $I_\Phi(x) = 1 - \Phi(b(\Phi))$ and define

$$y = (b(\Phi), |x(1)|, |x(2)|, \ldots), \qquad z = (b(\Phi), 0, \ldots). \tag{3.21}$$

Then $0 \le z \le y$, $\|z\|_\Phi = \|y\|_\Phi = 1$, and $\|y-z\|_\Phi = \|x\|_\Phi$. Therefore,

$$\delta_{m,\ell^\Phi}(1) \le 1 - \|y-z\|_\Phi = 1 - \|x\|_\Phi \tag{3.22}$$

and, by the arbitrariness of x, we have

$$\delta_{m,\ell^\Phi}(1) \le 1 - \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}. \tag{3.23}$$

In order to prove the opposite inequality it is enough to show that the inequality

$$\|y-z\|_\Phi \le \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\} \tag{3.24}$$

holds for any couple of elements y and z such that $0 \le z \le y$ and $\|z\|_\Phi = \|y\|_\Phi = 1$.

First assume that $y(i) < b(\Phi)$ for every $i \in \mathbb{N}$. Since there is at most one coordinate i_0 satisfying $\Phi^{-1}(1/2) < y(i_0) < b(\Phi)$, we can find $\lambda > 1$ such that $\lambda y(i) \le b(\Phi)$ for any $i \in \mathbb{N}$. Hence applying the assumption $\Phi \in \Delta_2(0)$, we get that $I_\Phi(\lambda y) < \infty$, whence $I_\Phi(y) = 1$. Since $0 \le z \le y$, then in a similar way as for y we obtain that $I_\Phi(z) = 1$. Since $\Phi \in \Delta_2(0)$, we have $a(\Phi) = 0$, whence we get that $z(i) = y(i)$ for any $i \in \mathbb{N}$. Therefore $\|y-z\|_\Phi = 0$, and inequality (3.24) is true.

Let now there exist $n \in \mathbb{N}$, for which $y(n) = b(\Phi)$. Since $\|z\|_\Phi = 1$ and $0 \le z \le y$, we get $z(n) = b(\Phi)$. Let us denote by $\overline{y}$ the element y if $I_\Phi(y) = 1$ or the element $(y(1), y(2), \ldots, y(n-1), b(\Phi), \overline{y}(n+1), y(n+2), \ldots)$, where $\overline{y}(n+1)$ is chosen in such a way that $I_\Phi(\overline{y}) = 1$ if $I_\Phi(y) < 1$. Then

$$\begin{aligned} \|y-z\|_\Phi &= \|(y-z)\chi_{\mathbb{N}\setminus\{n\}}\|_\Phi \le \|y\chi_{\mathbb{N}\setminus\{n\}}\|_\Phi \le \|\overline{y}\chi_{\mathbb{N}\setminus\{n\}}\|_\Phi \\ &\le \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}, \end{aligned} \tag{3.25}$$

which finishes the proof. □

For the sake of completeness we will give proofs of Lemmas 3.7 and 3.8 because we do not know the papers in which they were also proved for degenerated Orlicz functions, that is, for Orlicz functions Φ with $\Phi(b(\Phi)) < 1$.

Lemma 3.7. *Let $\Phi \in \Delta_2(0)$, $\Phi(b(\Phi)) < 1$, and $0 < a < b(\Phi)$. Then $I_\Phi(x_m) \to 1$ provided that $\|x_m\|_\Phi \to 1$ for any sequence (x_m) such that $x_m \in B(\ell^\Phi)$ and $|x_m(n)| \le a$ for all $m, n \in \mathbb{N}$.*

Proof. Assume that there exists a sequence (x_m) in $B(\ell^\Phi)$ such that $\|x_m\|_\Phi \to 1$, $|x_m(n)| \le a$ for any $m, n \in \mathbb{N}$, and $I_\Phi(x_m)$ does not tend to 1 as $n \to \infty$. Passing to a subsequence if necessary, we can assume that there exists $\delta > 0$ such that $I_\Phi(x_m) \le 1 - \delta$ for all $m \in \mathbb{N}$. Since $\Phi \in \Delta_2(0)$, we can find that $\eta > 1$ such that $\eta \le b(\Phi)/a$ and $\Phi(\eta u) \le (1/(1-\delta))\Phi(u)$ for $u \in [0, a]$. Therefore $I_\Phi(\eta x_m) \le (1/(1-\delta)) I_\Phi(x_m) = 1$, whence we have $\|x_m\|_\Phi \le 1/\eta < 1$, which is a contradiction. □

Lemma 3.8. *Assume that $\Phi \in \Delta_2(0)$ and $b(\Phi) < \infty$. Then for any sequence (x_m) such that $I_\Phi(x_m) \to 0$ there holds $\|x_m\|_\Phi \to 0$.*

Proof. Let us take an arbitrary but fixed sequence (x_m) such that $I_\Phi(x_m) \to 0$. We will show that $I_\Phi(\lambda x_m) \to 0$ for arbitrary $\lambda > 0$, whence we obtain that $\|x_m\|_\Phi \to 0$ (see [23]).

Take an arbitrary but fixed $\lambda > 0$ and $\varepsilon \in (0,1)$ and let n be the smallest natural number such that $\lambda \le 2^n$. Since $\Phi \in \Delta_2(0)$, there exists $K > 0$ such that $\Phi(2u) \le K\Phi(u)$ for $u \le b(\Phi)/2^2$. By $I_\Phi(x_m) \to 0$ we can find $m_0 \in \mathbb{N}$ such that

$$I_\Phi(x_m) \le \min\left\{\Phi\left(\frac{b(\Phi)}{2^{n+1}}\right), \frac{\varepsilon}{K^n}\right\} \tag{3.26}$$

for $m \ge m_0$. Hence $|x_m(n)| \le b(\Phi)/2^{n+1}$ for any $n \in \mathbb{N}$ and $m \ge m_0$, and finally

$$I_\Phi(\lambda x_m) \le I_\Phi(2^n x_m) \le K^n I_\Phi(x_m) \le K^n \frac{\varepsilon}{K^n} = \varepsilon \tag{3.27}$$

for $m \ge m_0$, which ends the proof. □

Theorem 3.9. *Let ℓ^Φ be an Orlicz sequence space. Then the following statements are true:*

(i) *If $\Phi \notin \Delta_2(0)$ or $\Phi(b(\Phi)) \le 1/2$, then $\varepsilon_{0,m}(\ell^\Phi) = 1$.*

(ii) *If $\Phi \in \Delta_2(0)$ and $1/2 < \Phi(b(\Phi)) < 1$, then*

$$\varepsilon_{0,m}\left(\ell^\Phi\right) = \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}. \tag{3.28}$$

(iii) *If $\Phi \in \Delta_2(0)$ and $\Phi(b(\Phi)) \ge 1$, then $\varepsilon_{0,m}(\ell^\Phi) = 0$.*

Proof. (i) If $\Phi \notin \Delta_2(0)$, then the Orlicz sequence space ℓ^Φ contains an order isomorphically isometric copy of ℓ^∞ (see [25, 26]), whence $\delta_{m,\ell^\Phi}(1) = 0$ and consequently $\varepsilon_{0,m}(\ell^\Phi) = 1$. Assume now that $\Phi(b(\Phi)) \le 1/2$. Defining

$$x = (b(\Phi), b(\Phi), 0, 0, \ldots), \qquad y = (b(\Phi), 0, 0, 0, \ldots), \tag{3.29}$$

we have that $0 \le y \le x$ and $x, y \in S_+(\ell_\Phi)$. Moreover $x - y = (0, b(\Phi), 0, 0, \ldots)$, so $\|x - y\| = 1$. Consequently $\delta_{m,\ell_\Phi}(1) = 0$, so $\varepsilon_{0,m}(\ell_\Phi) = 1$.

(ii) In the first part of the proof we will show that there exists $\varepsilon_0 \in (0,1)$ such that the inequality

$$\sup\left\{\|x\chi_{A'}\|_\Phi : x \in S_+\left(\ell^\Phi\right), A \subset \mathbb{N},\ \|x\chi_A\|_\Phi \ge \varepsilon\right\} \le \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(\varepsilon \cdot b(\Phi))\} \tag{3.30}$$

is true for every $\varepsilon \in [\varepsilon_0, 1)$. In order to do this, let $a = \Phi^{-1}(\max(1/2, (5/4)\Phi(b(\Phi)) - 1/4))$. Then obviously $\Phi^{-1}(1/2) \le a < b(\Phi)$. By virtue of Lemma 3.7, we can find $\varepsilon_1 \in (0,1)$ such that $I_\Phi(x) \ge \Phi(b(\Phi))$ if $\|x\|_\Phi \ge \varepsilon_1$, for every $x \in B(\ell^\Phi)$ satisfying $|x(i)| \le a$ for any $i \in \mathbb{N}$. Let us define the constant $\varepsilon_2 \in (0,1)$ by the equality $\varepsilon_2 \cdot b(\Phi) = a$. Since $\Phi \in \Delta_2(0)$, we can also find ε_3 from the interval $(0,1)$ such that the inequality

$$\Phi\left(\frac{u}{\varepsilon}\right) \leq \left(1 + \frac{1}{2}\right)\Phi(u) \tag{3.31}$$

holds for $\varepsilon \in [\varepsilon_3, 1)$ and $u \in [0, a]$. Finally, we put $\varepsilon_0 = \max(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Take now arbitrary $\varepsilon \in [\varepsilon_0, 1)$. We will show that $I_\Phi(x\chi_{A'}) \leq 1 - \Phi(\varepsilon \cdot b(\Phi))$ for any $x \in S_+(\ell^\Phi)$ and any set $A \subset \mathbb{N}$ such that $\|x\chi_A\|_\Phi \geq \varepsilon$, whence we will obtain inequality (3.30). We need to consider two cases.

Let $|x(i)| \leq a$ for every $i \in \mathbb{N}$. Then the definition of ε_0 ($\varepsilon_0 \geq \varepsilon_1$) yields that $I_\Phi(x\chi_A) \geq \Phi(b(\Phi))$. Hence $I_\Phi(x\chi_{A'}) \leq 1 - \Phi(b(\Phi)) < 1 - \Phi(\varepsilon \cdot b(\Phi))$.

Assume now that there exists exactly one $n \in \mathbb{N}$ such that $x(n) \in (a, b(\Phi)]$. Since

$$I_\Phi\left(x\chi_{\mathbb{N}\setminus\{n\}}\right) \leq 1 - \Phi(x(n)) < 1 - \Phi(a) \leq \frac{1}{2} < \Phi(b(\Phi)), \tag{3.32}$$

by the definition of ε_0 ($\varepsilon_0 \geq \varepsilon_1$), we get $\|x\chi_{\mathbb{N}\setminus\{n\}}\|_\Phi < \varepsilon$, whence $n \in A$. We have to consider two different subcases.

First, if $x(n) \in (a, \varepsilon \cdot b(\Phi))$, then $(x(n)/\varepsilon) < b(\Phi)$. Hence, by $\|x\chi_A\| \geq \varepsilon$, we get

$$\Phi\left(\frac{x(n)}{\varepsilon}\right) + \sum_{i \in A\setminus\{n\}} \Phi\left(\frac{x(i)}{\varepsilon}\right) \geq 1, \tag{3.33}$$

and consequently

$$\sum_{i \in A\setminus\{n\}} \Phi\left(\frac{x(i)}{\varepsilon}\right) \geq 1 - \Phi\left(\frac{x(n)}{\varepsilon}\right) > 1 - \Phi(b(\Phi)). \tag{3.34}$$

Since $x(i) \leq a$ for any $i \in \mathbb{N} \setminus \{n\}$, by the definition of ε_0 ($\varepsilon_0 \geq \varepsilon_3$) and inequality (3.31), we obtain

$$\sum_{i \in A\setminus\{n\}} \Phi(x(i)) \geq \frac{2}{3}(1 - \Phi(b(\Phi))). \tag{3.35}$$

Therefore

$$\begin{aligned} I_\Phi\left(x\chi_A\right) &= \Phi(x_n) + \sum_{i \in A\setminus\{n\}} \Phi(x(i)) \geq \Phi(a) + \frac{2}{3}(1 - \Phi(b(\Phi))) \\ &\geq \left(\frac{5}{4}\Phi(b(\Phi)) - \frac{1}{4}\right) + \left(\frac{2}{3} - \frac{2}{3}\Phi(b(\Phi))\right) \geq \Phi(b(\Phi)), \end{aligned} \tag{3.36}$$

whence $I_\Phi(x\chi_{A'}) \leq 1 - \Phi(b(\Phi)) < 1 - \Phi(\varepsilon b(\Phi))$.

Let now $x(n) \in [\varepsilon \cdot b(\Phi), b(\Phi))$. Then

$$I_\Phi(x\chi_{A'}) = 1 - I_\Phi(x\chi_A) \le 1 - \Phi(x(n)) \le 1 - \Phi(\varepsilon b(\Phi)). \tag{3.37}$$

It is worth noticing that in the above inequality we can obtain the equality for $A = \{n\}$ and $x(n) = \varepsilon \cdot b(\Phi)$.

In the second part of the proof we will show that

$$\lim_{\varepsilon \to 1^-} \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(\varepsilon \cdot b(\Phi))\} = \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}, \tag{3.38}$$

whence, by virtue of inequality (3.30) and Corollary 2.9, we will get

$$\varepsilon_{0,m}\left(\ell^\Phi\right) = \widehat{\varepsilon}_{0,m}\left(\ell^\Phi\right) \le \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\}. \tag{3.39}$$

Since, by Theorem 3.6, we have the inequality opposite to (3.39), the proof will be finished.

Let $\varepsilon \in [\varepsilon_0, 1)$. Then for arbitrary x satisfying $I_\Phi(x) = 1 - \Phi(\varepsilon \cdot b(\Phi))$ we can find y such that $0 \le y \le x$ and $I_\Phi(y) = 1 - \Phi(b(\Phi))$. By superadditivity of the Orlicz function Φ on $[0, \infty)$, we can write

$$\Phi(x(n)) = \Phi\big(x(n) - y(n) + y(n)\big) \ge \Phi\big(x(n) - y(n)\big) + \Phi\big(y(n)\big) \tag{3.40}$$

for all $n \in \mathbb{N}$, whence

$$I_\Phi(x - y) \le I_\Phi(x) - I_\Phi(y) = \Phi(b(\Phi)) - \Phi(\varepsilon \cdot b(\Phi)) < \frac{1}{2}. \tag{3.41}$$

By virtue of Lemma 3.8, there is $\sigma(\varepsilon) > 0$ such that $\|x - y\|_\Phi \le \sigma(\varepsilon)$, whence $\|x\|_\Phi \le \|y\|_\Phi + \sigma(\varepsilon)$. Consequently

$$\sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(\varepsilon \cdot b(\Phi))\} \le \sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\} + \sigma(\varepsilon). \tag{3.42}$$

Assuming now that $\varepsilon \to 1^-$ and applying again Lemma 3.8, we have that $\sigma(\varepsilon) \to 0$, which gives (3.38).

(iii) It is well known that the condition $\Phi \in \Delta_2(0)$ implies that $a(\Phi) = 0$, which together with the condition $\Phi(b(\Phi)) \ge 1$ gives that ℓ^Φ is uniformly monotone (see [27]), that is, $\varepsilon_{0,m}(\ell^\Phi) = 0$. □

Remark 3.10. The formulas given in Theorems 3.6 and 3.9(ii), respectively, are not completely constructive because they are not expressed in terms of the generating Orlicz functions only. However, finding better, that is, "more evident" formulas will be probably very difficult because these formulas can have different forms depending on the generating Orlicz function Φ. We will illustrate this phenomena in some examples below.

In Example 3.11 we will show that for some Orlicz functions Φ,

$$\sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\} = \frac{\Phi^{-1}(1 - \Phi(b(\Phi)))}{b(\Phi)}. \tag{3.43}$$

Example 3.11. Assume that $\Phi(u) = u^n$ for $u \in [0, b(\Phi)]$ and $\Phi(u) = \infty$ for $u \in (b(\Phi), \infty)$, where $n \in \mathbb{N}$ and $b(\Phi) \in (\sqrt[n]{1/2}, 1)$. Let us take an arbitrary x such that $I_\Phi(x) = 1 - \Phi(b(\Phi)) = 1 - (b(\Phi))^n$. We will consider two cases separately.

First assume that $\mu(\operatorname{supp} x) = 1$, that is, $|x| = \Phi^{-1}(1 - \Phi(b(\Phi)))e_i = \sqrt[n]{(1 - (b(\Phi))^n)}e_i$ for some $i \in \mathbb{N}$. Then

$$I_\Phi\left(\frac{x}{\Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)}\right) = \frac{\left(\sqrt[n]{1 - (b(\Phi))^n}\right)^n}{\left(\sqrt[n]{1 - (b(\Phi))^n}\right)^n / (b(\Phi))^n} = (b(\Phi))^n < 1. \tag{3.44}$$

Simultaneously, for $\lambda < \Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)$, we have that $|x(i)|/\lambda > b(\Phi)$, whence $I_\Phi(x/\lambda) = \infty$ and consequently $\|x\|_\Phi = \Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)$.

Assume now that $\mu(\operatorname{supp} x) \geq 2$. Then there exists $\delta_x > 0$ such that $|x(i)| \leq \Phi^{-1}(1 - \Phi(b(\Phi))) - \delta_x = \sqrt[n]{1 - (b(\Phi))^n} - \delta_x$ for any $i \in \operatorname{supp} x$. Then

$$I_\Phi\left(\frac{x}{\Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)}\right) = \frac{\sum_{i \in \operatorname{supp} x} |x(i)|^n}{\left(\sqrt[n]{1 - (b(\Phi))^n}\right)^n (b(\Phi))^n} = \frac{\left(1 - (b(\Phi))^n\right) \cdot (b(\Phi))^n}{1 - (b(\Phi))^n} < 1. \tag{3.45}$$

Since $\Phi \in \Delta_2(0)$, there exists $\lambda < \Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)$ such that $I_\Phi(x/\lambda) \leq 1$, so $\|x\|_\Phi < \Phi^{-1}(1 - \Phi(b(\Phi)))/b(\Phi)$.

In the next example we will find an Orlicz function Φ, for which

$$\sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\} = \frac{\Phi^{-1}((1 - \Phi(b(\Phi)))/2)}{\Phi^{-1}(1/2)}. \tag{3.46}$$

Example 3.12. Let

$$\Phi(u) = \begin{cases} u & \text{for } u \in \left[0, \dfrac{5}{50}\right), \\ 5u - \dfrac{2}{5} & \text{for } u \in \left[\dfrac{5}{50}, \dfrac{12}{50}\right], \\ \infty & \text{for } u > \dfrac{12}{50}. \end{cases} \tag{3.47}$$

Then $b(\Phi) = 12/50$, $\Phi(12/50) = 8/10$, $\Phi^{-1}(1-\Phi(12/50)) = \Phi^{-1}(2/10) = 6/50$, $\Phi^{-1}((1-\Phi(12/50))/2) = 1/10$, and $\Phi^{-1}(1/2) = 9/50$, so

$$\frac{\Phi^{-1}((1-\Phi(12/50))/2)}{\Phi^{-1}(1/2)} = \frac{5}{9}. \tag{3.48}$$

For x such that $\mu(\operatorname{supp}(x)) = 2$ and $|x(i)| = 1/10$ for $i \in \operatorname{supp}(x)$, we have that $I_\Phi(x) = 1 - \Phi(12/50) = 2/10$ and $I_\Phi(x/(5/9)) = 2 \cdot \Phi((9/5)\cdot(1/10)) = 1$, whence $\|x\|_\Phi = 1$.

Notice also that if $|x| = (6/50)e_i$ for some $i \in \mathbb{N}$, then $I_\Phi(x) = 2/10$ and $\|x\|_\Phi = 1/2 < 5/9$. Finally, let us take arbitrary x such that $I_\Phi(x) = 2/10$, $\mu(\operatorname{supp}(x)) \geq 2$, and $|x(i)| \neq 1/10$ for some $i \in \operatorname{supp}(x)$. It is easy to see that we can find $j \in \operatorname{supp}(x)$ for which $|x(j)| < 1/10$. Moreover, $\Phi(u) \geq u$ for any $u \geq 0$. Hence, denoting by φ the right-hand-side derivative of the Orlicz function Φ, we have

$$\begin{aligned}
I_\Phi\left(\frac{x}{5/9}\right) &= \sum_{i\in\operatorname{supp}(x)} \Phi\left(\frac{9}{5}x(i)\right) \\
&= \sum_{i\in\operatorname{supp}(x)} \left(\Phi\left(\frac{9}{5}x(i)\right) - \Phi(x(i))\right) + \sum_{i\in\operatorname{supp}(x)} \Phi(x(i)) \\
&= \sum_{i\in\operatorname{supp}(x)} \int_{x(i)}^{(9/5)x(i)} \varphi(t)dt + 0,2 < \sum_{i\in\operatorname{supp}(x)} \int_{x(i)}^{(9/5)x(i)} 5dt + 0,2 \\
&= 5\left(\frac{9}{5} - 1\right)\sum x_i + 0,2 \leq 1.
\end{aligned} \tag{3.49}$$

This inequality and $\Phi \in \Delta_2(0)$ imply that $\|x\|_\Phi < 5/9$.

In the last example we will show that for some Orlicz functions Φ,

$$\sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(b(\Phi))\} > \max\left\{\frac{\Phi^{-1}(1-\Phi(b(\Phi)))}{b(\Phi)}, \sup_{n\geq 2}\frac{\Phi^{-1}((1-\Phi(b(\Phi)))/n)}{\Phi^{-1}(1/n)}\right\}. \tag{3.50}$$

Example 3.13. Let

$$\Phi(u) = \begin{cases} u & \text{for } u \in \left[0, \dfrac{11}{100}\right], \\ 5u - \dfrac{44}{100} & \text{for } u \in \left(\dfrac{11}{100}, \dfrac{20}{100}\right], \\ 6u - \dfrac{64}{100} & \text{for } u \in \left(\dfrac{20}{100}, \dfrac{24}{100}\right], \\ \infty & \text{for } u > \dfrac{24}{100}. \end{cases} \tag{3.51}$$

Then $b(\Phi) = 24/100$, $\Phi(b(\Phi)) = 8/10$, and $\Phi^{-1}(1 - \Phi(24/100)) = \Phi^{-1}(2/10) = 16/125$, so

$$\frac{\Phi^{-1}(1 - \Phi(24/100))}{24/100} = \frac{8}{15}. \tag{3.52}$$

Since $(1 - \Phi(24/100))/n \le 1/10$ for $n \ge 2$, we have

$$\Phi^{-1}\left(\frac{1 - \Phi(24/100)}{n}\right) = \frac{1 - \Phi(24/100)}{n} = \frac{2}{10n} \tag{3.53}$$

for all $n \ge 2$. Obviously $\Phi^{-1}(1/n) = 1/n$ for $n \ge 10$, whence

$$\frac{\Phi^{-1}((1 - \Phi(24/100))/n)}{\Phi^{-1}(1/n)} = \frac{2}{10} \tag{3.54}$$

for the same n. By $\Phi^{-1}(1/n) \in (11/100, 2/10)$ for $n = 2, \ldots, 9$, we get $\Phi^{-1}(1/n) = (100 + 44n)/500n$ for the same n. Thus for those n (i.e., for $n = 2, \ldots, 9$), we get

$$\frac{\Phi^{-1}((1 - \Phi(24/100))/n)}{\Phi^{-1}(1/n)} = \frac{(2/10n)}{(100 + 44n)/500n} = \frac{25}{25 + 11n} \le \frac{25}{47}. \tag{3.55}$$

Finally,

$$\max\left\{\frac{\Phi^{-1}(1 - \Phi(24/100))}{24/100}, \sup_{n\ge 2}\frac{\Phi^{-1}((1 - \Phi(24/100))/n)}{\Phi^{-1}(1/n)}\right\} = \frac{8}{15}. \tag{3.56}$$

Simultaneously, for x such that $|x| = (11/100)e_i + (9/100)e_j$ for some $i, j \in \mathbb{N}$, we obtain that $I_\Phi(x) = 2/10$ and $\|x\|_\Phi = 111/208$, so $\sup\{\|x\|_\Phi : I_\Phi(x) = 1 - \Phi(24/100)\} > 8/15$.

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