Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 970579, 20 pages
doi:10.1155/2010/970579

Research Article

Equivalent Extensions to Caristi-Kirk's Fixed Point Theorem, Ekeland's Variational Principle, and Takahashi's Minimization Theorem

Zili Wu

Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren Ai Road, Dushu Lake Higher Education Town, Suzhou Industrial Park, Suzhou, Jiangsu 215123, China

Correspondence should be addressed to Zili Wu, ziliwu@email.com

Received 26 September 2009; Accepted 24 November 2009

Academic Editor: Mohamed A. Khamsi

With a recent result of Suzuki (2001) we extend Caristi-Kirk's fixed point theorem, Ekeland's variational principle, and Takahashi's minimization theorem in a complete metric space by replacing the distance with a τ-distance. In addition, these extensions are shown to be equivalent. When the τ-distance is l.s.c. in its second variable, they are applicable to establish more equivalent results about the generalized weak sharp minima and error bounds, which are in turn useful for extending some existing results such as the petal theorem.

1. Introduction

Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty]$ a proper lower semicontinuous (l.s.c.) bounded below function. Caristi-Kirk fixed point theorem [1, Theorem (2.1)$'$] states that there exists $x_0 \in Tx_0$ for a relation or multivalued mapping $T : X \to X$ if for each $x \in X$ with $\inf_X f < f(x)$ there exists $\overline{x} \in Tx$ such that

$$d(x, \overline{x}) + f(\overline{x}) \leq f(x), \tag{1.1}$$

(see also [2, Theorem 4.12] or [3, Theorem C]) while Ekeland's variational principle (EVP) [4, 5] asserts that for each $\epsilon \in (0, +\infty)$ and $u \in X$ with $f(u) \leq \inf_X f + \epsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$ and

$$f(x) + \epsilon d(v, x) > f(v) \quad \forall x \in X \text{ with } x \neq v. \tag{1.2}$$

EVP has been shown to have many equivalent formulations such as Caristi-Kirk fixed point theorem, the drop theorem [6], the petal theorem [3, Theorem F], Takahashi minimization theorem [7, Theorem 1], and two results about weak sharp minima and error bounds [8, Theorems 3.1 and 3.2]. Moreover, in a Banach space, it is equivalent to the Bishop-Phelps theorem (see [9]). EVP has played an important role in the study of nonlinear analysis, convex analysis, and optimization theory. For more applications, EVP and several equivalent results stated above have been extended by introducing more general distances. For example, Kada et al. have presented the concept of a w-distance in [10] to extend EVP, Caristi's fixed point theorem, and Takahashi minimization theorem. Suzuki has extended these three results by replacing a w-distance with a τ-distance in [11]. For more extensions of these theorems, with a w-distance being replaced by a τ-function and a Q-function, respectively, the reader is referred to [12, 13].

Theoretically, it is interesting to reveal the relationships among the above existing results (or their extensions). In this paper, while further extending the above theorems in a complete metric space with a τ-distance, we show that these extensions are equivalent. For the case where the τ-distance is l.s.c. in its second variable, we apply our generalizations to extend several existing results about the weak sharp minima and error bounds and then demonstrate their equivalent relationship. In particular, when the τ-distance reduces to the complete metric, our results turn out to be equivalent to EVP and hence to its existing equivalent formulations.

2. w-Distance and τ-Distance

For convenience, we recall the concepts of w-distance and τ-distance and some properties which will be used in the paper.

Definition 2.1 (see [10]). Let (X, d) be a metric space. A function $p : X \times X \to [0, +\infty)$ is called a w-distance on X if the following are satisfied:

(ω_1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $(x, y, z) \in X \times X \times X$;

(ω_2) for each $x \in X$, $p(x, \cdot) : X \to [0, +\infty)$ is l.s.c.;

(ω_3) for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$p(z, x) \leq \delta, \qquad p(z, y) \leq \delta \Longrightarrow d(x, y) \leq \epsilon. \tag{2.1}$$

From the definition, we see that the metric d is a w-distance on X. If X is a normed linear space with norm $\|\cdot\|$, then both p_1 and p_2 defined by

$$p_1(x, y) = \|y\|, \quad p_2(x, y) = \|x\| + \|y\| \quad \forall (x, y) \in X \times X \tag{2.2}$$

are w-distances on X. Note that $p_1(x, x) \neq 0 \neq p_2(x, x)$ for each $x \in X$ with $x \neq 0$. For more examples, we see [10].

It is easy to see that for any $\alpha \in (0, 1)$ and w-distance p, the function αp is also a w-distance. For any positive M and w-distance p on X, the function p_M defined by

$$p_M(x, y) := \min\{p(x, y), M\} \quad \forall (x, y) \in X \times X \tag{2.3}$$

is a bounded w-distance on X.

The following proposition shows that we can construct another w-distance from a given w-distance under certain conditions.

Proposition 2.2. *Let $x_0 \in X$, p a w-distance on X, and $h : [0,+\infty) \to [0,+\infty)$ a nondecreasing function. If, for each $r > 0$,*

$$\inf_{x\in X}\int_{p(x_0,x)}^{p(x_0,x)+r}\frac{dt}{1+h(t)} > 0, \tag{2.4}$$

then the function q defined by

$$q(x,y) := \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)}\frac{dt}{1+h(t)} \quad \text{for } (x,y) \in X\times X \tag{2.5}$$

is a w-distance. In particular, if p is bounded on $X\times X$, then q is a w-distance.

Proof. Since h is nondecreasing, for $(x,z) \in X\times X$,

$$\begin{aligned}
q(x,z) &= \int_{p(x_0,x)}^{p(x_0,x)+p(x,z)}\frac{dt}{1+h(t)} \le \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)+p(y,z)}\frac{dt}{1+h(t)}\\
&= \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)}\frac{dt}{1+h(t)} + \int_{p(x_0,x)+p(x,y)}^{p(x_0,x)+p(x,y)+p(y,z)}\frac{dt}{1+h(t)}\\
&\le \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)}\frac{dt}{1+h(t)} + \int_{p(x_0,y)}^{p(x_0,y)+p(y,z)}\frac{dt}{1+h(t)}\\
&= q(x,y)+q(y,z).
\end{aligned} \tag{2.6}$$

In addition, q is obviously lower semicontinuous in its second variable.

Now, for each $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$p(z,x) \le \delta_1, \qquad p(z,y) \le \delta_1 \Longrightarrow d(x,y) \le \epsilon. \tag{2.7}$$

Taking δ such that

$$0 < \delta < \inf_{x\in X}\int_{p(x_0,x)}^{p(x_0,x)+\delta_1}\frac{dt}{1+h(t)}, \tag{2.8}$$

we obtain that, for x, y, z in X with $q(z,x) \le \delta$ and $q(z,y) \le \delta$,

$$q(z,x) = \int_{p(x_0,z)}^{p(x_0,z)+p(z,x)}\frac{dt}{1+h(t)} \le \delta < \int_{p(x_0,z)}^{p(x_0,z)+\delta_1}\frac{dt}{1+h(t)}, \tag{2.9}$$

from which it follows that $p(z,x) \leq \delta_1$. Similarly, we have $p(z,y) \leq \delta_1$. Thus $d(x,y) \leq \epsilon$. Therefore, q is a w-distance on X.

Next, if p is bounded on $X \times X$, then there exists $M > 0$ such that

$$\int_{p(x_0,x)}^{p(x_0,x)+r} \frac{dt}{1+h(t)} \geq \frac{r}{1+h(M+r)} > 0 \quad \forall x \in X. \tag{2.10}$$

Thus q is also a w-distance on X. □

When p is unbounded on $X \times X$, the condition in Proposition 2.2 may not be satisfied. However, if h is a nondecreasing function satisfying

$$\int_{0}^{+\infty} \frac{dt}{1+h(t)} = +\infty, \tag{2.11}$$

then the function q in Proposition 2.2 is a τ-distance (see [11, Proposition 4]), a more general distance introduced by Suzuki in [11] as below.

Definition 2.3 (see [11]). $p : X \times X \to [0,+\infty)$ is said to be a τ-distance on X provided that

(τ_1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $(x,y,z) \in X \times X \times X$ and there exists a function $\eta : X \times [0,+\infty) \to [0,+\infty)$ such that

(τ_2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $(x,t) \in X \times [0,+\infty)$, and η is concave and continuous in its second variable;

(τ_3) $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} \sup\{\eta(z_n, p(z_n,x_m)) : n \leq m\} = 0$ imply

$$p(w,x) \leq \liminf_{n\to+\infty} p(w,x_n) \quad \forall w \in X; \tag{2.12}$$

(τ_4) $\lim_{n\to+\infty} \sup\{p(x_n,y_m) : n \leq m\} = 0$ and $\lim_{n\to+\infty} \eta(x_n,t_n) = 0$ imply

$$\lim_{n\to+\infty} \eta(y_n,t_n) = 0; \tag{2.13}$$

(τ_5) $\lim_{n\to+\infty} \eta(z_n, p(z_n,x_n)) = 0$ and $\lim_{n\to+\infty} \eta(z_n, p(z_n,y_n)) = 0$ imply

$$\lim_{n\to+\infty} d(x_n,y_n) = 0. \tag{2.14}$$

Suzuki has proved that a w-distance is a τ-distance [11, Proposition 4]. If a τ-distance p satisfies $p(z,x) = 0$ and $p(z,y) = 0$ for $(x,y,z) \in X \times X \times X$, then $x = y$ (see [11, Lemma 2]). For more properties of a τ-distance, the reader is referred to [11].

3. Fixed Point Theorems

From now on, we assume that (X,d) is a complete metric space and $f : X \to (-\infty,+\infty]$ is a proper l.s.c. and bounded below function unless specified otherwise. In this section, mainly motivated by fixed point theorems (for a single-valued mapping) in [10, 11, 14–16], we present two similar results which are applicable to multivalued mapping cases. The following theorem established by Suzuki's in [11] plays an important role in extending existing results from a single-valued mapping to a multivalued mapping.

Theorem 3.1 (see [11, Proposition 8]). *Let p be a τ-distance on X. Denote*

$$M(x) := \{y \in X : p(x,y) + f(y) \leq f(x)\} \quad \forall x \in X. \tag{3.1}$$

Then for each $u \in X$ with $M(u) \neq \emptyset$, there exists $x_0 \in M(u)$ such that $M(x_0) \subseteq \{x_0\}$. In particular, there exists $y_0 \in X$ such that $M(y_0) \subseteq \{y_0\}$.

Based on Theorem 3.1, [11, Theorem 3] asserts that a single-valued mapping $T : X \to X$ has a fixed point x_0 in X when $Tx \in M(x)$ holds for all $x \in X$ (which generalizes [10, Theorem 2] by replacing a w-distance with a τ-distance). We show that the conclusion can be strengthened under a slightly weaker condition (in which $Tx \cap M(x) \neq \emptyset$ holds on a subset of X instead) for a multivalued mapping T.

Theorem 3.2. *Let p be a τ-distance on X and $T : X \to X$ a multivalued mapping. Suppose that for some $\epsilon \in (0, +\infty]$ there holds $Tx \cap M(x) \neq \emptyset$ for each $x \in X$ with $\inf_X f \leq f(x) < \inf_X f + \epsilon$. Then there exists $x_0 \in X$ such that*

$$\{x_0\} = M(x_0) = \left\{x \in M(x_0) : x \in Tx,\ p(x,x) = 0,\ \inf_X f \leq f(x) < \inf_X f + \epsilon\right\}, \tag{3.2}$$

where $M(x_0) := \{y \in X : p(x_0,y) + f(y) \leq f(x_0)\}$.

Proof. For each $x \in X$ with $\inf_X f \leq f(x) < \inf_X f + \epsilon$, the set

$$M_x := \{y \in X : f(y) \leq f(x)\} \tag{3.3}$$

is a nonempty closed subset of X since f is lower semicontinuous and

$$\overline{x} \in M(x) := \{y \in X : p(x,y) + f(y) \leq f(x)\} \subseteq M_x \tag{3.4}$$

for some $\overline{x} \in Tx$. Thus (M_x, d) is a complete metric space. By Theorem 3.1, there exists $x_0 \in M(x)$ such that $M(x_0) \subseteq \{x_0\}$. Since

$$\inf_X f \leq f(x_0) \leq f(x) < \inf_X f + \epsilon, \tag{3.5}$$

there exists $\overline{x}_0 \in Tx_0$ such that $\overline{x}_0 \in M(x_0)$. Thus $M(x_0) = \{x_0\}$, $x_0 = \overline{x}_0 \in Tx_0$, and

$$0 \leq p(x_0, x_0) = p(x_0, \overline{x}_0) \leq f(x_0) - f(\overline{x}_0) = 0. \tag{3.6}$$

□

Clearly, [8, Thoerem 4.1] follows as a special case of Theorem 3.2 with $p = d$. In addition, when $\epsilon = +\infty$ and T is a single-valued mapping, Theorem 3.2 contains [11, Theorem 3]. The following simple example further shows that Theorem 3.2 is applicable to more cases.

Example 3.3. Consider the mapping $T : [0,+\infty) \to [0,+\infty)$ defined by

$$Tx = \begin{cases} \left[x - x^2, x - \dfrac{1}{2}x^2\right) & \text{for } x \in [0,1); \\ \{x + x^2\} & \text{for } x \in [1,+\infty) \end{cases} \tag{3.7}$$

and the function $f(x) = 2\sqrt{x}$ for $x \in [0,+\infty)$. Obviously $f(0) = \inf_{[0,+\infty)} f$. For any $\epsilon \in (0,1]$, $x \in [0,\epsilon)$, and $y \in [0,x]$, we have

$$|x - y| = x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) \leq f(x) - f(y), \tag{3.8}$$

so, applying Theorem 3.2 to the above T and f with $p(x,y) = |x - y|$ for $x, y \in X := [0,+\infty)$, we obtain $x_0 \in X$ as in Theorem 3.2.

Motivated by [16, Theorem 7] and [14, Theorem 2.3], we further extend Theorem 3.2 as follows.

Theorem 3.4. *Let p be a τ-distance on X and $T : X \to X$ a multivalued mapping. Let $\epsilon \in (0,+\infty]$ and $\varphi : f^{-1}(-\infty, \inf_X f + \epsilon] \to [0,+\infty)$ satisfy*

$$\gamma := \sup\left\{\varphi(x) : x \in f^{-1}\left(-\infty, \inf_X f + \min\{\epsilon,\eta\}\right]\right\} < +\infty, \tag{3.9}$$

for some $\eta > 0$. If for each $x \in X$ with $\inf_X f \leq f(x) < \inf_X f + \epsilon$, there exists $\overline{x} \in Tx$ such that

$$f(\overline{x}) \leq f(x), \qquad p(x,\overline{x}) \leq \varphi(x)\left[f(x) - f(\overline{x})\right], \tag{3.10}$$

then there exists $x_0 \in X$ such that

$$\{x_0\} = M_\gamma(x_0) = \left\{x \in M_\gamma(x_0) : x \in Tx,\ p(x,x) = 0,\ \inf_X f \leq f(x) < \inf_X f + \epsilon\right\}, \tag{3.11}$$

where $M_\gamma(x_0) := \{y \in X : p(x_0,y) \leq (\gamma+1)[f(x_0) - f(y)]\}$.

Proof. For each $x \in X$ with $\inf_X f \leq f(x) < \inf_X f + \min\{\epsilon,\eta\}$, by assumption, there exists $\overline{x} \in Tx$ such that

$$p(x,\overline{x}) \leq \varphi(x)\left[f(x) - f(\overline{x})\right] \leq (\gamma+1)\left[f(x) - f(\overline{x})\right], \tag{3.12}$$

based on the inequalities $0 \le \varphi(x)$ and $f(\overline{x}) \le f(x)$. Upon applying Theorem 3.2 to the lower semicontinuous function $(\gamma+1)f$ on $f^{-1}(-\infty, \inf_X f + \epsilon]$ which is complete, we arrive at the conclusion. □

Next result is immediate from Theorem 3.4.

Theorem 3.5. *Let p be a τ-distance on X, $g : [\inf_X f, \inf_X f + \epsilon] \to [0,+\infty)$ either nondecreasing or upper semicontinuous (u.s.c.), and $T : X \to X$ a multivalued mapping. If for some $\epsilon \in (0,+\infty]$ and each $x \in X$ with $\inf_X f \le f(x) < \inf_X f + \epsilon$, there exists $\overline{x} \in Tx$ such that*

$$f(\overline{x}) \le f(x), \qquad p(x,\overline{x}) \le g(f(x))[f(x) - f(\overline{x})], \tag{3.13}$$

then there exists $x_0 \in X$ such that

$$\{x_0\} = M_\gamma(x_0) = \left\{x \in M_\gamma(x_0) : x \in Tx,\ p(x,x) = 0,\ \inf_X f \le f(x) < \inf_X f + \epsilon\right\}, \tag{3.14}$$

where $M_\gamma(x_0) := \{y \in X : p(x_0,y) \le (\gamma+1)[f(x_0) - f(y)]\}$ with

$$\gamma := \sup\left\{g(s) : \inf_X f \le s \le \inf_X f + \min\{\epsilon, 1\}\right\}. \tag{3.15}$$

Proof. For $x \in f^{-1}(-\infty, \inf_X f + \epsilon]$, define $\varphi(x) = g(f(x))$. Then for the case where g is nondecreasing we have

$$\sup\left\{\varphi(x) : x \in f^{-1}\left(-\infty, \inf_X f + \min\{\epsilon,1\}\right]\right\} \le g\left(\inf_X f + \min\{\epsilon,1\}\right) < +\infty. \tag{3.16}$$

Thus the conclusion follows from Theorem 3.4.

For the case where g is u.s.c., we define $c : [\inf_X f, \inf_X f + \epsilon] \to [0,+\infty)$ by $c(t) := \sup\{g(s) : \inf_X f \le s \le t\}$. Since g is u.s.c., c is well defined and nondecreasing. Now, for some $\epsilon \in (0,+\infty]$ and each $x \in X$ with $\inf_X f \le f(x) < \inf_X f + \epsilon$ there exists $\overline{x} \in Tx$ satisfying

$$f(\overline{x}) \le f(x), \qquad p(x,\overline{x}) \le g(f(x))[f(x) - f(\overline{x})] \le c(f(x))[f(x) - f(\overline{x})], \tag{3.17}$$

so we can apply the conclusion in the previous paragraph to c to get the same conclusion. □

Remark 3.6. When $\epsilon = +\infty$ and T is a single-valued mapping, Theorem 3.4 reduces to [16, Theorem 7] while Theorem 3.5 to [16, Theorems 8 and 9]. If also $p(x,y) = d(x,y)$ for all $(x,y) \in X \times X$, then Theorem 3.5 reduces to [14, Theorem 2.3] (when g is nondecreasing) and [15, Theorem 3] (when g is upper semicontinuous). In the later case, it also extends [14, Theorem 2.4].

Furthermore, we will see that the relaxation of T from a single-valued mapping (as in several existing results stated before) to a multivalued one (as in Theorems 3.2–3.5) is more helpful for us to obtain more results in the next section.

4. Extensions of Ekeland's Variational Principle

As applications of Theorems 3.4 and 3.5, several generalizations of EVP will be presented in this section.

Theorem 4.1. *Let p be a τ-distance on X, $\epsilon \in (0,+\infty]$, $u \in X$ satisfy $f(u) \le \inf_X f + \epsilon$, and $\varphi : f^{-1}(-\infty, \inf_X f + \epsilon] \to (0,+\infty)$ satisfy*

$$\sup\left\{\varphi(x) : x \in f^{-1}\left(-\infty, \inf_X f + \min\{\epsilon, \eta\}\right]\right\} < +\infty, \tag{4.1}$$

for some $\eta > 0$. Then there exists $v \in X$ such that $f(v) \le f(u)$ and

$$p(v,x) > \varphi(v)\left[f(v) - f(x)\right] \quad \forall x \in X \text{ with } x \ne v. \tag{4.2}$$

Proof. Take $M_u := \{x \in X : f(x) \le f(u)\}$. Then (M_u, d) is a nonempty complete metric space. We claim that there must exist $v \in M_u$ such that

$$p(v,x) > \varphi(v)\left[f(v) - f(x)\right] \quad \forall x \in M_u \text{ with } x \ne v. \tag{4.3}$$

Otherwise for each $x \in M_u$ the set

$$Tx := \begin{cases} \{y \in M_u : y \ne x,\ p(x,y) \le \varphi(x)[f(x) - f(y)]\} & \text{if } f(x) < +\infty; \\ M_u \setminus \{x\} & \text{if } f(x) = +\infty \end{cases} \tag{4.4}$$

would be nonempty and $x \notin Tx$. As a mapping from M_u to M_u, T satisfies the conditions in Theorem 3.4, so there exists $x_0 \in M_u$ such that $x_0 \in Tx_0$. This is a contradiction.

Now, for each $x \in X \setminus M_u$, since $f(x) > f(u) \ge f(v)$ and $p(v,x) \ge 0$, inequality (4.3) still holds. □

It is worth noting that T in the above proof is a multivalued mapping to which Theorem 3.4 is directly applicable, in contrast to [11, Theorem 3] and [16, Theorem 7].

From the proof of Theorem 3.5, we see that the function φ defined by

$$\varphi(x) := \sup\left\{g(s) : \inf_X f \le s \le f(x)\right\} \tag{4.5}$$

satisfies the condition in Theorem 4.1 when $g : [\inf_X f, \inf_X f + \epsilon] \to (0,+\infty)$ is a nondecreasing or u.s.c. function. So, based on Theorem 4.1 or Theorem 3.5, we obtain next result (from which [11, Theorem 4] follows by taking $g = 1$).

Theorem 4.2. *Let p be a τ-distance on X, $\epsilon \in (0,+\infty]$, $u \in X$ satisfy $f(u) \le \inf_X f + \epsilon$, and $g : [\inf_X f, \inf_X f + \epsilon] \to (0,+\infty)$ either nondecreasing or u.s.c.. Denote*

$$\varphi(x) := \sup\left\{g(s) : \inf_X f \le s \le f(x)\right\} \quad \text{for } x \in f^{-1}\left(-\infty, \inf_X f + \epsilon\right]. \tag{4.6}$$

Then there exists $v \in X$ *such that* $f(v) \le f(u)$ *and*

$$p(v,x) > g\big(f(v)\big)\big[f(v) - f(x)\big] \quad \forall x \in X \text{ with } x \neq v. \tag{4.7}$$

If also $p(u,u) = 0$ *and* p*, is l.s.c. in its second variable, then there exists* $v \in X$ *satisfying the above property and the following inequality:*

$$p(u,v) \le \varphi(u)\big[f(u) - f(v)\big]. \tag{4.8}$$

Proof. Similar to the proof of Theorem 4.1, the first part of the conclusion can be derived from Theorem 3.5.

Now, let $p(u,u) = 0$ and p l.s.c. in its second variable. Then the set

$$M(u) := \{x \in X : p(u,x) + \varphi(u)f(x) \le \varphi(u)f(u)\} \tag{4.9}$$

is nonempty and complete. Note that $c(t) := \sup\{g(s) : \inf_X f \le s \le t\}$ is nondecreasing and $\varphi(x) = c(f(x))$. Applying the conclusion of the first part to the function f on $M(u)$, we obtain $v \in M(u)$ such that

$$p(v,x) > \varphi(v)\big[f(v) - f(x)\big] \tag{4.10}$$

for all $x \in M(u)$ with $x \neq v$. For $x \in X \setminus M(u)$, we still have the inequality. Otherwise, there would exist $x \in X \setminus M(u)$ such that $f(x) \le f(v)$ and

$$p(v,x) \le \varphi(v)\big[f(v) - f(x)\big]. \tag{4.11}$$

This with $v \in M(u)$ and the triangle inequality yield

$$\begin{aligned} p(u,x) &\le \varphi(u)\big[f(u) - f(v)\big] + \varphi(v)\big[f(v) - f(x)\big] \\ &\le \varphi(u)\big[f(u) - f(x)\big], \end{aligned} \tag{4.12}$$

that is, $x \in M(u)$, which is a contradiction. □

Remark 4.3. (i) For the case where g is nondecreasing, the function $\varphi(x)$ in the proof of Theorem 4.2 reduces to $g(f(x))$. From the proof we can further see that the nonemptiness and the closedness of $M(u)$ imply the existence of v in $M(u)$ such that $M(v) \subseteq \{v\}$.

(ii) If we apply Theorem 4.1 directly, then the factor $g(f(v))$ on the right-hand side of the inequality

$$p(v,x) > g\big(f(v)\big)\big[f(v) - f(x)\big] \tag{4.13}$$

in Theorem 4.2 can be replaced with $\varphi(v)$.

(iii) When $x_0 \in X$, p is a w-distance on X, and h is a nondecreasing function such that

$$\int_0^{+\infty} \frac{dt}{1+h(t)} = +\infty, \tag{4.14}$$

applying Theorem 4.2 to the τ-distance

$$\int_{p(x_0,x)}^{p(x_0,x)+p(x,y)} \frac{dt}{1+h(t)} \quad \text{for } (x,y) \in X \times X \tag{4.15}$$

and $g(t) = \lambda/\epsilon$, we arrive at the following conclusion, from which (by taking $p = d$) we can obtain [17, Theorem 1.1], a generalization of EVP.

Corollary 4.4. *Let $x_0 \in X$, p a w-distance on X, $\epsilon > 0$ and $u \in X$ satisfy $p(u,u) = 0$ and $f(u) \le \inf_X f + \epsilon$. Let $h : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function such that*

$$\int_0^{+\infty} \frac{dt}{1+h(t)} = +\infty. \tag{4.16}$$

Then for each $\lambda > 0$, there exists $v \in X$ such that $f(v) \le f(u)$,

$$\begin{gathered} \int_{p(x_0,u)}^{p(x_0,u)+p(u,v)} \frac{dt}{1+h(t)} \le \lambda, \\ f(x) + \frac{\epsilon}{\lambda} \cdot \frac{p(v,x)}{1+h(p(x_0,v))} > f(v) \quad \forall x \in X \textit{ with } x \ne v. \end{gathered} \tag{4.17}$$

Note that there exist nondecreasing functions h satisfying

$$\int_0^{+\infty} \frac{dt}{1+h(t)} < +\infty. \tag{4.18}$$

For example, $h(t) = t^2$ and $h(t) = e^t$. Clearly, Corollary 4.4 is not applicable to these examples. For these cases, we present another extension of EVP by using Theorem 4.1 and Proposition 2.2.

Theorem 4.5. *Let p be a w-distance on X, $\epsilon \in (0,+\infty]$, $u \in X$ satisfy $f(u) \le \inf_X f + \epsilon$, and $\varphi : f^{-1}(-\infty, \inf_X f + \epsilon] \to (0,+\infty)$ satisfying*

$$\sup\left\{\varphi(x) : x \in f^{-1}\left(-\infty, \inf_X f + \min\{\epsilon,\eta\}\right]\right\} < +\infty, \tag{4.19}$$

for some $\eta > 0$*. If* $h : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function and for some* $x_0 \in X$ *and each* $r > 0$ *there holds*

$$\inf_{x\in X} \int_{p(x_0,x)}^{p(x_0,x)+r} \frac{dt}{1+h(t)} > 0, \tag{4.20}$$

then there exists $v \in X$ *such that* $f(v) \le f(u)$ *and*

$$\frac{p(v,x)}{1+h(p(x_0,v))} > \varphi(v)\left[f(v) - f(x)\right] \quad \forall x \in X \text{ with } x \neq v. \tag{4.21}$$

Proof. Proposition 2.2 shows that the function q defined by

$$q(x,y) := \int_{p(x_0,x)}^{p(x_0,x)+p(x,y)} \frac{dt}{1+h(t)} \quad \text{for } (x,y) \in X \times X \tag{4.22}$$

is a w-distance. Applying Theorem 4.1 to the w-distance, the desired conclusion follows. □

Remark 4.6. We have obtained Theorem 4.5 from Theorem 4.1. Conversely, when p is a w-distance, Theorem 4.1 follows from Theorem 4.5 by taking $h(t) = 0$ for all $t \in [0, +\infty)$. In this case they are equivalent results. If also $p(x,y) \le M$ holds for some $M > 0$ and all $(x,y) \in X \times X$, Theorem 4.5 is obviously applicable. In particular, when we take $x_0 = u$ for certain point $u \in X$, the condition in Theorem 4.5 about h can be deleted.

Theorem 4.7. *Let* p *be a* w*-distance on* X*,* $\epsilon \in (0, +\infty]$*,* $g : [\inf_X f, \inf_X f + \epsilon] \to (0, +\infty)$ *either nondecreasing or u.s.c., and* $h : [0, +\infty) \to [0, +\infty)$ *nondecreasing. Denote*

$$\varphi(x) := \sup\left\{g(s) : \inf_X f \le s \le f(x)\right\} \quad \text{for } x \in f^{-1}\left(-\infty, \inf_X f + \epsilon\right]. \tag{4.23}$$

Then for $u \in X$ *with* $p(u,u) = 0$ *and*

$$\varphi(u)\left[f(u) - \inf_X f\right] < \min\left\{\epsilon, \int_0^{+\infty} \frac{dt}{1+h(t)}\right\}, \tag{4.24}$$

there exists $v \in X$ *such that*

$$\begin{gathered}\int_0^{p(u,v)} \frac{dt}{1+h(t)} \le \varphi(u)\left[f(u) - f(v)\right], \\ \frac{p(v,x)}{1+h(p(u,v))} > \varphi(v)\left[f(v) - f(x)\right] \quad \forall x \in X \text{ with } x \neq v.\end{gathered} \tag{4.25}$$

Proof. Let $a \geq 0$ satisfy

$$\int_0^a \frac{dt}{1+h(t)} = \varphi(u)\left[f(u) - \inf_X f\right],$$
$$p_1(x,y) := \min\left\{p(x,y), \varphi(u)\left[f(u) - \inf_X f\right] + 1 + a\right\}. \tag{4.26}$$

It is easy to see that p_1 is a bounded w-distance on X and hence

$$q_1(x,y) := \int_{p_1(u,x)}^{p_1(u,x)+p_1(x,y)} \frac{dt}{1+h(t)} \tag{4.27}$$

is a w-distance. By Theorem 4.2, there exists $v \in X$ such that

$$\frac{p_1(v,x)}{1+h(p_1(u,v))} \geq q_1(v,x) > \varphi(v)\left[f(v) - f(x)\right], \tag{4.28}$$

for all $x \in X$ with $x \neq v$ and

$$\int_0^{p_1(u,v)} \frac{1}{1+h(t)}\, dt = q_1(u,v) \leq \varphi(u)\left[f(u) - f(v)\right] \leq \varphi(u)\left[f(u) - \inf_X f\right], \tag{4.29}$$

from which we obtain $p_1(u,v) \leq a$ and hence $p_1(u,v) = p(u,v)$. Thus the desired conclusion follows. □

Upon taking $g = 1$ and $h = 0$ in Theorem 4.7 and replacing p with ϵp, we obtain (ii) of [10, Theorem 3], which is also an extension to EVP.

5. Nonconvex Minimization Theorems

In this section we mainly apply the extensions of EVP obtained in Section 4 to establish minimization theorems which generalize [11, Theorem 5] (an extension to [10, Theorem 1] and [7, Theorem 1]). From these results we also derive Theorem 3.2. Consequently, seven theorems established in Sections 3–5 are shown to be equivalent.

Firstly, we use Theorem 4.1 to prove the following result.

Theorem 5.1. *Let p be a τ-distance on X, $\epsilon \in (0,+\infty]$, and $\varphi : f^{-1}(-\infty, \inf_X f + \epsilon] \to (0,+\infty)$ satisfy*

$$\sup\left\{\varphi(x) : x \in f^{-1}\left(-\infty, \inf_X f + \min\{\epsilon,\eta\}\right]\right\} < +\infty, \tag{5.1}$$

for some $\eta > 0$*. If for each* $x \in X$ *with* $\inf_X f < f(x) < \inf_X f + \epsilon$ *there exists* $y \in X$ *such that* $y \neq x$ *and*

$$p(x, y) \leq \varphi(x)\left[f(x) - f(y)\right], \tag{5.2}$$

then there exists $x_0 \in X$ *such that* $f(x_0) = \inf_X f$*.*

Proof. Denote

$$M_x := \{y \in X : f(y) \leq f(x)\}, \quad \text{for } x \in X. \tag{5.3}$$

Let $x \in X$ (with $\inf_X f < f(x) < \inf_X f + \epsilon$) be fixed. Since f is l.s.c., the set (M_x, d) is nonempty and complete. Thus, by Theorem 4.1, there exists $v \in M_x$ such that

$$p(v, y) > \varphi(v)\left[f(v) - f(y)\right] \quad \forall y \in M_x \text{ with } y \neq v. \tag{5.4}$$

The point v must satisfy $f(v) = \inf_X f$. Otherwise, we suppose that

$$\inf_X f < f(v) \leq f(x) < \inf_X f + \epsilon. \tag{5.5}$$

By the assumption, there exists a point $\overline{v} \in X$ with $\overline{v} \neq v$ such that

$$p(v, \overline{v}) \leq \varphi(v)\left[f(v) - f(\overline{v})\right], \tag{5.6}$$

which implies $\overline{v} \in M_x$ and hence contradicts the inequality

$$p(v, \overline{v}) > \varphi(v)\left[f(v) - f(\overline{v})\right]. \tag{5.7}$$

□

Similarly, we can use Theorem 4.2 to establish the following result.

Theorem 5.2. *Let p be a* τ*-distance on X,* $\epsilon \in (0, +\infty]$*, and* $g : [\inf_X f, \inf_X f + \epsilon] \to (0, +\infty)$ *either nondecreasing or u.s.c.. If for each* $x \in X$ *with* $\inf_X f < f(x) < \inf_X f + \epsilon$ *there exists* $y \in X$ *such that* $y \neq x$ *and*

$$p(x, y) \leq g(f(x))\left[f(x) - f(y)\right], \tag{5.8}$$

then there exists $x_0 \in X$ *such that* $f(x_0) = \inf_X f$*.*

Example 5.3. Consider the function $f(x) = \sqrt{x}$ for $x \in [0, +\infty)$. Obviously, f attains its minimum at $x = 0$. For this simple example, we can also apply Theorem 5.2 to conclude that there exists $x_0 \in [0, +\infty)$ such that $f(x_0) = \inf_{[0,+\infty)} f$ since for any $\epsilon \in (0, +\infty)$ and each

$x \in (0,\epsilon]$ we have $y \in (0,x)$ such that

$$d(x,y) = |x-y| < 2\sqrt{x}(\sqrt{x}-\sqrt{y}) = g(f(x))[f(x)-f(y)], \tag{5.9}$$

where $g(x) = 2x$ for $x \in (0,\epsilon]$ and $g(0) = 1$.

Remark 5.4. Up to now, beginning with Theorem 3.1, we have established the following results with the proof routes:

$$\begin{gathered}\text{Theorem 3.2} \Longrightarrow \text{Theorem 3.4} \Longrightarrow \text{Theorem 3.5;}\\ \text{Theorem 3.4} \Longrightarrow \text{Theorem 4.1} \Longrightarrow \text{Theorem 5.1;}\\ \text{Theorem 3.5} \Longrightarrow \text{Theorem 4.2} \Longrightarrow \text{Theorem 5.2.}\end{gathered} \tag{5.10}$$

As a conclusion in this paper, the following result states that these seven theorems are equivalent.

Theorem 5.5. *Theorems 3.2–3.5, 4.1-4.2, and 5.1-5.2 are all equivalent.*

Proof. By Remark 5.4, it suffices to show that Theorems 5.1-5.2 both imply Theorem 3.2.

Suppose that for some $\epsilon \in (0,+\infty]$ and for each $x \in X$ with $\inf_X f \le f(x) < \inf_X f + \epsilon$ there exists $\overline{x} \in Tx$ such that $\overline{x} \in M(x)$, that is,

$$p(x,\overline{x}) \le f(x) - f(\overline{x}). \tag{5.11}$$

If there exists $x_0 \in X$ with $f(x_0) < \inf_X f + \epsilon$ such that $M(x_0) = \{x_0\}$, then, since there exists $\overline{x}_0 \in Tx_0$ such that $\overline{x}_0 \in M(x_0)$, $\overline{x}_0 = x_0$, $p(x_0,x_0) = 0$. In this case, Theorem 3.2 follows.

Next we claim that there must exist $x_0 \in X$ such that

$$M(x_0) = \{x_0\}, \qquad f(x_0) < \inf_X f + \epsilon. \tag{5.12}$$

Otherwise, suppose that $M(x) \ne \{x\}$ for each $x \in X$ with $f(x) < \inf_X f + \epsilon$. By Theorem 5.1 or Theorem 5.2 there exists $x_1 \in X$ such that $f(x_1) = \inf_X f$. Since $p(x_1,x) = 0$ for $x \in M(x_1)$, according to the property that $p(x_1,x) = 0$ and $p(x_1,y) = 0$ imply $x = y$, $M(x_1)$ is a singleton. This implies that there exists x_0 such that $M(x_1) = \{x_0\}$ and $f(x_0) = \inf_X f = f(x_1)$, from which and the triangle inequality we obtain

$$\emptyset \ne M(x_0) \subseteq M(x_1) \subseteq \{x_0\}. \tag{5.13}$$

This gives $M(x_0) = \{x_0\}$ and hence a contradiction to the assumption. □

6. Generalized ϵ-Conditions of Takahashi and Hamel

The condition in Theorem 5.2 is sufficient for f to attain minimum on X. In this section we show that such a condition implies more when the τ-distance p (on $X \times X$) is l.s.c. in its second variable. For convenience we introduce the following notions.

Definition 6.1. A function $f : X \to (-\infty, +\infty]$ is said to satisfy *the generalized ϵ-condition of Takahashi* (*Hamel*) if for some $\epsilon \in (0, +\infty]$, some nondecreasing function $g : [\inf_X f, \inf_X f + \epsilon] \to (0, +\infty)$, and each $x \in X$ with $\inf_X f < f(x) < \inf_X f + \epsilon$ there exists $y \in X$ ($y \in Z$) such that $y \neq x$ and

$$p(x, y) \leq g(f(x))[f(x) - f(y)], \tag{6.1}$$

where $Z = \{z \in X : f(z) = \inf_X f\}$. In particular, for the case $\epsilon = +\infty$ the generalized ϵ-condition of Takahashi (Hamel) is called *the generalized condition of Takahashi* (*Hamel*).

When $g = 1$, the above concepts, respectively, reduce to ϵ-condition of Takahashi (Hamel) and the condition of Takahashi (Hamel) in [8].

It is clear that for any $0 < \epsilon_1 < \epsilon_2$ the generalized ϵ_2-condition of Takahashi implies the generalized ϵ_1-condition of Takahashi and the generalized ϵ_2-condition of Hamel implies the generalized ϵ_1-condition of Hamel. For any $\epsilon \in (0, +\infty]$ the generalized ϵ-condition of Takahashi and the generalized ϵ-condition of Hamel are, respectively, weaker than that of Takahashi and of Hamel. For example, when $X = [0, +\infty)$, the function $f(x) = \sqrt{x}$ satisfies the generalized ϵ-conditions of Takahashi and Hamel for any $\epsilon \in (0, +\infty)$ but it does not satisfy that of Takahashi nor of Hamel. Furthermore, the generalized ϵ-condition of Hamel always implies that of Takahashi. Next result asserts that the converse is also true in a complete metric space.

Theorem 6.2. *Let p be a τ-distance on X such that $p(x, \cdot)$ is l.s.c. on X for each $x \in X$. For $\epsilon \in (0, +\infty]$, f satisfies the generalized ϵ-condition of Takahashi if and only if f satisfies the generalized ϵ-condition of Hamel.*

Proof. The sufficiency is obvious, so it suffices to prove the necessity. Let f satisfy the generalized ϵ-condition of Takahashi and let g be the corresponding nondecreasing function in the definition. Denote

$$M(x) := \{y \in X : p(x, y) + g(f(x))f(y) \leq g(f(x))f(x)\}, \quad \text{for } x \in X. \tag{6.2}$$

Then for the case $0 < \epsilon < +\infty$, it suffices to prove that the set $M(x) \cap Z$ is nonempty for each $x \in X$ with $\inf_X f < f(x) < \inf_X f + \epsilon$, where

$$Z = \left\{ z \in X : f(z) = \inf_X f \right\}. \tag{6.3}$$

Let $x \in X$ with $\inf_X f < f(x) < \inf_X f + \epsilon$ be fixed. Since f and $p(x, \cdot)$ are both l.s.c., the set $M(x)$ is nonempty and complete. Thus, by Theorem 4.1 or Theorem 4.2, there exists $\overline{x} \in M(x)$ such that

$$p(\overline{x}, y) > g(f(\overline{x}))[f(\overline{x}) - f(y)] \quad \forall y \in M(x) \text{ with } y \neq \overline{x}. \tag{6.4}$$

The point $\overline{x}$ must be in Z. Otherwise, if $\overline{x}$ were not in Z, then

$$\inf_X f < f(\overline{x}) \leq f(x) < \inf_X f + \epsilon. \tag{6.5}$$

By the assumption, there exists a point $\overline{y} \in X$ with $\overline{y} \neq \overline{x}$ such that

$$p(\overline{x}, \overline{y}) \leq g(f(\overline{x}))[f(\overline{x}) - f(\overline{y})], \tag{6.6}$$

from which and the inequality $g(f(\overline{x})) \leq g(f(x))$ we obtain

$$p(x, \overline{y}) \leq p(x, \overline{x}) + p(\overline{x}, \overline{y}) \leq g(f(x))[f(x) - f(\overline{y})], \tag{6.7}$$

that is, $\overline{y} \in M(x)$. And hence $p(\overline{x}, \overline{y}) > g(f(\overline{x}))[f(\overline{x}) - f(\overline{y})]$. This is a contradiction. Therefore, $\overline{x} \in M(x) \cap Z$.

Next, we suppose that f satisfies the generalized condition of Takahashi. For each $0 < \epsilon < +\infty$, the function f satisfies the generalized ϵ-condition of Takahashi, so f satisfies the generalized ϵ-condition of Hamel. This implies that Z is nonempty. For each $x \in X$ with $\inf_X f < f(x)$, if $f(x) < +\infty$, then $\inf_X f < f(x) < \inf_X f + \epsilon$ for some $0 < \epsilon < +\infty$. In this case we can find $z \in Z$ such that

$$p(x, z) \leq g(f(x))[f(x) - f(z)]. \tag{6.8}$$

If $f(x) = +\infty$, then this inequality holds for each $z \in Z$. Therefore f satisfies the generalized condition of Hamel. □

7. Generalized Weak Sharp Minima and Error Bounds

As stated in [8], the ϵ-condition of Takahashi is one of sufficient conditions for an inequality system to have weak sharp minima and error bounds. With Theorem 6.2 being established, the generalized ϵ-condition of Takahashi plays a similar role for the generalized weak sharp minima and error bounds introduced below.

For a proper l.s.c. and bounded below function $f : X \to (-\infty, +\infty]$, we say that f has *generalized local (global) weak sharp minima* if the set Z of minimizers of f on X is nonempty and if for some $\epsilon \in (0, +\infty)$ $(\epsilon = +\infty)$ and some nondecreasing function $g : [\inf_X f, \inf_X f + \epsilon] \to (0, +\infty)$ and each $x \in X$ with $\inf_X f < f(x) < \inf_X f + \epsilon$ there holds

$$p_Z(x) \leq g(f(x))\left[f(x) - \inf_X f\right], \tag{7.1}$$

where $p_Z(x) = \inf\{p(x, z) : z \in Z\}$.

Due to the equivalence stated in Theorem 6.2, the generalized ϵ-condition of Takahashi is sufficient for f to have generalized local (global) weak sharp minima.

Theorem 7.1. *Let p be a τ-distance on X such that $p(x, \cdot)$ is l.s.c. on X for each $x \in X$. If, for some $\epsilon \in (0, +\infty]$, f satisfies the generalized ϵ-condition of Takahashi, then the set Z of minimizers of f on X is nonempty and for every $x \in X$ with* $\inf_X f < f(x) < \inf_X f + \epsilon$ *and each $z \in Z$ there holds*

$$p_Z(x) \leq g(f(x))[f(x) - f(z)]. \tag{7.2}$$

Proof. The proof is immediate from Theorem 6.2. □

For an l.s.c. function $f : X \to (-\infty, +\infty]$, denote

$$S := \{x \in X : f(x) \leq 0\}, \qquad p_S(x) := \inf\{p(x, s) : s \in S\}. \tag{7.3}$$

We say that f (or S) has a *generalized local error bound* if there exist $\epsilon \in (0, +\infty)$ and a nondecreasing function $g : [0, \epsilon) \to (0, +\infty)$ such that

$$p_S(x) \leq g\big(f(x)_+\big) f(x)_+ \quad \forall x \in X \text{ with } f(x) < \epsilon, \tag{7.4}$$

where $f(x)_+ = \max\{0, f(x)\}$. The function f is said to have a *generalized global error bound* if the above statement is true for $\epsilon = +\infty$.

When $p = d$ and $g = 1$, the study of generalized error bounds has received growing attention in the mathematical programming (see [18] and the references therein). Now, using Theorem 7.1, we present the following sufficient condition for an l.s.c. inequality system to have generalized error bounds.

Theorem 7.2. *Let p be a τ-distance on X such that $p(x, \cdot)$ is l.s.c. on X for each $x \in X$ and $f : X \to (-\infty, +\infty]$ be a proper l.s.c. function. Let $\epsilon_1 \in (0, +\infty]$ and $g : [0, \epsilon_1) \to (0, +\infty)$ be a nondecreasing function. Suppose for each $\epsilon \in (0, \epsilon_1]$, the set $f^{-1}(-\infty, \epsilon)$ is nonempty and for each $x \in f^{-1}(0, \epsilon)$ there exists a point $y \in f^{-1}[0, \epsilon)$ such that $y \neq x$ and*

$$p(x, y) \leq g\big(f(x)\big)\big[f(x) - f(y)\big]. \tag{7.5}$$

Then $S := \{x \in X : f(x) \leq 0\}$ is nonempty and

$$p_S(x) \leq g\big(f(x)_+\big) f(x)_+ \quad \forall x \in f^{-1}(-\infty, \epsilon_1). \tag{7.6}$$

Proof. Let $\epsilon_1 \in (0, +\infty]$ be given. Since $f(\cdot)_+$ is l.s.c. and bounded below with $S = \{x \in X : f(x)_+ = 0\}$ and $\inf_X f_+ \geq 0$, by Theorem 7.1, it suffices to prove

$$S = Z := \left\{z \in X : f(z)_+ = \inf_X f_+\right\}, \tag{7.7}$$

that is, $\inf_X f_+ = 0$. This must be true. Otherwise, if $\inf_X f_+ > 0$, then for $0 < \epsilon < \min\{\epsilon_1, \inf_X f_+\}$ the set $f^{-1}(-\infty, \epsilon)$ would be empty. This contradicts the assumption. □

Remark 7.3. Note that the nonemptiness of S in Theorem 7.2 is not a part of assumption but a part of conclusion. In addition, the condition in Theorem 7.2 implies that f_+ satisfies the generalized ϵ-condition of Takahashi, that is,

$$M_g(x) := \{y \in X : p(x, y) \leq g\big(f(x)\big)\big[f(x) - f(y)\big]\} \not\subseteq \{x\}, \tag{7.8}$$

for each $x \in X$ with $\inf_X f_+ < f(x) < \inf_X f_+ + \epsilon$. However, once $M_g(x)$ is nonempty, there exists $x_0 \in M_g(x)$ such that $M_g(x_0) \subseteq \{x_0\}$ as stated below.

Theorem 7.4. *Let p be a τ-distance such that $p(x,\cdot)$ is l.s.c. on X for each $x \in X$ and $g : [0,+\infty) \to (0,+\infty)$ be a nondecreasing function. Denote*

$$M_g(x) := \{y \in X : p(x,y) + g(f(x))f(y) \le g(f(x))f(x)\} \quad \forall x \in X. \tag{7.9}$$

Then for each $u \in X$ with $M_g(u) \neq \emptyset$, there exists $x_0 \in M_g(u)$ such that $M_g(x_0) \subseteq \{x_0\}$. In particular, there exists $y_0 \in X$ such that $M_g(y_0) \subseteq \{y_0\}$.

Proof. Since both p and f are l.s.c., for $u \in X$ with $M_g(u) \neq \emptyset$, $(M_g(u), d)$ is nonempty complete metric space. Suppose that for each $x \in M_g(u)$ there held $M_g(x) \not\subseteq \{x\}$. Then for each $x \in M_g(u)$ there exists $\overline{x} \in M_g(x)$ such that $\overline{x} \neq x$. Define

$$F(x) := f(x) - \inf_{M_g(u)} f \quad \text{for } x \in M_g(u) \tag{7.10}$$

and denote $S := \{x \in M_g(u) : F(x) = 0\}$. Then

$$S = \left\{ x \in M_g(u) : f(x) = \inf_{M_g(u)} f \right\}. \tag{7.11}$$

By Theorem 7.2, the set S is nonempty.

Now for $x \in S$, since $f(x) < +\infty$ (no matter whether $f(u) < +\infty$ or $f(u) = +\infty$), there exists $\overline{x} \in M_g(x)$ such that $\overline{x} \neq x$ and

$$0 \le p(x,\overline{x}) \le g(f(x))\left[f(x) - f(\overline{x})\right] \le 0 \tag{7.12}$$

from which we obtain $p(x,\overline{x}) = 0$ and $f(\overline{x}) = f(x)$. Similarly, we have $\overline{\overline{x}} \in M_g(\overline{x})$ such that $\overline{\overline{x}} \neq \overline{x}$ and $p(\overline{x},\overline{\overline{x}}) = 0$. This, with $p(x,\overline{x}) = 0$, implies $p(x,\overline{\overline{x}}) = 0$. Thus $\overline{x} = \overline{\overline{x}}$, which is a contradiction. □

Remark 7.5. When $g = 1$ and p is a τ-distance such that $p(x,\cdot)$ is l.s.c. on X for each $x \in X$, we can obtain Theorem 3.1 by applying Theorem 7.4 to the function $f - \inf_X f$. As more applications, the following two propositions are immediate from Theorem 7.4 by taking $g = 1$, $f(\cdot) = p(b,\cdot)/\gamma$, and $f(\cdot) = p(\cdot,b)/\gamma$, respectively, on (X,d).

Proposition 7.6. *Let X be a complete nonempty subset of a metric space (E,d), $a \in X$, $b \in E \setminus X$, and let p be a τ-distance on E such that $p(x,\cdot)$ is l.s.c. on X for each $x \in X$. Denote*

$$P_\gamma(a,b) := \{x \in E : \gamma p(a,x) + p(b,x) \le p(b,a)\}, \quad \textit{for } \gamma \in (0,+\infty). \tag{7.13}$$

Suppose that $X \cap P_\gamma(a,b)$ is nonempty for some $\gamma \in (0,+\infty)$. If $p(x,x) = 0$ for all $x \in X \cap P_\gamma(a,b)$, then there exists $x_0 \in X \cap P_\gamma(a,b)$ such that

$$X \cap P_\gamma(x_0,b) = \{x_0\}. \tag{7.14}$$

Proposition 7.7. *Let X be a complete nonempty subset of a metric space (E, d), $a \in X$, $b \in E \setminus X$, and let p be a τ-distance on E. Denote*

$$Q_\gamma(a,b) := \{x \in E : \gamma p(a,x) + p(x,b) \le p(a,b)\}, \quad \text{for } \gamma \in (0,+\infty). \tag{7.15}$$

Suppose that p is l.s.c. in its both variables and $X \cap Q_\gamma(a,b)$ is nonempty for some $\gamma \in (0,+\infty)$. If $p(x,x) = 0$ for all $x \in X \cap Q_\gamma(a,b)$, then there exists $x_0 \in X \cap Q_\gamma(a,b)$ such that $X \cap Q_\gamma(x_0,b) = \{x_0\}$. In particular, if $p(a,a) = 0$ and $p(x,x) = 0$ for all $x \in X \cap Q_1(a,b)$, then there exists $x_0 \in X$ such that $p(a,b) = p(a,x_0) + p(x_0,b)$ and

$$\{x \in X : p(x_0,b) = p(x_0,x) + p(x,b)\} = \{x_0\}. \tag{7.16}$$

Remark 7.8. Upon taking $p(x,y) = d(x,y)$ in Propositions 7.6 and 7.7, we obtain [3, Theorem F] which is equivalent to EVP in a complete metric space. In this case EVP implies Theorem 3.1.

Finally, following the statement in Theorem 5.5, on the condition that the τ-distance $p(x,\cdot)$ is l.s.c. on X for each $x \in X$, Theorems 3.1–3.5, 4.1-4.2, 5.1-5.2, 6.2, and 7.1–7.4 turn out to be equivalent since we have further shown that

$$\begin{aligned} &\text{Theorem 4.2} \Longrightarrow \text{Theorem 6.2} \Longrightarrow \text{Theorem 7.1} \\ &\Longrightarrow \text{Theorem 7.2} \Longrightarrow \text{Theorem 7.4} \Longrightarrow \text{Theorem 3.1} \end{aligned} \tag{7.17}$$

in Sections 6 and 7. In particular, each theorem stated above is equivalent to Theorem 4.5 (as stated in Remark 4.6) when p is a w-distance on X, to [3, Theorem F] and EVP when $p = d$ (see Remark 7.8), and to the Bishop-Phelps Theorem in a Banach space when p is the corresponding norm. Therefore, we can conclude our paper as below.

Theorem 7.9. *Let (X, d) be a complete metric space and p a τ-distance on X such that $p(x,\cdot)$ is l.s.c. for each $x \in X$. Then*

(i) *Theorems 3.1–3.5, 4.1-4.2, 5.1-5.2, 6.2, and 7.1-7.4 are all equivalent;*

(ii) *when p is a w-distance on X, each theorem in (i) is equivalent to Theorem 4.5;*

(iii) *when $p = d$, each theorem in (i) is equivalent to EVP.*

References

[1] J. Caristi, "Fixed point theorems for mappings satisfying inwardness conditions," *Transactions of the American Mathematical Society*, vol. 215, pp. 241–251, 1976.
[2] J.-P. Aubin and J. Siegel, "Fixed points and stationary points of dissipative multivalued maps," *Proceedings of the American Mathematical Society*, vol. 78, no. 3, pp. 391–398, 1980.
[3] J.-P. Penot, "The drop theorem, the petal theorem and Ekeland's variational principle," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 10, no. 9, pp. 813–822, 1986.
[4] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 47, pp. 324–353, 1974.
[5] I. Ekeland, "Nonconvex minimization problems," *Bulletin of the American Mathematical Society*, vol. 1, no. 3, pp. 443–474, 1979.

[6] J. Danes, "A geometric theorem useful in nonlinear functional analysis," *Bollettino della Unione Matematica Italiana*, vol. 6, pp. 369–375, 1972.

[7] W. Takahashi, "Existence theorems generalizing fixed point theorems for multivalued mappings," in *Fixed Point Theory and Applications*, M. A. Théra and J. B. Baillon, Eds., vol. 252 of *Pitman Research Notes in Mathematics Series*, pp. 397–406, Longman Scientific & Technical, Harlow, UK, 1991.

[8] Z. Wu, "Equivalent formulations of Ekeland's variational principle," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 5, pp. 609–615, 2003.

[9] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis*, Springer, New York, NY, USA, 2005.

[10] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.

[11] T. Suzuki, "Generalized distance and existence theorems in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.

[12] L.-J. Lin and W.-S. Du, "Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 360–370, 2006.

[13] S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, "Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 1, pp. 126–139, 2008.

[14] J. S. Bae, "Fixed point theorems for weakly contractive multivalued maps," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 690–697, 2003.

[15] J. S. Bae, E. W. Cho, and S. H. Yeom, "A generalization of the Caristi-Kirk fixed point theorem and its applications to mapping theorems," *Journal of the Korean Mathematical Society*, vol. 31, no. 1, pp. 29–48, 1994.

[16] T. Suzuki, "Generalized Caristi's fixed point theorems by Bae and others," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 502–508, 2005.

[17] C.-K. Zhong, "On Ekeland's variational principle and a minimax theorem," *Journal of Mathematical Analysis and Applications*, vol. 205, no. 1, pp. 239–250, 1997.

[18] K. F. Ng and X. Y. Zheng, "Error bounds for lower semicontinuous functions in normed spaces," *SIAM Journal on Optimization*, vol. 12, no. 1, pp. 1–17, 2001.