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Research Article

Classification of Normal Sequences

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Base sequences $\mathrm{BS}(m,n)$ are quadruples $(A;B;C;D)$ of $\{\pm1\}$-sequences, with A and B of length m and C and D of length n, such that the sum of their nonperiodic autocorrelation functions is a δ-function. Normal sequences $\mathrm{NS}(n)$ are base sequences $(A;B;C;D)\in\mathrm{BS}(n,n)$ such that $A=B$. We introduce a definition of equivalence for normal sequences $\mathrm{NS}(n)$ and construct a canonical form. By using this canonical form, we have enumerated the equivalence classes of $\mathrm{NS}(n)$ for $n\le 40$.

1. Introduction

By a *binary* respectively *ternary sequence* we mean a sequence $A=a_1,a_2,\ldots,a_m$ whose terms belong to $\{\pm1\}$ respectively $\{0,\pm1\}$. To such a sequence, we associate the polynomial $A(z)=a_1+a_2z+\cdots+a_mz^{m-1}$. We refer to the Laurent polynomial $N(A)=A(z)A(z^{-1})$ as the *norm* of A. *Base sequences* $(A;B;C;D)$ are quadruples of binary sequences, with A and B of length m and C and D of length n, and such that

$$N(A)+N(B)+N(C)+N(D)=2(m+n). \tag{1.1}$$

The set of such sequences will be denoted by $\mathrm{BS}(m,n)$.

In this paper, we consider only the case where $m=n$ or $m=n+1$. The base sequences $(A;B;C;D)\in\mathrm{BS}(n,n)$ are *normal* if $A=B$. We denote by $\mathrm{NS}(n)$ the set of normal sequences of length n, that is, those contained in $\mathrm{BS}(n,n)$. It is well known [1] that for normal sequences $2n$ must be a sum of three squares. In particular, $\mathrm{NS}(14)$ and $\mathrm{NS}(30)$ are empty. Exhaustive computer searches have shown that $\mathrm{NS}(n)$ are empty also for $n=6,17,21,22,23,24$ (see [2]) and $n=27,28,31,33,34,\ldots,39$ (see [3–6]).

Table 1: Number of equivalence classes of NS(n).

n	Equ	Gol	Spo	n	Equ	Gol	Spo
1	1	1		21			
2	1	1		22			
3	1		1	23			
4	1	1		24			
5	1		1	25	4		4
6				26	2	2	
7	4		4	27			
8	7	6	1	28			
9	3		3	29	2		2
10	5	4	1	30			
11	2		2	31			
12	4		4	32	516	480	36
13	3		3	33			
14				34			
15	2		2	35			
16	52	48	4	36			
17				37			
18	1		1	38			
19	1		1	39			
20	36	34	2	40	304	304	

The base sequences $(A;B;C;D) \in \mathrm{BS}(n+1,n)$ are *near-normal* if $b_i = (-1)^{i-1}a_i$ for all $i \leq n$. For near-normal sequences n must be even or 1. We denote by $\mathrm{NN}(n)$ the set of near-normal sequences in $\mathrm{BS}(n+1,n)$.

Normal sequences were introduced by Yang in [1] as a generalization of Golay sequences. Let us recall that *Golay sequences* $(A;B)$ are pairs of binary sequences of the same length, n, and such that $N(A)+N(B) = 2n$. We denote by $\mathrm{GS}(n)$ the set of Golay sequences of length n. It is known that they exist when $n = 2^a 10^b 26^c$ where a,b,c are arbitrary nonnegative integers. There exist two embeddings $\mathrm{GS}(n) \rightarrow \mathrm{NS}(n)$: the first defined by $(A;B) \rightarrow (A;A;B;B)$ and the second by $(A;B) \rightarrow (B;B;A;A)$. We say that these normal sequences (and those equivalent to them) are of *Golay type*. For the definition of equivalence of normal sequences see Section 3. However, as observed by Yang, there exist normal sequences which are not of Golay type. We refer to them as *sporadic* normal sequences. From the computational results reported in this paper (see Table 1) it appears that there may be only finitely many sporadic normal sequences. For example, all 304 equivalence classes in NS(40) are of Golay type. The smallest length for which the existence question of normal sequences is still unresolved is $n = 41$.

Base sequences, and their special cases such as normal and near-normal sequences, play an important role in the construction of Hadamard matrices [7, 8]. For instance, the discovery of a Hadamard matrix of order 428 (see [9]) used a $\mathrm{BS}(71,36)$, constructed specially for that purpose.

Examples of normal sequences $\mathrm{NS}(n)$ have been constructed in [1, 2, 5, 7, 10]. For various applications, it is of interest to classify the normal sequences of small length. Our main goal is to provide such classification for $n \leq 40$. The classification of near-normal sequences $\mathrm{NN}(n)$ for $n \leq 40$ and base sequences $\mathrm{BS}(n+1, n)$ for $n \leq 30$ has been carried out in our papers [5, 6, 11] and [10, 12], respectively.

We give examples of normal sequences of lengths $n = 1, \ldots, 5$:

$$
\begin{array}{llll}
A = +; & A = +,+; & A = +,+,-; & A = +,+,-,+; \\
A = +; & A = +,+; & A = +,+,-; & A = +,+,-,+; \\
C = +; & C = +,-; & C = +,+,+; & C = +,+,+,-; \\
D = +; & D = +,-; & D = +,-,+; & D = +,+,+,-; \\
\end{array}
$$
$$
\begin{array}{l}
A = +,+,+,-,+; \\
A = +,+,+,-,+; \\
C = +,+,+,-,-; \\
D = +,-,+,+,-.
\end{array}
\tag{1.2}
$$

When displaying a binary sequence, we often write + for +1 and − for −1. We have written the sequence A twice to make the quads visible (see Section 2).

If $(A; A; C; D) \in \mathrm{NS}(n)$ then $(A, +; A, -; C; D) \in \mathrm{BS}(n+1, n)$. This has been used in our previous papers to view normal sequences $\mathrm{NS}(n)$ as a subset of $\mathrm{BS}(n+1, n)$. For classification purposes it is more convenient to use the definition of $\mathrm{NS}(n)$ as a subset of $\mathrm{BS}(n, n)$, which is closer to Yang's original definition [1].

In Section 2, we recall the basic properties of base sequences $\mathrm{BS}(m, n)$. The quad decomposition and our encoding scheme for $\mathrm{BS}(n+1, n)$ used in our previous papers also work for $\mathrm{NS}(n)$, but not for arbitrary base sequences in $\mathrm{BS}(n, n)$. The quad decomposition of normal sequences $\mathrm{NS}(n)$ is somewhat simpler than that of base sequences $\mathrm{BS}(n+1, n)$. We warn the reader that the encodings for the first two sequences of $(A; A; C; D) \in \mathrm{NS}(n)$ and $(A, +; A, -; C; D) \in \mathrm{BS}(n+1, n)$ are quite different.

In Section 3, we introduce the elementary transformations of $\mathrm{NS}(n)$. We point out that the elementary transformation (E4) is quite nonintuitive. It originated in our paper [5] where we classified near-normal sequences of small length. Subsequently, it has been extended and used to classify (see [10, 12]) the base sequences $\mathrm{BS}(n+1, n)$ for $n \leq 30$. We use these elementary transformations to define an equivalence relation and equivalence classes in $\mathrm{NS}(n)$. We also introduce the canonical form for normal sequences, and, by using it, we were able to compute the representatives of the equivalence classes for $n \leq 40$.

In Section 4, we introduce an abstract group, G_{NS}, of order 512 which acts naturally on all sets $\mathrm{NS}(n)$. Its definition depends on the parity of n. The orbits of this group are just the equivalence classes of $\mathrm{NS}(n)$.

In Section 5, we tabulate the results of our computations giving the list of representatives of the equivalence classes of $\mathrm{NS}(n)$ for $n \leq 40$. The representatives are written in the encoded form which is explained in the next section.

The summary is given in Table 1. The column "Equ" gives the number of equivalence classes in $\mathrm{NS}(n)$. Note that most of the known normal sequences are of Golay type. The column "Gol" respectively "Spo" gives the number of equivalence classes which are of Golay type respectively sporadic. (Blank entries are zeros.)

2. Quad Decomposition and the Encoding Scheme

Let $A = a_1, a_2, \ldots, a_n$ be an integer sequence of length n. To this sequence, we associate the polynomial

$$A(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, \tag{2.1}$$

viewed as an element of the Laurent polynomial ring $\mathbf{Z}[x, x^{-1}]$ (as usual, $\mathbf{Z}$ denotes the ring of integers). The *nonperiodic autocorrelation function* N_A of A is defined by

$$N_A(i) = \sum_{j \in \mathbf{Z}} a_j a_{i+j}, \quad i \in \mathbf{Z}, \tag{2.2}$$

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbf{Z}$ and $N_A(i) = 0$ for $i \geq n$. The *norm* of A is the Laurent polynomial $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = \sum_{i \in \mathbf{Z}} N_A(i) x^i. \tag{2.3}$$

Hence, if $(A; B; C; D) \in \mathrm{BS}(m, n)$ then

$$N_A(i) + N_B(i) + N_C(i) + N_D(i) = 0, \quad i \neq 0. \tag{2.4}$$

The negation, $-A$, of A is the sequence

$$-A = -a_1, -a_2, \ldots, -a_n. \tag{2.5}$$

The *reversed* sequence A' and the *alternated* sequence A^* of the sequence A are defined by

$$\begin{aligned} A' &= a_n, a_{n-1}, \ldots, a_1, \\ A^* &= a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1} a_n. \end{aligned} \tag{2.6}$$

Observe that $N(-A) = N(A') = N(A)$ and $N_{A^*}(i) = (-1)^i N_A(i)$ for all $i \in \mathbf{Z}$. By A, B we denote the concatenation of the sequences A and B.

Let $(A; A; C; D) \in \mathrm{NS}(n)$. For convenience, we set $n = 2m$ ($n = 2m + 1$) for n even (odd). We decompose the pair $(C; D)$ into quads

$$\begin{bmatrix} c_i & c_{n+1-i} \\ d_i & d_{n+1-i} \end{bmatrix}, \quad i = 1, 2, \ldots, m, \tag{2.7}$$

and, if n is odd, the central column $\left[\begin{smallmatrix} c_{m+1} \\ d_{m+1} \end{smallmatrix}\right]$. Similar decomposition is valid for the pair $(A; A)$.

The possibilities for the quads of base sequences $\mathrm{BS}(n+1,n)$ are described in detail in [10]. In the case of normal sequences we have 8 possibilities for the quads of $(C;D)$:

$$1=\begin{bmatrix}+&+\\+&+\end{bmatrix},\quad 2=\begin{bmatrix}+&+\\-&-\end{bmatrix},\quad 3=\begin{bmatrix}-&+\\-&+\end{bmatrix},\quad 4=\begin{bmatrix}+&-\\-&+\end{bmatrix},$$
$$5=\begin{bmatrix}-&+\\+&-\end{bmatrix},\quad 6=\begin{bmatrix}+&-\\+&-\end{bmatrix},\quad 7=\begin{bmatrix}-&-\\+&+\end{bmatrix},\quad 8=\begin{bmatrix}-&-\\-&-\end{bmatrix}, \tag{2.8}$$

but only 4 possibilities, namely, 1, 3, 6, and 8, for the quads of $(A;A)$. In [10], we referred to these eight quads as BS-quads. The additional eight Golay quads were also needed for the classification of base sequences $\mathrm{BS}(n+1,n)$. Unless stated otherwise, the word "quad" will refer to BS-quads.

We say that a quad is *symmetric* if its two columns are the same, and otherwise we say that it is *skew*. The quads $1,2,7,8$ are symmetric and $3,4,5,6$ are skew. We say that two quads have the *same symmetry type* if they are both symmetric or both skew.

There are 4 possibilities for the central column:

$$0=\begin{bmatrix}+\\+\end{bmatrix},\quad 1=\begin{bmatrix}+\\-\end{bmatrix},\quad 2=\begin{bmatrix}-\\+\end{bmatrix},\quad 3=\begin{bmatrix}-\\-\end{bmatrix}. \tag{2.9}$$

We encode the pair $(A;A)$ by the symbol sequence

$$p_1p_2\cdots p_m,\ \text{respectively,}\ p_1p_2\cdots p_mp_{m+1}, \tag{2.10}$$

when n is even respectively odd. Here, p_i is the label of the ith quad for $i\le m$ and p_{m+1} is the label of the central column (when n is odd). Similarly, we encode the pair $(C;D)$ by the symbol sequence

$$q_1q_2\cdots q_m,\ \text{respectively,}\ q_1q_2\cdots q_mq_{m+1}. \tag{2.11}$$

For example, the five normal sequences displayed in the introduction are encoded as $(0;0),(1;6),(60;11),(16;61)$, and $(160;640)$, respectively.

3. The Equivalence Relation

We start by defining five types of *elementary transformations* of normal sequences $(A;A;C;D)\in\mathrm{NS}(n)$

(E1) Negate both sequences $A;A$ or one of $C;D$.

(E2) Reverse both sequences $A;A$ or one of $C;D$.

(E3) Interchange the sequences $C;D$.

(E4) Replace the pair $(C;D)$ with the pair $(\widetilde{C};\widetilde{D})$ which is defined as follows: if (2.11) is the encoding of $(C;D)$, then the encoding of $(\widetilde{C};\widetilde{D})$ is $\tau(q_1)\tau(q_2)\cdots\tau(q_m)$ or

$\tau(q_1)\tau(q_2)\cdots\tau(q_m)q_{m+1}$ depending on whether n is even or odd, where τ is the transposition (45). In other words, the encoding of $(\widetilde{C};\widetilde{D})$ is obtained from that of $(C;D)$ by replacing simultaneously each quad symbol 4 with the symbol 5, and vice versa. For the proof of the equality $N_{\widetilde{C}} + N_{\widetilde{D}} = N_C + N_D$ see [10].

(E5) Alternate all four sequences $A;A;C;D$.

We say that two members of $\mathrm{NS}(n)$ are *equivalent* if one can be transformed to the other by applying a finite sequence of elementary transformations. One can enumerate the equivalence classes by finding suitable representatives of the classes. For that purpose we introduce the canonical form.

Definition 3.1. Let $S = (A;A;C;D) \in \mathrm{NS}(n)$ and let (2.10) respectively (2.11) be the encoding of the pair $(A;A)$ respectively $(C;D)$. We say that S is in the *canonical form* if the following twelve conditions hold.

(i) For n even $p_1 = 1$, and for $n > 1$ odd $p_1 \in \{1,6\}$.

(ii) The first symmetric quad (if any) of $(A;A)$ is 1.

(iii) The first skew quad (if any) of $(A;A)$ is 6.

(iv) If n is odd and all quads of $(A;A)$ are skew, then $p_{m+1} = 0$.

(v) If n is odd and $i < m$ is the smallest index such that the consecutive quads p_i and p_{i+1} have the same symmetry type, then $p_{i+1} \in \{1,6\}$. If there is no such index and p_m is symmetric, then $p_{m+1} = 0$.

(vi) $q_1 \in \{1,6\}$ if $n > 1$.

(vii) The first symmetric quad (if any) of $(C;D)$ is 1.

(viii) The first skew quad (if any) of $(C;D)$ is 6.

(ix) If i is the least index such that $q_i \in \{2,7\}$ then $q_i = 2$.

(x) If i is the least index such that $q_i \in \{4,5\}$ then $q_i = 4$.

(xi) If n is odd and $q_i \neq 2$, for all $i \leq m$, then $q_{m+1} \neq 2$.

(xii) If n is odd and $q_i \neq 1$, for all $i \leq m$, then $q_{m+1} = 0$.

We can now prove that each equivalence class has a member which is in the canonical form. The uniqueness of this member will be proved in the next section.

Proposition 3.2. *Each equivalence class $\mathcal{E} \subseteq NS(n)$ has at least one member having the canonical form.*

Proof. Let $S = (A;A;C;D) \in \mathcal{E}$ be arbitrary and let (2.10) respectively (2.11) be the encoding of $(A;A)$ respectively $(C;D)$. By applying the elementary transformations (E1), we can assume that $a_1 = c_1 = d_1 = +1$. If $n = 1$, S is in the canonical form. So, let $n > 1$ from now on. Note that now the first quads, p_1 and q_1, necessarily belong to $\{1,6\}$ and that $p_1 \neq q_1$ by (2.4). In the case when n is even and $p_1 = 6$ we apply the elementary transformation (E5). Note that (E5) preserves the quads p_1 and q_1. Thus the conditions (i) and (vi) for the canonical form are satisfied.

The conditions (ii), (iii), and (iv) are pairwise disjoint, so at most one of them may be violated. To satisfy (ii), it suffices (if necessary) to apply to the pair $(A;A)$ the transformation (E2). To satisfy (iii) or (iv), it suffices (if necessary) to apply to the pair $(A; A)$ the transformations (E1) and (E2).

For (v), assume that p_i and p_{i+1} have the same symmetry type and that i is the smallest such index. Also assume that $p_{i+1} \notin \{1, 6\}$, that is, $p_{i+1} \in \{3, 8\}$.

We first consider the case where $p_1 = 1$ and p_i and p_{i+1} are symmetric. By our assumption, we have $p_{i+1} = 8$, and, by the minimality of i, i must be odd. We first apply (E2) to the pair $(A; A)$ and then apply (E5). The quads p_j for $j \leq i$ remain unchanged. On the other hand, (E2) fixes p_{i+1} because it is symmetric, while, (E5) replaces $p_{i+1} = 8$ with 1 because $i + 1$ is even. We have to make sure that previously established conditions are not spoiled. Only condition (iii) may be affected. If so, we must have $i = 1$ and we simply apply (E2) again.

Next, we consider the case where again $p_1 = 1$ while p_i and p_{i+1} are now skew. Thus $p_{i+1} = 3$ and i is even. We again apply (E2) to the pair $(A; A)$ and then apply (E5). The quads p_j for $j \leq i$ again remain unchanged. On the other hand (E2) replaces $p_{i+1} = 3$ with 6 while (E5) fixes it because $i + 1$ is odd. Note that in this case none of the conditions (i–iv) and (vi) will be spoiled.

The remaining two cases (where $p_1 = 6$) can be treated in a similar fashion. Now assume that any two consecutive quads p_i, p_{i+1} have different symmetry types and that the last quad, p_m, is symmetric. Assume also that $p_{m+1} \neq 0$, that is, $p_{m+1} = 3$. If $p_1 = 1$ then m is odd and we just apply (E5). Otherwise $p_1 = 6$ and m is even and we apply the elementary transformations (E1) and (E2) to the pair $(A; A)$ and then apply (E5). After this change, the conditions (i–vi) will be satisfied.

To satisfy (vii), in view of (vi) we may assume that $q_1 = 6$. If the first symmetric quad in $(C; D)$ is 2 respectively 7, we reverse and negate C respectively D. If it is 8, we reverse and negate both C and D. Now, the first symmetric quad will be 1.

To satisfy (viii), (if necessary) reverse C or D, or both. To satisfy (ix), (if necessary) interchange C and D. To satisfy (x), (if necessary) apply the elementary transformation (E4). Note that in this process we do not violate the previously established properties.

To satisfy (xi), (if necessary) switch C and D and apply (E4) to preserve (x). To satisfy (xii), (if necessary) replace C with $-C'$ or D with $-D'$, or both.

Hence, S is now in the canonical form. □

We end this section by a remark on Golay-type normal sequences. Let $(A; B) \in \mathrm{GS}(n)$, with $n = 2m > 2$. While the Golay sequences $(A; B)$ and $(B; A)$ are always considered as equivalent (see [13]) the normal sequences $(A; A; B; B)$ and $(B; B; A; A)$ may be nonequivalent. It is easy to show that, in fact, these two normal sequences are equivalent if and only if the binary sequences A and B^* are equivalent, that is, if and only if $B^* \in \{A; -A; A'; -A'\}$.

The equivalence classes of Golay sequences of length ≤ 40 have been enumerated in [13]. This was accomplished by defining the canonical form and listing the canonical representatives of the equivalence classes. These representatives are written there in encoded form as $\delta_1\delta_2 \cdots \delta_m$ obtained by decomposing $(A; B)$ into m quads. These are Golay quads and should not be confused with the BS-quads defined in Section 2. If $(A; B) \in \mathrm{GS}(n)$ is one of the representatives, it is obvious that $B^* \neq -A$ and $B^* \neq -A'$, and it is easy to see that also $B^* \neq A$. Thus. if B^* is equivalent to A we must have $B^* = A'$. Finally, one can show that the equality $B^* = A'$ holds if and only if $\delta_i \equiv i \pmod 2$ for each index i. For another meaning of the latter condition see [13, Proposition 5.1]. Thus an equivalence class of Golay sequences $\mathrm{GS}(n)$ with canonical representative $(A;B)$ provides either one or two equivalence classes of $\mathrm{NS}(n)$. The former case occurs if and only if $\delta_i \equiv i \pmod 2$ for each index i.

By using this criterion, it is straightforward to list the equivalence classes of $\mathrm{NS}(n)$ of Golay type for $n \leq 40$. For instance, if $n = 8$ there are five equivalence classes of Golay sequences. Their representatives are (see [13]) 3218, 3236, 3254, 3272, and 3315. Only the last representative violates the above condition. Hence, we have exactly $4 + 2 = 6$ equivalence classes of Golay type in NS(8).

4. The Symmetry Group of NS(n)

We will construct a group G_{NS} of order 512 which acts on $\mathrm{NS}(n)$. Our (redundant) generating set for G_{NS} will consist of 9 involutions. Each of these generators is an elementary transformation, and we use this information to construct G_{NS}, that is, to impose the defining relations. We denote by $S = (A;A;C;D)$ an arbitrary member of $\mathrm{NS}(n)$.

To construct G_{NS}, we start with an elementary abelian group E of order 64 with generators ν, ρ, and ν_i, ρ_i, $i \in \{3,4\}$. It acts on $\mathrm{NS}(n)$ as follows:

$$\begin{aligned}
\nu S &= (-A;-A;C;D), &\quad \rho S &= (A';A';C;D),\\
\nu_3 S &= (A;A;-C;D), &\quad \rho_3 S &= (A;A;C';D),\\
\nu_4 S &= (A;A;C;-D), &\quad \rho_4 S &= (A;A;C;D').
\end{aligned} \tag{4.1}$$

Next, we introduce the involutory generator σ. We declare that σ commutes with ν and ρ, and that $\sigma\nu_3 = \nu_4\sigma$ and $\sigma\rho_3 = \rho_4\sigma$. The group $H = \langle E,\sigma\rangle$ is the direct product of two groups: $H_1 = \langle \nu,\rho\rangle$ of order 4 and $H_2 = \langle \nu_3,\rho_3,\sigma\rangle$ of order 32. The action of E on $\mathrm{NS}(n)$ extends to H by defining $\sigma S = (A;A;D;C)$.

We add a new generator θ which commutes elementwise with H_1, commutes with $\nu_3\rho_3, \nu_4\rho_4$, and σ, and satisfies $\theta\rho_3 = \rho_4\theta$. Let us denote this enlarged group by $\widetilde{H}$. It has the direct product decomposition

$$\widetilde{H} = \langle H,\theta\rangle = H_1 \times \widetilde{H}_2, \tag{4.2}$$

where the second factor is itself a direct product of two copies of the dihedral group D_8 of order 8:

$$\widetilde{H}_2 = \langle \rho_3,\rho_4,\theta\rangle \times \langle \nu_3\rho_3,\nu_4\rho_4,\theta\sigma\rangle. \tag{4.3}$$

The action of H on $\mathrm{NS}(n)$ extends to $\widetilde{H}$ by letting θ act as the elementary transformation (E5).

Finally, we define G_{NS} as the semidirect product of $\widetilde{H}$ and the group of order 2 with generator α. By definition, α commutes with ν,ν_3,ν_4 and satisfies

$$\begin{aligned}
\alpha\rho\alpha &= \rho\nu^{n-1},\\
\alpha\rho_j\alpha &= \rho_j\nu_j^{n-1}, \quad j = 3,4;\\
\alpha\theta\alpha &= \theta\sigma^{n-1}.
\end{aligned} \tag{4.4}$$

The action of $\widetilde{H}$ on $\mathrm{NS}(n)$ extends to G_{NS} by letting α act as the elementary transformation (E5), that is, we have $\alpha S = (A^*; B^*; C^*; D^*)$.

We point out that the definition of the subgroup $\widetilde{H}$ is independent of n and its action on $\mathrm{NS}(n)$ has a quadwise character. By this we mean that the value of a particular quad, say p_i, of $S \in \mathrm{NS}(n)$ and $h \in \widetilde{H}$ determine uniquely the quad p_i of hS. In other words, $\widetilde{H}$ acts on the quads and the set of central columns such that the encoding of hS is given by the symbol sequences

$$h(p_1)h(p_2)\cdots, \qquad h(q_1)h(q_2)\cdots. \tag{4.5}$$

On the other hand, the definition of the full group G_{NS} depends on the parity of n, and only for n odd it has the quad-wise character.

An important feature of the quad-action of $\widetilde{H}$ is that it preserves the symmetry type of the quads. If n is odd, this is also true for G_{NS}.

The following proposition follows immediately from the construction of G_{NS} and the description of its action on $\mathrm{NS}(n)$.

Proposition 4.1. *The orbits of G_{NS} in $NS(n)$ are the same as the equivalence classes.*

The main tool that one uses to enumerate the equivalence classes of $\mathrm{NS}(n)$ is the following theorem.

Theorem 4.2. *For each equivalence class $\mathcal{E} \subseteq NS(n)$ there is a unique $S = (A; A; C; D) \in \mathcal{E}$ having the canonical form.*

Proof. In view of Proposition 3.2, we just have to prove the uniqueness assertion. Let

$$S^{(k)} = \left(A^{(k)}; A^{(k)}; C^{(k)}; D^{(k)}\right) \in \mathcal{E}, \quad (k = 1, 2) \tag{4.6}$$

be in the canonical form. We have to prove that in fact $S^{(1)} = S^{(2)}$.

By Proposition 4.1, we have $gS^{(1)} = S^{(2)}$ for some $g \in G_{\mathrm{NS}}$. We can write g as $g = \alpha^s h$ where $s \in \{0, 1\}$ and $h = h_1 h_2$ with $h_1 \in H_1$ and $h_2 \in \widetilde{H}_2$. Let $p_1^{(k)} p_2^{(k)} \cdots$ be the encoding of the pair $(A^{(k)}; A^{(k)})$ and $q_1^{(k)} q_2^{(k)} \cdots$ the encoding of the pair $(C^{(k)}; D^{(k)})$. The symbols (i–xii) will refer to the corresponding conditions of Definition 3.1.

We prove first preliminary claims (a–c).

(a) $p_1^{(1)} = p_1^{(2)}$ and, consequently, $q_1^{(1)} = q_1^{(2)}$.

For n even this follows from (i). Let n be odd. When we apply the generator α to any $S \in \mathrm{NS}(n)$, we do not change the first quad of $(A; A)$. It follows that the quads $p_1^{(1)}$ and $p_1^{(2)} = g(p_1^{(1)}) = h_1(p_1^{(1)})$ have the same symmetry type. The claim now follows from (i).

Clearly, we are done with the case $n = 2$.

If $n = 3$ it is easy to see that we must have $p_1^{(1)} = p_1^{(2)} = 6$ and $q_1^{(1)} = q_1^{(2)} = 1$. By (iv), for the central column symbols, we have $p_2^{(1)} = p_2^{(2)} = 0$. Then (2.4) for $i = 1$ implies that $q_2^{(k)} \in \{1, 2\}$ for $k = 1, 2$. By (xi) we must have $q_2^{(1)} = q_2^{(2)} = 1$. Hence $S^{(1)} = S^{(2)}$ in that case.

Thus from now on we may assume that $n > 3$.

(b) If n is even then, $s = 0$.

Table 2: Class representatives for $n \leq 15$.

		$n = 1$			
1	0 0				
		$n = 2$			
1	6 1				
		$n = 3$			
1	60 11				
		$n = 4$			
1	16 61				
		$n = 5$			
1	160 640				
		$n = 7$			
1	1660 6122	2	6113 1623	3	6160 1262
4	6163 1261				
		$n = 8$			
1	1163 6618	2	1613 6168	3	1613 6443
4	1638 6116	5	1661 6183	6	1686 6131
7	1866 6311				
		$n = 9$			
1	16133 64140	2	16163 64150	3	61180 16640
		$n = 10$			
1	11863 66311	2	16166 64156	3	16613 61838
4	16616 61831	5	18863 63311		
		$n = 11$			
1	611680 164231	2	616163 126232		
		$n = 12$			
1	161383 641261	2	163868 612243	3	186338 631422
4	186631 631422				
		$n = 13$			
1	1616133 6414853	2	6116680 1286320	3	6168160 1613441
		$n = 15$			
1	61613163 12676761	2	61683860 12626262		

By (i), $p_1^{(1)} = p_1^{(2)} = 1$. Note that the first quads of $(A; A)$ in S and in αS have different symmetry types for any $S \in \mathcal{E}$. As the quad $h(1)$ is symmetric, the equality $\alpha^s hS^{(1)} = S^{(2)}$ forces s to be 0.

As an immediate consequence of (b), we point out that, if n is even, a quad $p_i^{(1)}$ is symmetric iff $p_i^{(2)}$ is, and the same is true for the quads $q_i^{(1)}$ and $q_i^{(2)}$.

(c) $p_2^{(1)} = p_2^{(2)}$.

We first observe that $p_2^{(1)}$ and $p_2^{(2)}$ have the same symmetry type. If n is even this follows from (b) since then $g = h$. If n is odd then under the quad action on p_2, each of α, ν, ρ preserves the symmetry type of p_2. Now the assertion (c) follows from (ii) and (iii) if $p_1^{(1)}$ and $p_2^{(1)}$ have different symmetry types, and from (v) otherwise.

We will now prove that $A^{(1)} = A^{(2)}$.

Table 3: Class representatives for $16 \le n \le 29$.

	$n = 16$		
1	11186366 66631811	2	11186636 66631181
3	11631866 66186311	4	11633381 66181163
5	11636618 66188836	6	11638133 66183688
7	11661836 66116381	8	11663681 66111863
9	11666318 66118136	10	11668163 66113618
11	11816333 66361888	12	11816663 66361118
13	16131686 61686131	14	16133831 61681613
15	16136168 61688386	16	16138313 61683868
17	16161386 61616831	18	16163861 61611683
19	16163861 64124328	20	16166138 61618316
21	16166138 64127156	22	16168613 61613168
23	16381331 61166813	24	16381661 61166183
25	16388338 61163816	26	16388668 61163186
27	16611368 61836886	28	16611638 61836116
29	16618361 61833883	30	16618631 61833113
31	16831313 61386868	32	16833838 61381616
33	16836161 61384242	34	16836161 61388383
35	16838686 61383131	36	16838863 61344313
37	16861613 61316168	38	16863868 61311686
39	16866131 61318313	40	16868386 61313831
41	18116333 63661888	42	18116663 63661118
43	18631133 63186688	44	18633388 63181166
45	18636611 63188833	46	18638866 63183311
47	18661163 63116618	48	18663688 63111866
49	18666311 63118133	50	18668836 63113381
51	18886366 63331811	52	18886636 63331181
	$n = 18$		
1	161633881 641242146		
	$n = 19$		
1	1168186360 6643551210		
	$n = 20$		
1	1166131836 6611686381	2	1166861836 6611316381
3	1181616633 6636161188	4	1186161633 6631616188
5	1186868366 6631313811	6	1188686366 6633131811
7	1611663138 6441827614	8	1613383113 6168161368
9	1613383186 6168161331	10	1616138631 6164224786
11	1616311386 6161866831	12	1616681386 6161136831
13	1616831361 6161386883	14	1616833886 6161381631
15	1616836113 6161388368	16	1616838638 6161383116

Table 3: Continued.

17	1638133138 6116681316	18	1638133161 6116681383
19	1638883818 6183331633	20	1661813881 6116361666
21	1661863138 6183311316	22	1661863161 6183311383
23	1683381313 6138836868	24	1683611313 6138166868
25	1683831361 6138386883	26	1683833886 6138381631
27	1683836113 6138388368	28	1683838638 6138383116
29	1686613113 6131831368	30	1686613186 6131831331
31	1863161133 6318616688	32	1863831133 6318386688
33	1881616663 6336161118	34	1886161663 6331616118
35	1886868336 6331313881	36	1888686336 6333131881
	$n = 25$		
1	1616138313163		6414148485143
2	1616161383163		6414148584143
3	1616161386163		6414148585143
4	1616168613163		6414158585143
	$n = 29$		
1	161383131316830		641414841515843
2	161686161313860		641515851514853

Assume first that n is even. Then $p_1^{(1)} = p_1^{(2)} = 1$ by (i), $s = 0$ by (b), and the equality $h_1(p_1^{(1)}) = p_1^{(2)}$ implies that $h_1(1) = 1$. Thus $h_1 \in \{1, \rho\}$. Let i be the smallest index (if any) such that the quad $p_i^{(1)}$ is skew. Then $p_i^{(1)} = p_i^{(2)} = 6$ by (iii). Hence $h_1(6) = 6$ and so $h_1 = 1$ and $A^{(1)} = A^{(2)}$ follows. On the other hand, if all quads $p_i^{(1)}$ are symmetric, then all these quads are fixed by h_1 and so $A^{(1)} = A^{(2)}$.

Next assume that n is odd. Then $p_1^{(1)} = p_2^{(1)} \in \{1, 6\}$ by (i). Let $i < m$ be the smallest index (if any) such that the quads $p_i^{(1)}$ and $p_{i+1}^{(1)}$ have the same symmetry type.

We first consider the case $p_1^{(1)} = 1$. Since n is odd, α fixes the quad p_1, and so h_1 must fix the quad 1. Thus we again have $h_1 \in \{1, \rho\}$.

If i is even then, by minimality of i, both $p_i^{(1)}$ and $p_{i+1}^{(1)}$ are skew. By (v), we have $p_{i+1}^{(1)} = p_{i+1}^{(2)} = 6$. Since i is even, α fixes p_{i+1} and so we must have $h_1(6) = 6$. It follows that $h_1 = 1$. As $i > 1$, the quad $p_2^{(1)}$ is skew and by (iii) we have $p_2^{(1)} = p_2^{(2)} = 6$. Since α maps p_2 to its negative, we must have $s = 0$. Consequently, $A^{(1)} = A^{(2)}$.

If i is odd then both $p_i^{(1)}$ and $p_{i+1}^{(1)}$ are symmetric. By (v) we have $p_{i+1}^{(1)} = p_{i+1}^{(2)} = 1$. Since i is odd, α maps p_{i+1} to its negative. Since ρ fixes the symmetric quads, we conclude that $1 = g(1) = \alpha^s h_1(1) = \alpha^s(1)$ and so $s = 0$. If all quads $p_i^{(1)}$ are symmetric, then they are all fixed by g and so $A^{(1)} = A^{(2)}$. Otherwise, let j be the smallest index such that $p_j^{(1)}$ is skew. By (iii) we have $p_j^{(1)} = p_j^{(2)} = 6$, and $6 = p_j^{(2)} = g(p_j^{(1)}) = g(6) = h_1(6)$ implies that $h_1 = 1$. Thus $A^{(1)} = A^{(2)}$.

We now consider the case $p_1^{(1)} = 6$. Since n is odd, α fixes the quad p_1, and so h_1 must fix the quad 6. Thus we have $h_1 \in \{1, \nu\rho\}$.

Table 4: Sporadic classes for $n = 32$.

1	1111636366331881 6666181845542277
2	1111663318816363 6666455411882727
3	1166186333886318 6641231814721176
4	1166186366113681 6641231858635567
5	1166813633883681 6614328141271167
6	1166813666116318 6614328185365576
7	1613161361683831 6168616842525747
8	1616168313861313 6412651765826487
9	1616168338613838 6412623728284126
10	1616168361386161 6412623756567358
11	1616383883163861 6412214634822843
12	1616386113133168 6412434384672376
13	1616386186866831 6412282832157623
14	1616613813136831 6412565684677623
15	1616613886863168 6412717132152376
16	1616616116833861 6412785365172843
17	1616831613868686 6412348265823512
18	1616831638616161 6412376243437358
19	1616831661383838 6412376271714126
20	1638163886681331 6142241631477413
21	1638163886681331 6241142632488423
22	1661166113688631 6142758368527413
23	1661166113688631 6241857367518423
24	1683161638383861 6138642142161717
25	1683161661616138 6138642183575656
26	1683383813863131 6138421671711253
27	1683383886136868 6138164234348746
28	1683616113866868 6138428321218256
29	1683616186133131 6138834235351743
30	1683838338386138 6138342816574646
31	1683838361613861 6138342842831212
32	1686168638686131 6131613142475752
33	1818633611886666 6363445518812222
34	1818666636638811 6363111144552772
35	1863116636816611 6341268841334537
36	1863116663183388 6341268814221826

If i is even then, by minimality of i, both $p_i^{(1)}$ and $p_{i+1}^{(1)}$ are symmetric. By (v) we have $p_{i+1}^{(1)} = p_{i+1}^{(2)} = 1$. Since i is even, α fixes p_{i+1} and so we must have $h_1(1) = 1$. It follows that $h_1 = 1$. As $i > 1$, the quad $p_2^{(1)}$ is symmetric and by (ii) we have $p_2^{(1)} = p_2^{(2)} = 1$. Since α maps p_2 to its negative, we must have $s = 0$. Consequently, $A^{(1)} = A^{(2)}$.

If i is odd then both $p_i^{(1)}$ and $p_{i+1}^{(1)}$ are skew. By (v) we have $p_{i+1}^{(1)} = p_{i+1}^{(2)} = 6$. Since i is odd, α maps p_{i+1} to its negative. Since $\nu\rho$ fixes the skew quads, we conclude that $6 = g(6) = \alpha^s h_1(6) = \alpha^s(6)$ and so $s = 0$. If all quads $p_i^{(1)}$, $i \leq m$, are skew, then they are all fixed by g and $p_{m+1}^{(1)} = p_{m+1}^{(2)} = 0$ by (iv). Now $0 = p_{m+1}^{(2)} = h_1(p_{m+1}^{(1)}) = h_1(0)$ entails that $h_1 = 1$ and so $A^{(1)} = A^{(2)}$. Otherwise let j be the smallest index such that $p_j^{(1)}$ is symmetric. By (ii) we have $p_j^{(1)} = p_j^{(2)} = 1$, and $1 = p_j^{(2)} = g(p_j^{(1)}) = h_1(1)$ implies that $h_1 = 1$. Thus $A^{(1)} = A^{(2)}$.

It remains to consider the case where any two consecutive quads $p_i^{(1)}$ and $p_{i+1}^{(1)}$, $i < m$, have different symmetry types. Say, the quads $p_i^{(1)}$, $i \leq m$, are skew for even i and symmetric for odd i. By (i) and (iii) we have $p_1^{(1)} = p_1^{(2)} = 1$ and $p_2^{(1)} = p_2^{(2)} = 6$. Then h_1 must fix the quad 1, and so $h_1 \in \{1, \rho\}$. Since $6 = p_2^{(2)} = g(p_1^{(2)}) = g(6) = \alpha^s h_1(6)$, we must have $s = 0$ and $h_1 = 1$ or $s = 1$ and $h_1 = \rho$. In the former case, we obviously have $A^{(1)} = A^{(2)}$. In the latter case, all quads $p_i^{(1)}$, $i \leq m$, are fixed by g. Moreover, if m is even also the central column p_{m+1} is fixed by g and so $A^{(1)} = A^{(2)}$. On the other hand, if m is odd, then the quad $p_m^{(1)}$ is symmetric and the second part of the condition (v) implies that $p_{m+1}^{(1)} = p_{m+1}^{(2)} = 0$. Hence again $A^{(1)} = A^{(2)}$.

Similar proof can be used if the quads $p_i^{(1)}$, $i \leq m$, are symmetric for even i and skew for odd i. This completes the proof of the equality $A^{(1)} = A^{(2)}$. The proof of the equality $(C^{(1)}; D^{(1)}) = (C^{(2)}; D^{(2)})$ is the same as in [5]. □

5. Representatives of the Equivalence Classes

We have, computed a set of representatives for the equivalence classes of normal sequences NS(n) for all $n \leq 40$. Each representative is given in the canonical form which is made compact by using our standard encoding. The encoding is explained in detail in Section 2. This compact notation is used primarily in order to save space, but also to avoid introducing errors during decoding. For each n, the representatives are listed in the lexicographic order of the symbol sequences (2.10) and (2.11).

In Tables 2 and 3, we list the codes for the representatives of the equivalence classes of NS(n) for $n \leq 15$ and $16 \leq n \leq 29$, respectively. As there are 516 and 304 equivalence classes in NS(32) and NS(40), respectively, we list in Table 4 only the 36 representatives of the sporadic classes of NS(32). The cases

$$n = 6, 14, 17, 21, \ldots, 24, 27, 28, 30, 31, 33, 34, \ldots, 39 \tag{5.1}$$

are omitted since then NS(n) $= \emptyset$. We also omit $n = 40$ because in that case there are no sporadic classes. The Golay-type equivalence classes of normal sequences can be easily enumerated (as explained in Section 3) by using the tables of representatives of the equivalence classes of Golay sequences [13].

Note that in the case $n = 1$, there are no quads and both zeros in Table 2 represent central columns.

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