Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 907275, 16 pages
doi:10.1155/2010/907275

Research Article

Strong Convergence of a Generalized Iterative Method for Semigroups of Nonexpansive Mappings in Hilbert Spaces

Husain Piri and Hamid Vaezi

Faculty of Mathematical Sciences, University of Tabriz, Tabriz 51664, Iran

Correspondence should be addressed to Husain Piri, husain.piri@gmail.com

Received 20 April 2010; Accepted 18 June 2010

Academic Editor: A. T. M. Lau

Using δ-strongly accretive and λ-strictly pseudocontractive mapping, we introduce a general iterative method for finding a common fixed point of a semigroup of non-expansive mappings in a Hilbert space, with respect to a sequence of left regular means defined on an appropriate space of bounded real-valued functions of the semigroup. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality.

1. Introduction

Let H be a real Hilbert space. A mapping T of H into itself is called non-expansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in H$. By $\mathrm{Fix}(T)$, we denote the set of fixed points of T (i.e., $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$).

Mann [1] introduced an iteration procedure for approximation of fixed points of a non-expansive mapping T on a Hilbert space as follows. Let $x_0 \in H$ and

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \qquad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. See also [2].

On the other hand, Moudafi [3] introduced the viscosity approximation method for fixed point of non-expansive mappings (see [4] for further developments in both Hilbert and Banach spaces). Let f be a contraction on a Hilbert space H (i.e., $\|fx - fy\| \leq \alpha\|x - y\|$, for all $x, y \in H$ and $0 \leq \alpha < 1$). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.2}$$

where α_n is sequence in $(0, 1)$. It is proved in [3, 4] that, under appropriate condition imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution x^* in $\mathrm{Fix}(T)$ of the variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in \mathrm{Fix}(T). \tag{1.3}$$

Assume that A is strongly positive, that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.4}$$

In [4] (see also [5]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.5}$$

converges strongly to the unique solution of the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, u \rangle, \tag{1.6}$$

provided that the sequence $\{\alpha_n\}$ satisfies certain conditions. Marino and Xu [6] combined the iterative (1.5) with the viscosity approximation method (1.2) and considered the following general iterative methods:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where $0 < \gamma < \overline{\gamma}/\alpha$. They proved that if $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(C_1) $\alpha_n \to 0$,

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C_3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} (\alpha_{n+1}/\alpha_n) = 1$,

then, the sequence $\{x_n\}$ generated by (1.7) converges strongly, as $n \to \infty$, to the unique solution of the variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.8}$$

which is the optimality condition for minimization problem

$$\min_{x\in \operatorname{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$).

Let E^* be the topological dual of a Banach space E. The value of $j \in E^*$ at $x \in E$ will be denoted by $\langle x, j\rangle$ or $j(x)$. With each $x \in E$, we associate the set

$$J(x) = \left\{ j \in E^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2 \right\}. \tag{1.10}$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for each $x \in E$. The multivalued mapping J from E into E^* is said to be the (normalized) duality mapping. A Banach space E is said to be smooth if the duality mapping J is single valued. As it is well known, the duality mapping is the identity when E is a Hilbert space; see [7].

Let δ and λ be two positive real numbers such that $\delta, \lambda < 1$. Recall that a mapping F with domain $D(F)$ and range $R(F)$ in E is called δ-strongly accretive if, for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \geq \delta \|x-y\|^2. \tag{1.11}$$

Recall also that a mapping F is called λ-strictly pseudo-contractive if, for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(x-y) - (Fx - Fy)\|^2. \tag{1.12}$$

It is easy to see that (1.12) can be rewritten as

$$\langle (I-F)x - (I-F)y, j(x-y)\rangle \geq \lambda\|(I-F)x - (I-F)y\|^2, \tag{1.13}$$

see [8].

In this paper, motivated and inspired by Atsushiba and Takahashi [9], Lau et al. [10], Marino and Xu [6] and Xu [4, 11], we introduce the iterative below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n}(x_n), \qquad n \geq 0, \tag{1.14}$$

where F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, f is a contraction on a Hilbert space H with coefficient $0 < \alpha < 1$, γ is a positive real number such that $\gamma < 1 - \sqrt{(1-\delta)/\lambda}/\alpha$, and $\varphi = \{T_t : t \in S\}$ is a non-expansive semigroup on H such that the set $\operatorname{Fix}(\varphi)$ of common fixed point of φ is nonempty, X is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t(x), y\rangle$ is an element of X for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on X. Our purpose in this paper is to introduce this general iterative algorithm for approximating a common fixed points of semigroups of non-expansive mappings which solves some variational inequality. We will prove that if $\{\mu_n\}$ is left regular and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C_1) and (C_2), then $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\varphi)$, which solves the variational inequality:

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{1.15}$$

Various applications to the additive semigroup of nonnegative real numbers and commuting pairs of non-expansive mappings are also presented. It is worth mentioning that we obtain our result without assuming condition (C_3).

2. Preliminaries

Let S be a semigroup and let $B(S)$ be the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st), \quad (r_s f)(t) = f(ts), \quad \forall t \in S. \tag{2.1}$$

Let X be a subspace of $B(S)$ containing 1, and let X^* be its dual. An element μ in X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp., right invariant), that is, $l_s(X) \subset X$ (resp., $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp., right) amenable if X has a left (resp., right) invariant mean. X is amenable if X is both left and right amenable. As it is well known, $B(S)$ is amenable when S is a commutative semigroup; see [12]. A net $\{\mu_\alpha\}$ of means on X is said to be left regular if

$$\lim_{\alpha} \| {l_s}^* \mu_\alpha - \mu_\alpha \| = 0, \tag{2.2}$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\varphi = \{T_t : t \in S\}$ of mapping from C into itself is said to be a non-expansive semigroup on C if T_t is non-expansive and $T_{ts} = T_t T_s$ for each $t, s \in S$. We denote by $\mathrm{Fix}(\varphi)$ the set of common fixed points of φ, that is,

$$\mathrm{Fix}(\varphi) = \bigcap_{t \in S} \{x \in C : T_t x = x\}. \tag{2.3}$$

The open ball of radius r centered at 0 is denoted by B_r. For subset D of E, by $\overline{\mathrm{co}} D$, we denote the closed convex hull of D. Weak convergence is denoted by $\rightharpoonup$, and strong convergence is denoted by $\to$.

Lemma 2.1 (see [12, 13]). *Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact, and let X be a subspace of $B(S)$ containing all functions $t \to \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle, \tag{2.4}$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t) d\mu(t) \in \overline{\text{co}}\{f(t) : t \in S\}. \tag{2.5}$$

One can write f_μ by $\int f(t) d\mu(t)$.

Lemma 2.2 (see [13]). *Let C be a closed convex subset of a Hilbert space H, $\varphi = \{T_t : t \in S\}$ a semigroup from C into C such that $\text{Fix}(\varphi) \neq \emptyset$, the mapping $t \to \langle T_t x, y \rangle$ an element of X for each $x \in C$ and $y \in H$, and μ a mean on X. If one writes $T_\mu(x)$ instead of $\int T_t x d\mu(t)$, then the following holds.*

(i) *T_μ is non-expansive mapping from C into C.*

(ii) *$T_\mu(x) = x$ for each $x \in \text{Fix}(\varphi)$.*

(iii) *$T_\mu(x) \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$.*

(iv) *If μ is left invariant, then T_μ is a non-expansive retraction from C onto $\text{Fix}(\varphi)$.*

Let C be a nonempty subset of a normed space E, and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\|x - y_0\| = d(x, C), \tag{2.6}$$

where $d(x, C) = \inf_{y \in C} \|x - y\|$. The number $d(x, C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximation from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}. \tag{2.7}$$

This defines a mapping P_C from X into 2^C and is called metric (the nearest point) projection onto C.

Lemma 2.3 (see [7]). *Let C be a nonempty convex subset of a smooth Banach space E and let $x \in X$ and $y \in C$. Then, the following is equivalent.*

(i) *y is the best approximation to x.*

(ii) *y is a solution of the variational inequality*

$$\langle y - z, J(x - y) \rangle \geq 0, \quad \forall z \in C. \tag{2.8}$$

Let C be a nonempty subset of a Banach space E and $T : C \to E$ a mapping. Then T is said to be demiclosed at $v \in E$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C, \quad Tx_n \longrightarrow v, \quad \text{imply } Tu = v. \tag{2.9}$$

Lemma 2.4 (see [14]). *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is non-expansive. Then, the mapping $I - T$ is demiclosed at zero.*

The following lemma is well known.

Lemma 2.5. *Let H be a real Hilbert space. Then, for all $x, y \in H$*

(i) $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(ii) $\|x - y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.6 (see [11]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \qquad n \geq 0, \tag{2.10}$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

The following lemma will be frequently used throughout the paper. For the sake of completeness, we include its proof.

Lemma 2.7. *Let E be a real smooth Banach space and $F : E \to E$ a mapping.*

(i) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then, $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.*

(ii) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then, for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda}\,)$.*

Proof. (i) From (1.11) and (1.13), we obtain

$$\lambda\left\|(I-F)x - (I-F)y\right\|^2 \leq \left\|x - y\right\|^2 - \langle Fx - Fy, J(x-y)\rangle \leq (1-\delta)\left\|x - y\right\|^2. \tag{2.11}$$

Because $\delta + \lambda > 1 \Leftrightarrow \sqrt{(1-\delta)/\lambda} \in (0,1)$, we have

$$\left\|(I-F)x - (I-F)y\right\| \leq \sqrt{\frac{1-\delta}{\lambda}}\left\|x - y\right\|, \tag{2.12}$$

and, therefore, $I - F$ is contractive with constant$\sqrt{(1-\delta)/\lambda}$.

(ii) Because $I - F$ is contractive with constant$\sqrt{(1-\delta)/\lambda}$, for each fixed number $\tau \in (0,1)$, we have

$$\begin{aligned}\|x - y - \tau(F(x) - F(y))\| &= \|(1-\tau)(x-y) + \tau[(I-F)x - (I-F)y]\| \\ &\le (1-\tau)\|x-y\| + \tau\|(I-F)x - (I-F)y\| \\ &\le (1-\tau)\|x-y\| + \tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \\ &= \left(1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|.\end{aligned} \tag{2.13}$$

This shows that $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda})$. □

Throughout this paper, F will denote a δ-strongly accretive and λ-strictly pseudo-contractive mapping with $\delta + \lambda > 1$, and f is a contraction with coefficient $0 < \alpha < 1$ on a Hilbert space H. We will also always use γ to mean a number in $(0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$.

3. Strong Convergence Theorem

The following is our main result.

Theorem 3.1. *Let $\varphi = \{T_t : t \in S\}$ be a non-expansive semigroup on a real Hilbert space H such that* $\mathrm{Fix}(\varphi) \neq \emptyset$. *Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \to o$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$ and $\{x_n\}$ be generated by the iteration algorithm* (1.14). *Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(\varphi)$, which is a unique solution of the variational inequality* (1.15). *Equivalently, one has*

$$P_{\mathrm{Fix}(\varphi)}(I - F + \gamma f)x^* = x^*. \tag{3.1}$$

Proof. First, we claim that $\{x_n\}$ is bounded. Let $p \in \mathrm{Fix}(\varphi)$; by Lemmas 2.2 and 2.7 we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - p\| \\ &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)p - \alpha_n F(p)\| \\ &\le \alpha_n\|\gamma f(x_n) - F(p)\| + \|(I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)p\| \\ &\le \alpha_n\|\gamma f(x_n) - \gamma f(p)\| \\ &\quad + \alpha_n\|\gamma f(p) - F(p)\| + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|T\mu_n(x_n) - p\| \\ &\le \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\alpha\right)\right)\|x_n - p\| + \alpha_n\|\gamma f(p) - F(p)\|\end{aligned}$$

$$
\begin{aligned}
&= \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)\right)\|x_n-p\| \\
&\quad + \frac{\alpha_n\left(1-\sqrt{(1-\delta)/\lambda}-\gamma\alpha\right)}{\left(1-\gamma\alpha-\sqrt{(1-\delta)/\lambda}\right)}\|\gamma f(p)-F(p)\| \\
&\leq \max\left\{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)^{-1}\|\gamma f(p)-F(p)\|, \|x_n-p\|\right\}.
\end{aligned} \tag{3.2}
$$

By induction,

$$
\|x_n-p\| \leq \max\left\{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)^{-1}\|\gamma f(p)-F(p)\|, \|x_0-p\|\right\} = M_0. \tag{3.3}
$$

Therefore, $\{x_n\}$ is bounded and so is $\{f(x_n)\}$.

Set $D = \{y \in H : \|y-p\| \leq M_0\}$. We remark that D is φ-invariant bounded closed convex set and $\{x_n\} \subset D$. Now we claim that

$$
\limsup_{n\to\infty} \sup_{y\in D}\left\|T_{\mu_n}(y)-T_t(T_{\mu_n}(y))\right\| = 0, \qquad \forall t \in S. \tag{3.4}
$$

Let $\epsilon > 0$. By [15, Theorem 1.2], there exists $\delta > 0$ such that

$$
\overline{\text{co}}F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D), \qquad \forall t \in S. \tag{3.5}
$$

Also by [15, Corollary 1.1], there exists a natural number N such that

$$
\left\|\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y) - T_t\left(\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y)\right)\right\| \leq \delta, \tag{3.6}
$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l_{t^i}^*\mu_n\| \leq \delta/(M_0 + \|p\|)$ for $n \geq n_0$ and $i = 1, 2, \ldots, N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\left\|T_{\mu_n}(y) - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}(y)\,d\mu_n(s)\right\| \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left|\langle T_{\mu_n}(y), z\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}(y)\,d\mu_n(s), z\right\rangle\right| \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left|\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_s(y), z\rangle - \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{t^is}(y), z\rangle\right| \\
&\leq \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T_s(y), z\rangle - (l_{t^i}^*\mu_n)_s\langle T_s(y), z\rangle\right| \\
&\leq \max_{i=0,1,2,\ldots,N}\|\mu_n - l_{t^i}^*\mu_n\|(M_0 + \|p\|) \leq \delta, \quad \forall n \geq n_0.
\end{aligned} \tag{3.7}$$

By Lemma 2.2 we have

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}(y)\,d\mu_n(s) \in \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s(y)) : s \in S\right\}. \tag{3.8}$$

It follows from (3.5), (3.6), (3.7), and (3.8) that

$$T_{\mu_n}(y) \in \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^is}(y) : s \in S\right\} + B_\delta \subset \overline{\text{co}}F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D), \tag{3.9}$$

for all $y \in D$ and $n \geq n_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D}\|T_t(T_{\mu_n}(y)) - T_{\mu_n}(y)\| \leq \epsilon. \tag{3.10}$$

Since $\epsilon > 0$ is arbitrary, we get (3.4). In this stage, we will show that

$$\lim_{n\to\infty}\|x_n - T_t(x_n)\| = 0, \quad \forall t \in S. \tag{3.11}$$

Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$, which satisfies (3.5). Take

$$L_0 = \left[\left(1 + \gamma\alpha + \sqrt{\frac{1-\delta}{\lambda}}\right)M_0 + \|\gamma f(p) - F(p)\|\right]. \tag{3.12}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and (3.4) there exists $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta/L_0$ and $T_{\mu_n}(x_n) \in F_\delta(T_t)$, for all $n \geq n_0$. By Lemma 2.7, we have

$$\begin{aligned}
&\alpha_n\|\gamma f(x_n) - FT_{\mu_n}(x_n)\| \\
&\quad\leq \alpha_n\left(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - F(T_{\mu_n}(x_n))\|\right) \\
&\quad\leq \alpha_n(\gamma\alpha\|x_n - p\| + \|\gamma f(p) - F(p)\|) \\
&\qquad + \alpha_n\left(\|(I-F)p - (I-F)T_{\mu_n}(x_n)\| + \|p - T_{\mu_n}(x_n)\|\right) \\
&\quad\leq \alpha_n\left(1 + \sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)\|x_n - p\| + \alpha_n\|\gamma f(p) - F(p)\| \\
&\quad\leq \alpha_n\left[\left(1 + \sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)M_0 + \|\gamma f(p) - F(p)\|\right] \\
&\quad\leq \alpha_n L_0 \leq \delta,
\end{aligned} \tag{3.13}$$

for all $n \geq n_0$. Therefore, we have

$$x_{n+1} = T_{\mu_n}(x_n) + \alpha_n\left[\gamma f(x_n) + F(T_{\mu_n}(x_n))\right] \in F_\delta(T) + B_\delta \subset F_\epsilon(T_t), \tag{3.14}$$

for all $n \geq n_0$. This shows that

$$\|x_n - T_t(x_n)\| \leq \epsilon, \qquad \forall n \geq n_0. \tag{3.15}$$

Since $\epsilon > 0$ is arbitrary, we get (3.11).

Let $Q = P_{\mathrm{Fix}(\varphi)}$. Then $Q(I - F - \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned}
&\|Q(I - F + \gamma \mathrm{f})(x) - Q(I - F + \gamma f)(y)\| \\
&\quad\leq \|(I - F + \gamma f)(x) - (I - F + \gamma f)(y)\| \\
&\quad\leq \|(I-F)(x) - (I-F)(y)\| + \gamma\|f(x) - f(y)\| \\
&\quad\leq \left(\sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)\|x - y\|,
\end{aligned} \tag{3.16}$$

and hence $Q(I - F - \gamma f)$ is a contraction due to $(\sqrt{(1-\delta)/\lambda} + \gamma\alpha) \in (0,1)$.

Therefore, by Banach contraction principal, $P_{\mathrm{Fix}(\varphi)}(\gamma f + I - F)$ has a unique fixed point x^*. Then using Lemma 2.3, x^* is the unique solution of the variational inequality

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in \mathrm{Fix}(\varphi). \tag{3.17}$$

We show that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle \leq 0. \tag{3.18}$$

Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle = \lim_{k\to\infty}\langle \gamma f(x^*) - F(x^*), x_{n_k} - x^*\rangle. \tag{3.19}$$

Because $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup z$. In terms of Lemma 2.4 and (3.11), we conclude that $z \in \mathrm{Fix}(\varphi)$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle = \langle \gamma f(x^*) - F(x^*), z - x^*\rangle \leq 0. \tag{3.20}$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. By Lemmas 2.5 and 2.7 we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - x^*\right\|^2 \\
&\quad = \left\|\alpha_n\gamma f(x_n) - \alpha_n F(x^*) + (I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)x^*\right\|^2 \\
&\quad = \left\|(I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)x^*\right\|^2 + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
&\quad \leq \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
&\quad \leq \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\qquad + 2\alpha_n\langle \gamma f(x^*) - F(x^*), x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.21}$$

On the other hand

$$\begin{aligned}
&\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\quad \leq \gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \leq \gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 \\
&\qquad + \gamma\alpha\|x_n - x^*\|\sqrt{2\left|\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle\right|}\sqrt{\alpha_n}.
\end{aligned} \tag{3.22}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, we can take a constant $G_0 > 0$ such that

$$\gamma\alpha\|x_n - x^*\|\sqrt{2\left|\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle\right|} < G_0, \qquad \forall n \in \mathbb{N}. \tag{3.23}$$

So from the above, we reach the following:

$$\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \le \gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 + G_0\sqrt{\alpha_n}. \tag{3.24}$$

Substituting (3.24) in (3.21), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad\le \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n G_0\sqrt{\alpha_n} + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
&\quad= \left(1 - 2\alpha_n\left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right)\|x_n - x^*\|^2 \\
&\qquad + \alpha_n\left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - x^*\|^2 + 2G_0\sqrt{\alpha_n} + 2\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle\right].
\end{aligned} \tag{3.25}$$

It follows that

$$\|x_{n+1} - x^*\|^2 \le \left(1 - \alpha_n\left[2\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha\gamma\right) + 2\alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right)\|x_n - x^*\|^2 + \alpha_n\beta_n, \tag{3.26}$$

where

$$\beta_n = \left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - x^*\|^2 + 2G_0\sqrt{\alpha_n} + 2\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle\right]. \tag{3.27}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, by (3.18), we get

$$\limsup_{n\to\infty} \beta_n \le 0. \tag{3.28}$$

Consequently, applying Lemma 2.6, to (3.26), we conclude that $x_n \to x^*$. □

Corollary 3.2. *Let* X*,* φ*,* $\{\mu_n\}$*, and* $\{\alpha_n\}$ *be as in Theorem 3.1. Suppose that* A *a strongly positive bounded linear operator on* H *with coefficient* $\overline{\gamma} > 1/2$ *and* $0 < \zeta < (1 - \sqrt{2 - 2\overline{\gamma}})/\alpha$*. Let* $\{x_n\}$ *be defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \zeta f(x_n) + (I - \alpha_n A) T_{\mu_n}(x_n), \quad n \geq 0. \tag{3.29}$$

Then, $\{x_n\}$ *converges strongly, as* $n \to \infty$*, to* $x^* \in \mathrm{Fix}(\varphi)$*, which is a unique solution of the variational inequality*

$$\langle (A - \zeta f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{3.30}$$

Proof. Because A is strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$, we have

$$\langle Ax - Ay, x - y \rangle \geq \overline{\gamma} \|x - y\|^2. \tag{3.31}$$

Therefore, A is $\overline{\gamma}$-strongly accretive. On the other hand,

$$\begin{aligned} &\|(I - A)x - (I - A)y\|^2 \\ &\quad = \langle (x - y) - (Ax - Ay), (x - y) - (Ax - Ay) \rangle \\ &\quad = \langle x - y, x - y \rangle - 2\langle Ax - Ay, x - y \rangle + \langle Ax - Ay, Ax - Ay \rangle \\ &\quad \leq \|x - y\|^2 - 2\langle Ax - Ay, x - y \rangle + \|A\| \|x - y\|^2. \end{aligned} \tag{3.32}$$

Since A is strongly positive if and only if $(1/\|A\|)A$ is strongly positive, we may assume, with no loss of generality, that $\|A\| = 1$, so that

$$\langle Ax - Ay, x - y \rangle \leq \|x - y\|^2 - \frac{1}{2} \|(I - A)x - (I - A)y\|^2. \tag{3.33}$$

This shows that A is $1/2$-strictly pseudo-contractive. Now apply Theorem 3.1 to conclude the result. □

Corollary 3.3. *Let* X*,* φ*,* $\{\mu_n\}$ *and* $\{\alpha_n\}$ *be as in Theorem 3.1. Suppose* $u, x_0 \in H$ *and define a sequence* $\{x_n\}$ *by the iterative algorithm*

$$x_{n+1} = \alpha_n u + (I - \alpha_n F) T_{\mu_n}(x_n), \quad n \geq 0. \tag{3.34}$$

Then, $\{x_n\}$ *converges strongly, as* $n \to \infty$*, to a* $x^* \in \mathrm{Fix}(\varphi)$*, which is a unique solution of the variational inequality*

$$\langle Fx^* - u, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{3.35}$$

Proof. It is sufficient to take $f = u$ and$\gamma = 1$ in Theorem 3.1. □

4. Some Application

Corollary 4.1. *Let S and T be non-expansive mappings on a Hilbert space H with $ST = TS$ such that* $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. *Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$ and define a sequence $\{x_n\}$ by the iterative algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^i (x_n), \quad n \geq 0. \tag{4.1}$$

Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ which solves the variational inequality:

$$\langle (F - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T). \tag{4.2}$$

Proof. Let $T(i,j) = S^i T^j$ for each $i, j \in \mathbb{N} \cup \{0\}$. Then $\{T(i,j) : i, j \in \mathbb{N} \cup \{0\}\}$ is a semigroup of non-expansive mappings on H. Now, for each $n \in \mathbb{N}$ and $i, j \in B((\mathbb{N} \cup \{0\})^2)$, we define $\mu_n(f) = (1/n^2) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(i,j)$. Then, $\{\mu_n\}$ is regular sequence of means [16]. Next, for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n}(x) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j (x). \tag{4.3}$$

Therefore, applying Theorem 3.1, the result follows. □

Corollary 4.2. *Let $\varphi = \{T_t : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of non-expansive mappings on a Hilbert space H such that* $\mathrm{Fix}(\varphi) \neq \emptyset$. *Let α_n be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$ and $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. Let $\{x_n\}$ be a sequence defined by the iterative algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) \frac{1}{t_n} \int_0^{t_n} T_s(x_n) ds, \quad n \geq 0, \tag{4.4}$$

where $\{t_n\}$ is an increasing sequence in $(0, \infty)$ such that $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} (t_n / t_{n+1}) = 1$. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(\varphi)$, which solves the variational inequality

$$\langle (F - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{4.5}$$

Proof. For $n \in \mathbb{N}$, we define $\mu_n(f) = (1/t_n) \int_0^{t_n} f(t) dt$ for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denotes the space of all real-valued bounded continuous functions on R^+ with supremum norm. Then, $\{\mu_n\}$ is regular sequence of means [16]. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = (1/t_n) \int_0^{t_n} T_s(x) ds$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.3. *Let $\varphi = \{T_t : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of non-expansive mappings on a Hilbert space H such that $\mathrm{Fix}(\varphi) \neq \emptyset$. Let α_n be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$ and $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. Let $\{x_n\}$ be a sequence defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) r_n \int_0^{\infty} \exp(-r_n s) T_s x_n ds, \quad n \geq 0, \tag{4.6}$$

where $\{r_n\}$ is an decreasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} r_n = 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(\varphi)$, which solves the variational inequality

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{4.7}$$

Proof. For $n \in \mathbb{N}$, we define $\mu_n(f) = r_n \int_0^{\infty} \exp(-r_n t) f(t) dt$ for each $f \in C(\mathbb{R}_+)$. Then $\{\mu_n\}$ is regular sequence of means [16]. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = r_n \int_0^{\infty} \exp(-r_n t) T_t(x) dt$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.4. *Let T be a non-expansive mapping on a Hilbert space H such that $\mathrm{Fix}(T) \neq \emptyset$. Let α_n be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and let $Q = \{q_{n,m}\}$ be a strongly regular matrix. Let $x_0 \in H$ and $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. Let $\{x_n\}$ be a sequence defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) \sum_{m=0}^{\infty} q_{n,m} T^m x_n, \quad n \geq 0. \tag{4.8}$$

Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(T)$ which solves the variational inequality

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{4.9}$$

Proof. For each $n \in \mathbb{N}$, we define

$$\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m), \tag{4.10}$$

for each $f \in B(\mathbb{N} \cup \{0\})$. Since Q is a strongly regular matrix, for each m, we have $q_{n,m} \to 0$, as $n \to \infty$; see [17]. Then, it is easy to see that $\{\mu_n\}$ is regular sequence of means. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = \sum_{m=0}^{\infty} q_{n,m} T^m(x)$. Now, apply Theorem 3.1 to conclude the result. □

Acknowledgments

The authors thank the referee(s) for the helpful comments, which improved the presentation of this paper. This paper is dedicated to Professor Anthony To Ming Lau. This paper is based on final report of the research project of the Ph.D. thesis which is done with financial support of research office of the University of Tabriz.

References

[1] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.

[2] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[3] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[4] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[5] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[6] G. Marino and H. K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[7] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Topological Fixed Point Theory and Its Applications, Springer, New York, NY, USA, 2009.

[8] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. III*, Springer, New York, NY, USA, 1985.

[9] S. Atsushiba and W. Takahashi, "Approximating common fixed points of nonexpansive semigroups by the Mann iteration process," *Annales Universitatis Mariae Curie-Skłodowska A*, vol. 51, no. 2, pp. 1–16, 1997.

[10] A. T. Lau, H. Miyake, and W. Takahashi, "Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 4, pp. 1211–1225, 2007.

[11] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[12] A. T. Lau, N. Shioji, and W. Takahashi, "Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces," *Journal of Functional Analysis*, vol. 161, no. 1, pp. 62–75, 1999.

[13] W. Takahashi, "A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space," *Proceedings of the American Mathematical Society*, vol. 81, no. 2, pp. 253–256, 1981.

[14] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[15] R. E. Bruck, "On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces," *Israel Journal of Mathematics*, vol. 38, no. 4, pp. 304–314, 1981.

[16] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000, Fixed Point Theory and Its Application.

[17] N. Hirano, K. Kido, and W. Takahashi, "Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 12, no. 11, pp. 1269–1281, 1988.