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Research Article

On Fixed Point Theorems for Multivalued Contractions

Zeqing Liu,[1] Wei Sun,[1] Shin Min Kang,[2] and Jeong Sheok Ume[3]

[1] *Department of Mathematics, Liaoning Normal University, Dalian, Liaoning 116029, China*
[2] *Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*
[3] *Department of Applied Mathematics, Changwon National University, Changwon 641-773, Republic of Korea*

Correspondence should be addressed to Shin Min Kang, smkang@gnu.ac.kr

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Three concepts of multivalued contractions in complete metric spaces are introduced, and the conditions guaranteeing the existence of fixed points for the multivalued contractions are established. The results obtained in this paper extend genuinely a few fixed point theorems due to Ćirić (2009) Feng and Liu (2006) and Klim and Wardowski (2007). Five examples are given to explain our results.

1. Introduction and Preliminaries

Let (X, d) be a metric space, and let $CL(X)$, $CB(X)$, and $C(X)$ denote the families of all nonempty closed, all nonempty closed and bounded, and all nonempty compact subsets of X, respectively. For any $U, V \in CL(X)$ and $x \in X$, let $d(x, U) = \inf\{d(x, y) : y \in U\}$ and

$$H(U,V) = \begin{cases} \max\left\{\sup\limits_{u\in U} d(u,V), \sup\limits_{v\in V} d(v,U)\right\}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

Such a mapping H is called *a generalized Hausdorff metric in* $CL(X)$ *induced by* d.

Throughout this paper, we assume that $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{N}$ denote the sets of all real numbers, nonnegative real numbers, and positive integers, respectively.

The existence of fixed points for various multivalued contractive mappings had been studied by many authors under different conditions. For details, we refer the reader to [1–7] and the references therein. In 1969, Nadler Jr [7] extended the famous Banach Contraction Principle from single-valued mapping to multivalued mapping and proved the following fixed point theorem for the multivalued contraction.

Theorem 1.1 (see [7]). *Let (X, d) be a complete metric space, and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$H(T(x), T(y)) \le cd(x, y), \quad \forall x, y \in X. \tag{1.2}$$

Then, T has a fixed point.

In 1989, Mizoguchi and Takahashi [6] generalized the Nadler fixed point theorem and got a fixed point theorem for the multivalued contraction as follows.

Theorem 1.2 (see [6]). *Let (X, d) be a complete metric space, and let T be a mapping from X into $CB(X)$. Assume that there exists a map $\varphi : (0, +\infty) \to [0, 1)$ such that*

$$\begin{gathered}\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+, \\ H(T(x), T(y)) \le \varphi(d(x, y))d(x, y), \quad \forall x, y \in X.\end{gathered} \tag{1.3}$$

Then, T has a fixed point.

In 2006, Feng and Liu [3] obtained a new extension of the Nadler fixed point theorem and proved the following fixed point theorem.

Theorem 1.3 (see [3]). *Let (X, d) be a complete metric space, and let T be a multivalued mapping from X into $CL(X)$. If there exist constants $b, c \in (0, 1), c < b$, such that for any $x \in X$ there is $y \in T(x)$ satisfying*

$$bd(x, y) \le f(x), \qquad f(y) \le cd(x, y), \tag{1.4}$$

then T has a fixed point in X provided a function $f(x) = d(x, T(x))$, $x \in X$ is lower semicontinuous.

In 2007, Klim and Wardowski [5] improved the result of Feng and Liu and proved the following results.

Theorem 1.4 (see [5]). *Let (X, d) be a complete metric space, and let T be a multivalued mapping from X into $CL(X)$. Assume that*

(a) *the mapping $f : X \to \mathbb{R}^+$, defined by $f(x) = d(x, T(x))$, $x \in X$, is lower semi-continuous;*

(b) *there exist $b \in (0, 1)$ and $\varphi : \mathbb{R}^+ \to [0, b)$ satisfying*

$$\limsup_{r \to t^+} \varphi(r) < b, \quad t \in \mathbb{R}^+, \tag{1.5}$$

and for any $x \in X$ *there is* $y \in T(x)$ *satisfying*

$$bd(x,y) \le d(x,Tx), \qquad f(y) \le \varphi(d(x,y))d(x,y). \tag{1.6}$$

Then, T has a fixed point in X.

Theorem 1.5 (see [5]). *Let* (X,d) *be a complete metric space, and let* T *be a mapping from* X *into* $C(X)$*. Assume that*

(a) *the mapping* $f : X \to \mathbb{R}^+$*, defined by* $f(x) = d(x,T(x))$*,* $x \in X$*, is lower semi-continuous;*

(b) *there exists a function* $\varphi : \mathbb{R}^+ \to [0,1)$ *satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+, \tag{1.7}$$

and for any $x \in X$ *there is* $y \in T(x)$ *satisfying*

$$d(x,y) = f(x), \qquad f(y) \le \varphi(d(x,y))d(x,y). \tag{1.8}$$

Then, T has a fixed point in X.

In 2008 and 2009, Ćirić [1, 2] introduced new multivalued nonlinear contractions and established a few nice fixed point theorems for the multivalued nonlinear contractions, one of which is as follows.

Theorem 1.6 (see [2]). *Let* (X,d) *be a complete metric space, and let* T *be a mapping from* X *into* $CL(X)$*. Assume that*

(a) *the mapping* $f : X \to \mathbb{R}^+$*, defined by* $f(x) = d(x,T(x))$*,* $x \in X$*, is lower semi-continuous;*

(b) *there exists a function* $\varphi : \mathbb{R}^+ \to [a,1)$*,* $0 < a < 1$*, satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+; \tag{1.9}$$

and for any $x \in X$ *there is* $y \in T(x)$ *satisfying*

$$\sqrt{\varphi(f(x))}d(x,y) \le f(x), \qquad f(y) \le \varphi(f(x))d(x,y). \tag{1.10}$$

Then T has a fixed point in X.

The aim of this paper is both to introduce three new multivalued contractions in complete metric spaces and to prove the existence of fixed points for the multivalued contractions under weaker conditions than the ones in [2, 3, 5]. Five nontrivial examples are given to show that the results presented in this paper generalize substantially and unify the corresponding fixed point theorems of Ćirić [2], Feng and Liu [3], and Klim and Wardowski [5] and are different from those results of Mizoguchi and Takahashi [6] and Nadler Jr [7].

Next we recall and introduce the following result in [4] and some notions and terminologies.

Lemma 1.7 (see [4]). *Let (X, d) be a complete metric space and $D \in CL(X)$. Then, for each $x \in X$ and $k > 1$ there exists an element $a \in D$ such that*

$$d(x, a) \leq kd(x, D). \tag{1.11}$$

In the rest of this paper, for a multivalued mapping $T : X \to CL(X)$, we put

$$A = \begin{cases} [0, \operatorname{diam}(X)], & \text{if } \operatorname{diam}(X) < +\infty, \\ [0, +\infty), & \text{if } \operatorname{diam}(X) = +\infty, \end{cases}$$
$$B = \begin{cases} [0, \sup f(X)], & \text{if } \sup f(X) < +\infty, \\ [0, +\infty), & \text{if } \sup f(X) = +\infty, \end{cases} \tag{1.12}$$

where $\operatorname{diam}(X) = \sup\{d(x, y) : x, y \in X\}$ and $f(x) = d(x, T(x))$, for all $x \in X$. The function $f : X \to \mathbb{R}^+$ is said to be *T-orbitally lower semicontinuous at $z \in X$* if $\{x_n\}_{n\geq 0} \subset X$ is an arbitrary orbit of T with $\lim_{n\to\infty} x_n = z$ impling that $f(z) \leq \liminf_{n\to\infty} f(x_n)$.

2. Main Results

In this section, we establish three fixed point theorems for three new multivalued contractions in complete metric spaces.

Theorem 2.1. *Let T be a multivalued mapping from a complete metric space (X, d) into $CL(X)$ such that*

$$\begin{gathered} \text{for each } x \in X \text{ there exists } y \in T(x) \text{ satisfying} \\ \alpha(f(x))d(x, y) \leq f(x) \text{ and } f(y) \leq \beta(f(x))d(x, y), \end{gathered} \tag{2.1}$$

where $\alpha : B \to (0, 1]$ and $\beta : B \to [0, 1)$ satisfy that

$$\liminf_{r\to 0^+} \alpha(r) > 0, \quad \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in [0, \sup f(X)). \tag{2.2}$$

Then,

(a1) *for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\geq 0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$;*

(a2) *z is a fixed point of T in X if and only if the function f is T-orbitally lower semi-continuous at z.*

Proof. Put

$$\gamma(t) = \frac{\beta(t)}{\alpha(t)}, \quad \forall t \in [0, \sup f(X)). \tag{2.3}$$

It follows from (2.1) that for each $x_0 \in X$ there exists $x_1 \in T(x_0)$ satisfying

$$\alpha(f(x_0))d(x_0, x_1) \leq f(x_0), \qquad f(x_1) \leq \beta(f(x_0))d(x_0, x_1), \tag{2.4}$$

which together with (2.3) yield that

$$f(x_1) \leq \beta(f(x_0))\frac{f(x_0)}{\alpha(f(x_0))} = \gamma(f(x_0))f(x_0). \tag{2.5}$$

Continuing this process, we choose easily an orbit $\{x_n\}_{n\geq 0}$ of T satisfying

$$\begin{gathered} x_{n+1} \in T(x_n), \quad \alpha(f(x_n))d(x_n, x_{n+1}) \leq f(x_n), \\ f(x_{n+1}) \leq \beta(f(x_n))d(x_n, x_{n+1}), \quad \forall n \geq 0, \end{gathered} \tag{2.6}$$

which imply that

$$f(x_{n+1}) \leq \beta(f(x_n))d(x_n, x_{n+1}) \leq \beta(f(x_n))\frac{f(x_n)}{\alpha(f(x_n))} = \gamma(f(x_n))f(x_n), \quad \forall n \geq 0. \tag{2.7}$$

Now, we claim that

$$\lim_{n\to\infty} f(x_n) = 0. \tag{2.8}$$

Notice that the ranges of α, β, (2.2), and (2.3) ensure that

$$0 \leq \gamma(t) < 1, \quad \forall t \in [0, \sup f(X)). \tag{2.9}$$

Using (2.7) and (2.9), we conclude that $\{f(x_n)\}_{n\geq 0}$ is a nonnegative and nonincreasing sequence, which means that

$$\lim_{n\to\infty} f(x_n) = a \tag{2.10}$$

for some $a \geq 0$. Suppose that $a > 0$. Taking limits superior as $n \to \infty$ in (2.7) and using (2.2), (2.3), (2.9), and (2.10), we get that

$$\begin{aligned} a &= \limsup_{n\to\infty} f(x_{n+1}) \leq \limsup_{n\to\infty} \left[\gamma(f(x_n)) f(x_n)\right] \\ &\leq \limsup_{n\to\infty} \gamma(f(x_n)) \limsup_{n\to\infty} f(x_n) = a \limsup_{n\to\infty} \gamma(f(x_n)) < a, \end{aligned} \tag{2.11}$$

which is a contradiction. Thus, $a = 0$; that is, (2.8) holds.

Next, we show that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Let

$$b = \limsup_{n\to\infty} \gamma(f(x_n)), \qquad c = \liminf_{n\to\infty} \alpha(f(x_n)). \tag{2.12}$$

It follows from (2.2), (2.3), (2.9), and (2.12) that

$$0 \leq b < 1, \qquad c > 0. \tag{2.13}$$

Let $p \in (0, c)$ and $q \in (b, 1)$. In light of (2.12) and (2.13), we deduce that there exists some $n_0 \in \mathbb{N}$ such that

$$\gamma(f(x_n)) < q, \quad \alpha(f(x_n)) > p, \quad \forall n \geq n_0, \tag{2.14}$$

which together with (2.6) and (2.7) yield that

$$f(x_{n+1}) \leq q f(x_n), \quad d(x_n, x_{n+1}) \leq \frac{f(x_n)}{p}, \quad \forall n \geq n_0, \tag{2.15}$$

which imply that

$$f(x_{n+1}) \leq q^{n+1-n_0} f(x_{n_0}), \quad d(x_n, x_{n+1}) \leq \frac{f(x_{n_0})}{p} q^{n-n_0}, \quad \forall n \geq n_0, \tag{2.16}$$

which give that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \frac{f(x_{n_0})}{p} \sum_{k=n}^{m-1} q^{k-n_0} \leq \frac{f(x_{n_0})}{p(1-q)} q^{n+1-n_0}, \quad \forall m > n \geq n_0, \tag{2.17}$$

which implies that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence because $q < 1$. It follows from completeness of (X, d) that there is some $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Suppose that f is T-orbitally lower semi-continuous in z. It follows from (2.8) that

$$0 \leq d(z, T(z)) = f(z) \leq \liminf_{n\to\infty} f(x_n) = 0, \tag{2.18}$$

which means that $z \in T(z)$ because $T(z)$ is closed.

Suppose that z is a fixed point of T in X. For any orbit $\{y_n\}_{n\geq 0}$ of T with $y_{n+1} \in T(y_n)$ and $\lim_{n\to\infty} y_n = z$, we have

$$f(z) = d(z, T(z)) = 0 \leq \liminf_{n\to\infty} d(y_n, T(y_n)) = \liminf_{n\to\infty} f(y_n), \tag{2.19}$$

that is, f is T-orbitally lower semi-continuous in z. This completes the proof. □

Theorem 2.2. *Let T be a multivalued mapping from a complete metric space (X, d) into $CL(X)$ such that*

$$f(y) \leq \varphi(f(x)) d(x, y), \quad \forall (x, y) \in X \times Tx, \tag{2.20}$$

where $\varphi : B \to (0, 1)$ satisfies that

$$\liminf_{r\to 0^+} \varphi(r) > 0, \quad \limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in [0, \sup f(X)). \tag{2.21}$$

Then,

(a1) *for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\geq 0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$;*

(a2) *z is a fixed point of T in X if and only if the function f is T-orbitally lower semi-continuous at z.*

Proof. It follows from Lemma 1.7 that for each $x \in X$ there exists $y \in Tx$ with

$$d(x, y) \leq \frac{f(x)}{\sqrt{\varphi(f(x))}}, \tag{2.22}$$

which together with (2.20) and (2.21) ensures that (2.1) and (2.2) hold with $\alpha = \sqrt{\varphi}$ and $\beta = \varphi$. Thus, Theorem 2.2 follows from Theorem 2.1. This completes the proof. □

Theorem 2.3. *Let T be a multivalued mapping from a complete metric space (X, d) into $CL(X)$ such that*

$$\begin{gathered} \text{for each } x \in X \text{ there exists } y \in T(x) \text{ satisfying} \\ \alpha(d(x, y)) d(x, y) \leq f(x) \text{ and } f(y) \leq \beta(d(x, y)) d(x, y), \end{gathered} \tag{2.23}$$

where $\alpha : A \to (0, 1]$ and $\beta : A \to [0, 1)$ satisfy that

$$\liminf_{r\to t^+} \alpha(r) > 0, \quad \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in [0, \operatorname{diam}(X)), \tag{2.24}$$

and one of α and β is nondecreasing. Then,

(a1) *for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\geq 0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$;*

(a2) *z is a fixed point of T in X if and only if the function f is T-orbitally lower semi-continuous at z.*

Proof. Put

$$\gamma(t) = \frac{\beta(t)}{\alpha(t)}, \quad \forall t \in [0, \operatorname{diam}(X)). \tag{2.25}$$

It follows from the ranges of α, β, (2.24), and (2.25) that

$$0 \leq \gamma(t) < 1, \quad \forall t \in [0, \operatorname{diam}(X)). \tag{2.26}$$

As in the proof of Theorem 2.1, we select an orbit $\{x_n\}_{n\geq 0}$ of T satisfying

$$\begin{gathered} x_{n+1} \in T(x_n), \quad \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq f(x_n), \\ f(x_{n+1}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}), \quad \forall n \geq 0, \end{gathered} \tag{2.27}$$

which imply that

$$f(x_{n+1}) \leq \beta(d(x_n, x_{n+1}))\frac{f(x_n)}{\alpha(d(x_n, x_{n+1}))} = \gamma(d(x_n, x_{n+1}))f(x_n), \quad \forall n \geq 0, \tag{2.28}$$

$$d(x_{n+1}, x_{n+2}) \leq \frac{f(x_{n+1})}{\alpha(d(x_{n+1}, x_{n+2}))} \leq \frac{\beta(d(x_n, x_{n+1}))}{\alpha(d(x_{n+1}, x_{n+2}))}d(x_n, x_{n+1}), \quad \forall n \geq 0. \tag{2.29}$$

Using (2.26) and (2.28), we conclude easily that $\{f(x_n)\}_{n\geq 0}$ is a nonnegative and nonincreasing sequence. Consequently there exists some $\theta \geq 0$ satisfying

$$\lim_{n\to\infty} f(x_n) = \theta. \tag{2.30}$$

Now we claim that

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}), \quad \forall n \geq 0. \tag{2.31}$$

Suppose to the contrary, that is, there exists some $n_0 \geq 0$ satisfying $d(x_{n_0+1}, x_{n_0+2}) > d(x_{n_0}, x_{n_0+1})$. Note that one of α and β is nondecreasing. In view of (2.25), (2.26), and (2.29), we get that

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) < d(x_{n_0+1}, x_{n_0+2}) &\leq \frac{\beta(d(x_{n_0}, x_{n_0+1}))}{\alpha(d(x_{n_0+1}, x_{n_0+2}))} d(x_{n_0}, x_{n_0+1}) \\ &\leq \max\{\gamma(d(x_{n_0}, x_{n_0+1})), \gamma(d(x_{n_0+1}, x_{n_0+2}))\} d(x_{n_0}, x_{n_0+1}) \\ &< d(x_{n_0}, x_{n_0+1}), \end{aligned} \tag{2.32}$$

which is a contradiction. Thus, (2.31) holds. Therefore, there exists some $\eta \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \eta^+. \tag{2.33}$$

Next, we show that $\theta = 0$. Suppose that $\theta > 0$. Taking limits superior as $n \to \infty$ in (2.28) and using (2.24), (2.25), (2.30), and (2.33), we get that

$$\begin{aligned} \theta = \limsup_{n\to\infty} f(x_{n+1}) &\leq \limsup_{n\to\infty} \gamma(d(x_n, x_{n+1})) \limsup_{n\to\infty} f(x_n) \\ &= \theta \limsup_{n\to\infty} \gamma(d(x_n, x_{n+1})) < \theta, \end{aligned} \tag{2.34}$$

which is impossible. Thus, $\theta = 0$. Let

$$b = \limsup_{n\to\infty} \gamma(d(x_n, x_{n+1})), \qquad c = \liminf_{n\to\infty} \alpha(d(x_n, x_{n+1})). \tag{2.35}$$

It follows from (2.24), (2.25), (2.33), and (2.35) that

$$0 \leq b < 1, \quad c > 0. \tag{2.36}$$

Let $p \in (0, c)$ and $q \in (b, 1)$. By means of (2.35) and (2.36), we infer that there exists some $n_0 \in N$ such that

$$\gamma(d(x_n, x_{n+1})) < q, \quad \alpha(d(x_n, x_{n+1})) > p, \quad \forall n \geq n_0, \tag{2.37}$$

which together with (2.27) and (2.28) yield that

$$f(x_{n+1}) \leq q f(x_n), \quad d(x_n, x_{n+1}) \leq \frac{f(x_n)}{p}, \quad \forall n \geq n_0. \tag{2.38}$$

The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

3. Remarks and Examples

In this section, we construct five examples to illustrate the superiority and applications of the results presented in this paper.

Remark 3.1. In case $\alpha(t) = b$ and $\beta(t) = c$ for all $t \in [0, \sup f(X))$, where b and c are constants in $(0,1)$ with $c < b$, then Theorem 2.1 reduces to a result, which is a generalization of Theorem 1.3. The following example reveals that Theorem 2.1 extends both essentially Theorem 1.3 and is different from Theorems 1.1 and 1.2.

Example 3.2. Let $X = [0,1]$ be endowed with the Euclidean metric $d = |\cdot|$, and let $T : X \to CL(X)$ be defined by

$$T(x) = \begin{cases} \left\{ \dfrac{x^2}{3} \right\}, & x \in \left[0, \dfrac{23}{48}\right) \cup \left(\dfrac{23}{48}, 1\right], \\ \left\{ \dfrac{1}{12}, \dfrac{131}{432} \right\}, & x = \dfrac{23}{48}. \end{cases} \tag{3.1}$$

It is easy to see that

$$f(x) = d(x, T(x)) = \begin{cases} x - \dfrac{x^2}{3}, & x \in \left[0, \dfrac{23}{48}\right) \cup \left(\dfrac{23}{48}, 1\right], \\ \dfrac{19}{108}, & x = \dfrac{23}{48} \end{cases} \tag{3.2}$$

is T-orbitally lower semi-continuous in X and $B = [0, 2/3]$. Define $\alpha : B \to (0,1]$ and $\beta : B \to [0,1)$ by

$$\alpha(t) = \begin{cases} \dfrac{1}{8}, & t \in \left[0, \dfrac{1}{13}\right), \\ \dfrac{3t}{2}, & t \in \left[\dfrac{1}{13}, \dfrac{2}{3}\right], \end{cases}$$
$$\beta(t) = \max\left\{ \dfrac{1}{9}, \dfrac{13t}{9} \right\}, \quad t \in \left[0, \dfrac{2}{3}\right]. \tag{3.3}$$

Obviously,

$$\liminf_{r \to 0^+} \alpha(r) = \frac{1}{8} > 0. \tag{3.4}$$

For $t \in [0, 1/13)$, we have

$$\limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} = \limsup_{r \to t^+} \frac{1}{9} \cdot \frac{8}{1} = \frac{8}{9} < 1 \tag{3.5}$$

For $t \in [1/13, 2/3)$, we infer that

$$\limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} = \limsup_{r \to t^+} \frac{13r}{9} \cdot \frac{2}{3r} = \frac{26}{27} < 1. \tag{3.6}$$

For $x \in [0, 23/48) \cup (23/48, 1]$, there exists $y = x^2/3 \in T(x)$ satisfying

$$\begin{aligned} \alpha(f(x))d(x,y) &= \alpha\left(x - \frac{x^2}{3}\right)\left(x - \frac{x^2}{3}\right) \le x - \frac{x^2}{3} = f(x), \\ f(y) = d(y, T(y)) &= \left|\frac{x^2}{3} - \frac{x^4}{27}\right| = \frac{1}{3}\left(x + \frac{x^2}{3}\right)\left(x - \frac{x^2}{3}\right) \\ &\le \max\left\{\frac{1}{9}, \frac{13}{9}\left(x - \frac{x^2}{3}\right)\right\} d(x,y) \\ &= \beta(f(x))d(x,y). \end{aligned} \tag{3.7}$$

For $x = 23/48$, there exists $y = 1/12 \in T(x)$ satisfying

$$\begin{aligned} \alpha(f(x))d(x,y) &= \frac{19}{72} \cdot \frac{19}{48} < \frac{19}{108} = f(x), \\ f(y) = d(y, T(y)) = d\left(\frac{1}{12}, \frac{1}{432}\right) &= \frac{35}{432} < \frac{4693}{46656} = \beta(f(x))d(x,y). \end{aligned} \tag{3.8}$$

That is, the conditions of Theorem 2.1 are fulfilled. It follows from Theorem 2.1 that T has a fixed point in X. However, we cannot invoke each of Theorems 1.1–1.3 to show that the mapping T has a fixed point in X.

In fact, for any $b \in (0,1)$, we consider two possible cases as follows.

Case 1. Let $b \in (0, 4/9]$. Take $x_0 = 1$ and $y_0 \in T(x_0) = \{1/3\}$. Notice that $c \in (0,1)$ and $c < b$. It is clear that

$$d(y_0, T(y_0)) = \frac{8}{27} = \frac{4}{9} \cdot \frac{2}{3} \not\le \frac{2c}{3} = cd(x_0, y_0); \tag{3.9}$$

Case 2. Let $b \in (4/9, 1)$. Put $x_0 = 23/48$ and $y_0 = 1/12 \in Tx_0 = \{1/12, 131/432\}$. It follows that

$$bd(x_0, y_0) = \frac{19b}{48} \not\le \frac{4}{9} \cdot \frac{19}{48} = \frac{19}{108} = f(x_0). \tag{3.10}$$

Set $x_0 = 23/48$ and $y_0 = 131/432 \in Tx_0 = \{1/12, 131/432\}$. Note that $c \in (0,1)$ and $c < b$. It is easy to verify that

$$d(y_0, T(y_0)) = \left| \frac{131}{432} - \frac{1}{3} \cdot \frac{131^2}{432^2} \right| = \frac{152615}{559872} \not\leq \frac{19c}{108} = cd(x_0, y_0). \tag{3.11}$$

That is, the conditions of Theorem 1.3 do not hold.

Put $x_0 = 1/2$ and $y_0 = 23/48$. Clearly

$$\begin{aligned} H(T(x_0), T(y_0)) &= \frac{95}{432} \not\leq \frac{c}{48} = cd(x_0, y_0), \quad \forall c \in [0,1), \\ H(T(x_0), T(y_0)) &= \frac{95}{432} \not\leq \frac{\varphi(d(x_0, y_0))}{48} = \varphi(d(x_0, y_0)) d(x_0, y_0) \end{aligned} \tag{3.12}$$

for any $\varphi : (0, +\infty) \to [0,1)$ with $\limsup_{r \to t^+} \varphi(r) < 1$, for all $t \in \mathbb{R}^+$. That is, the conditions of Theorems 1.1 and 1.2 do not hold.

Remark 3.3. If $\alpha(t) = \sqrt{\varphi(t)}$ and $\beta(t) = \varphi(t)$ for all $t \in [0, \sup f(X))$, then Theorem 2.1 changes into a result, which is an extension of Theorem 1.6. The following example demonstrates that Theorem 2.1 generalizes substantially Theorem 1.6.

Example 3.4. Let $X = [0,1]$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$T(x) = \begin{cases} \left\{ \dfrac{x^2}{2} \right\}, & x \in \left[0, \dfrac{7}{8}\right) \cup \left(\dfrac{7}{8}, 1\right], \\ \left\{ \dfrac{1}{2}, \dfrac{2}{5} \right\}, & x = \dfrac{7}{8}. \end{cases} \tag{3.13}$$

It is easy to see that

$$f(x) = d(x, T(x)) = \begin{cases} x - \dfrac{x^2}{2}, & x \in \left[0, \dfrac{7}{8}\right) \cup \left(\dfrac{7}{8}, 1\right], \\ \dfrac{3}{8}, & x = \dfrac{7}{8} \end{cases} \tag{3.14}$$

is T-orbitally lower semi-continuous in X and $B = [0, \sup f(X)] = [0, 1/2]$. Define $\alpha : B \to (0,1]$ and $\beta : B \to [0,1)$ by

$$\alpha(t) = \frac{15}{19}, \quad \beta(t) = \frac{3}{4}, \quad t \in \left[0, \frac{1}{2}\right]. \tag{3.15}$$

Obviously,

$$\liminf_{r\to 0^+}\alpha(r)=\frac{15}{19}>0,$$
$$\limsup_{r\to t^+}\frac{\beta(r)}{\alpha(r)}=\limsup_{r\to t^+}\frac{3}{4}\cdot\frac{19}{15}=\frac{57}{60}<1,\quad t\in\left[0,\frac{1}{2}\right).\tag{3.16}$$

For $x\in[0,7/8)\cup(7/8,1]$, there exists $y=x^2/2\in T(x)=\{x^2/2\}$ satisfying

$$\alpha(f(x))d(x,y)=\alpha\left(x-\frac{x^2}{2}\right)\left(x-\frac{x^2}{2}\right)\leq x-\frac{x^2}{2}=f(x),$$
$$f(y)=d(y,T(y))=\left|\frac{x^2}{2}-\frac{x^4}{8}\right|=\frac{1}{2}\left(x+\frac{x^2}{2}\right)\left(x-\frac{x^2}{2}\right)\tag{3.17}$$
$$\leq\frac{3}{4}\cdot\left(x-\frac{x^2}{2}\right)=\beta(f(x))d(x,y).$$

For $x=7/8$, there exists $y=2/5\in T(x)$ satisfying

$$\alpha(f(x))d(x,y)=\frac{15}{19}\cdot\frac{19}{40}=\frac{3}{8}=f(x),$$
$$f(y)=d(y,T(y))=d\left(\frac{2}{5},\frac{2}{25}\right)=\frac{8}{25}<\frac{57}{160}=\frac{3}{4}\cdot\frac{19}{40}=\beta(f(x))d(x,y).\tag{3.18}$$

That is, the conditions of Theorem 2.1 are fulfilled. It follows from Theorem 2.1 that T has a fixed point in X. However, Theorem 1.6 is inapplicable in ensuring the existence of fixed points for the mapping T in X because there does not exist $\varphi:\mathbb{R}^+\to[a,1)$ and $a\in(0,1)$ satisfying the assumptions of Theorem 1.6. In fact, for any $\varphi:\mathbb{R}^+\to[a,1)$ and $a\in(0,1)$, we consider the following two possible cases.

Case 1. Let $a\in(225/361,1)$. Put $x_0=7/8$. Note that $\varphi(f(x_0))\subseteq[a,1)$. If $y_0=2/5\in Tx_0=\{1/2,2/5\}$, we see that

$$\sqrt{\varphi(f(x_0))}d(x_0,y_0)=\frac{19}{40}\sqrt{\varphi\left(\frac{3}{8}\right)}\not\leq\frac{3}{8}=f(x_0).\tag{3.19}$$

If $y_0=1/2\in Tx_0=\{1/2,2/5\}$, then we infer that

$$d(y_0,T(y_0))=\left|\frac{1}{2}-\frac{1}{8}\right|=\frac{3}{8}\not\leq\frac{3}{8}\varphi\left(\frac{3}{8}\right)=\varphi(f(x_0))d(x_0,y_0).\tag{3.20}$$

Case 2. Let $a \in (0, 225/361]$. Put $x_0 = 7/8$. Note that $\varphi(f(x_0)) \subseteq [a, 1)$. Suppose that $\varphi(f(x_0)) \in [a, 225/361]$. Let $y_0 = 1/2 \in T(x_0) = \{1/2, 2/5\}$. It is clear that

$$d(y_0, T(y_0)) = \left|\frac{1}{2} - \frac{1}{8}\right| = \frac{3}{8} = d(x_0, y_0) \nleq \varphi(f(x_0)) d(x_0, y_0). \tag{3.21}$$

Take $y_0 = 2/5 \in T(x_0) = \{1/2, 2/5\}$. Obviously,

$$d(y_0, T(y_0)) = \left|\frac{2}{5} - \frac{2}{25}\right| = \frac{8}{25} \nleq \frac{19}{40}\varphi\left(\frac{3}{8}\right) = \varphi(f(x_0)) d(x_0, y_0). \tag{3.22}$$

Suppose that $\varphi(f(x_0)) \in (225/361, 1)$. As in the proof of Case 1 stated first, we conclude similarly the conclusion of Case 1 stated after then. Therefore, the assumptions of Theorem 1.6 do not hold.

Remark 3.5. The following example is an application of Theorem 2.2.

Example 3.6. Let $X = [0, 1]$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$ be defined by

$$T(x) = \begin{cases} \left\{\dfrac{x^2}{3}\right\}, & x \in \left[0, \dfrac{23}{48}\right) \cup \left(\dfrac{23}{48}, 1\right], \\ \left\{\dfrac{1}{12}, \dfrac{1}{24}, \dfrac{7}{48}\right\}, & x = \dfrac{23}{48}. \end{cases} \tag{3.23}$$

It is easy to see that

$$f(x) = \begin{cases} d(x, T(x)) = x - \dfrac{x^2}{3}, & x \in \left[0, \dfrac{23}{48}\right) \cup \left(\dfrac{23}{48}, 1\right], \\ \dfrac{1}{3}, & x = \dfrac{23}{48} \end{cases} \tag{3.24}$$

is T-orbitally lower semi-continuous in X and $B = [0, 2/3]$. Define $\varphi : B \to (0, 1)$ by

$$\varphi(t) = \max\left\{\frac{1}{9}, \frac{13t}{9}\right\}, \quad t \in \left[0, \frac{2}{3}\right]. \tag{3.25}$$

Clearly,

$$\begin{gathered} \liminf_{r \to 0^+} \varphi(r) = \liminf_{r \to 0^+} \max\left\{\frac{1}{9}, \frac{13t}{9}\right\} = \frac{1}{9} > 0, \\ \limsup_{r \to t^+} \varphi(r) < 1, \quad t \in \left[0, \frac{2}{3}\right). \end{gathered} \tag{3.26}$$

For $(x,y) \in ([0,1] \setminus \{23/48\}) \times T(x)$, we infer that

$$\begin{aligned} f(y) = d(y,T(y)) &= \left|\frac{x^2}{3} - \frac{x^4}{27}\right| = \frac{1}{3}\left(x + \frac{x^2}{3}\right)\left(x - \frac{x^2}{3}\right) \\ &\le \max\left\{\frac{1}{9}, \frac{13}{9}\left(x - \frac{x^2}{3}\right)\right\}\left(x - \frac{x^2}{3}\right) \\ &= \varphi(f(x))d(x,y). \end{aligned} \tag{3.27}$$

For $(x,y) = (23/48, 1/12) \in X \times T(x)$, we conclude that

$$f(y) = d(y,T(y)) = \left|\frac{1}{12} - \frac{1}{3}\cdot\frac{1}{12^2}\right| = \frac{35}{432} < \frac{247}{1296} = \frac{13}{27}\cdot\frac{19}{48} = \varphi(f(x))d(x,y). \tag{3.28}$$

For $(x,y) = (23/48, 1/24) \in X \times T(x)$, we obtain that

$$f(y) = d(y,T(y)) = \left|\frac{1}{24} - \frac{1}{3}\cdot\frac{1}{24^2}\right| = \frac{71}{1728} < \frac{273}{1296} = \frac{13}{27}\cdot\frac{21}{48} = \varphi(f(x))d(x,y). \tag{3.29}$$

For $(x,y) = (23/48, 7/48) \in X \times T(x)$, we get that

$$f(y) = d(y,T(y)) = \left|\frac{7}{48} - \frac{1}{3}\cdot\frac{7^2}{48^2}\right| = \frac{959}{6912} < \frac{13}{81} = \frac{13}{27}\cdot\frac{1}{3} = \varphi(f(x))d(x,y). \tag{3.30}$$

Therefore, all assumptions of Theorem 2.2 are satisfied. It follows from Theorem 2.2 that T has a fixed point in X.

Remark 3.7. If $\alpha(t) = 1$ and $\beta(t) = \varphi(t)$ for all $t \in [0, \operatorname{diam}(X))$, then Theorem 2.3 comes down to a result, which extends Theorem 1.5. The following example shows that Theorem 2.3 is a genuine generalization of Theorem 1.5.

Example 3.8. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$. Define $T : X \to CL(X)$, $\alpha : \mathbb{R}^+ \to (0,1]$, and $\beta : \mathbb{R}^+ \to [0,1)$ by

$$\begin{aligned} &T(x) = \left\{\frac{x}{3}\right\} \cup [3x, +\infty), \quad x \in X, \\ &\alpha(t) = 1, \quad \beta(t) = \frac{4}{9}, \quad t \in \mathbb{R}^+, \end{aligned} \tag{3.31}$$

respectively. Clearly, $A = [0, +\infty)$ and

$$f(x) = d(x,Tx) = x - \frac{x}{3} = \frac{2x}{3}, \quad \forall x \in X, \tag{3.32}$$

is continuous in X. Note that for each $x \in X$ there exists $y = x/3 \in T(x)$ such that

$$\alpha(d(x,y))d(x,y) = 1 \cdot \left(x - \frac{x}{3}\right) = \frac{2x}{3} = f(x),$$
$$f(y) = d(y,Ty) = \frac{x}{3} - \frac{x}{9} = \frac{2x}{9} \leq \frac{4}{9} \cdot \frac{2x}{3} = \beta(d(x,y))d(x,y). \tag{3.33}$$

It is easy to verify that the assumptions of Theorem 2.3 are satisfied. Consequently, Theorem 2.3 guarantees that T has a fixed point in X. But T does not satisfy the conditions of Theorem 1.5 because $T(x)$ is not compact for all $x \in X$.

Remark 3.9. In case $\alpha(t) = b$ and $\beta(t) = \varphi(t)$ for all $t \in [0, \operatorname{diam}(X))$, then Theorem 2.3 reduces to a result, which extends Theorem 1.4. The following example reveals that Theorem 2.3 generalizes properly Theorem 1.4.

Example 3.10. Let (X, d) and T be as in example 3.1. Clearly,

$$f(x) = d(x,T(x)) = \begin{cases} x - \dfrac{x^2}{3}, & x \in \left[0, \dfrac{23}{48}\right) \cup \left(\dfrac{23}{48}, 1\right], \\[2ex] \dfrac{19}{108}, & x = \dfrac{23}{48} \end{cases} \tag{3.34}$$

is T-orbitally lower semi-continuous in X and $A = [0,1]$. Define $\alpha : A \to (0,1]$ and $\beta : A \to [0,1)$ by

$$\alpha(t) = \begin{cases} \dfrac{1}{10}, & t \in \left[0, \dfrac{3}{22}\right), \\[2ex] t, & t \in \left[\dfrac{3}{22}, 1\right], \end{cases}$$
$$\beta(t) = \max\left\{\frac{1}{11}, \frac{2t}{3}\right\}, \quad t \in [0,1]. \tag{3.35}$$

It is easy to verify that α is nondecreasing and

$$0 < \liminf_{r \to t^+} \alpha(r) = \begin{cases} \dfrac{1}{10}, & t \in \left[0, \dfrac{3}{22}\right), \\[2ex] t, & t \in \left[\dfrac{3}{22}, 1\right). \end{cases} \tag{3.36}$$

For $t \in [0, 3/22)$, we have

$$\limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} = \limsup_{r \to t^+} \frac{1}{11} \cdot \frac{10}{1} = \frac{10}{11} < 1. \tag{3.37}$$

For $t \in [3/22, 1)$, we infer that

$$\limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} = \limsup_{r \to t^+} \frac{2r}{3} \cdot \frac{1}{r} = \frac{2}{3} < 1. \tag{3.38}$$

For $x \in [0, 23/48) \cup (23/48, 1]$, there exists $y = x^2/3 \in T(x)$ satisfying

$$\begin{aligned} \alpha(d(x,y))d(x,y) &= \alpha\left(x - \frac{x^2}{3}\right)\left(x - \frac{x^2}{3}\right) \le x - \frac{x^2}{3} = f(x), \\ f(y) = d(y, T(y)) &= \left|\frac{x^2}{3} - \frac{x^4}{27}\right| = \frac{1}{3}\left(x + \frac{x^2}{3}\right)\left(x - \frac{x^2}{3}\right) \\ &\le \max\left\{\frac{1}{11}, \frac{2}{3}\left(x - \frac{x^2}{3}\right)\right\} d(x,y) \\ &= \beta(d(x,y))d(x,y). \end{aligned} \tag{3.39}$$

For $x = 23/48$, there exists $y = 1/12 \in T(x)$, satisfying

$$\begin{aligned} \alpha(d(x,y))d(x,y) &= \frac{19}{48} \cdot \frac{19}{48} < \frac{19}{108} = f(x), \\ f(y) = d(y, T(y)) = d\left(\frac{1}{12}, \frac{1}{432}\right) &= \frac{35}{432} < \frac{361}{3456} = \frac{19}{72} \cdot \frac{19}{48} = \beta(d(x,y))d(x,y). \end{aligned} \tag{3.40}$$

That is, the conditions of Theorem 2.3 are fulfilled. It follows from Theorem 2.3 that T has a fixed point in X. However, we cannot use Theorem 1.4 to show that the mapping T has a fixed point in X since there does not exist $b \in (0,1)$ and $\varphi : \mathbb{R}^+ \to [0, b)$ satisfying the assumptions in Theorem 1.4. In fact, for any $b \in (0,1)$ and $\varphi : \mathbb{R}^+ \to [0, b)$, we consider two possible cases as follows.

Case 1. Let $b \in (0, 4/9]$. Take $x_0 = 1$ and $y_0 \in T(x_0) = \{1/3\}$. Note that $\varphi(d(x_0, y_0)) < b$. It is clear that

$$d(y_0, T(y_0)) = \frac{8}{27} = \frac{4}{9} \cdot \frac{2}{3} \nleq \varphi(d(x_0, y_0))\frac{2}{3} = \varphi(d(x_0, y_0))d(x_0, y_0). \tag{3.41}$$

Case 2. Let $b \in (4/9, 1)$. Put $x_0 = 23/48$ and $y_0 = 1/12 \in Tx_0 = \{1/12, 131/432\}$. It follows that

$$bd(x_0, y_0) = \frac{19b}{48} \nleq \frac{4}{9} \cdot \frac{19}{48} = \frac{19}{108} = f(x_0); \tag{3.42}$$

Let $x_0 = 23/48$ and $y_0 = 131/432 \in Tx_0 = \{1/12, 131/432\}$. Note that $\varphi(d(x_0, y_0)) < b$. It is easy to verify that

$$d(y_0, T(y_0)) = \left|\frac{131}{432} - \frac{1}{3} \cdot \frac{131^2}{432^2}\right| = \frac{152615}{559872} \nleq \varphi(d(x_0, y_0))\frac{19}{108} = \varphi(d(x_0, y_0))d(x_0, y_0). \quad (3.43)$$

Therefore, the assumptions of Theorem 1.4 are not satisfied.

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