Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2013, Article ID 520610, 8 pages
http://dx.doi.org/10.1155/2013/520610

Research Article

Bounds for the Largest Laplacian Eigenvalue of Weighted Graphs

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Received 29 November 2012; Accepted 14 March 2013

Academic Editor: Jun-Ming Xu

Let G be weighted graphs, as the graphs where the edge weights are positive definite matrices. The Laplacian eigenvalues of a graph are the eigenvalues of Laplacian matrix of a graph G. We obtain two upper bounds for the largest Laplacian eigenvalue of weighted graphs and we compare these bounds with previously known bounds.

1. Introduction

Let $G = (V, E)$ be simple graphs, as graphs which have no loops or parallel edges such that V is a finite set of vertices and E is a set of edges.

A weighted graph is a graph each edge of which has been assigned to a square matrix called the weight of the edge. All the weight matrices are assumed to be of same order and to be positive matrix. In this paper, by "weighted graph" we mean "a weighted graph with each of its edges bearing a positive definite matrix as weight," unless otherwise stated.

The notations to be used in paper are given in the following.

Let G be a weighted graph on n vertices. Denote by $w_{i,j}$ the positive definite weight matrix of order p of the edge ij, and assume that $w_{ij} = w_{ji}$. We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j:j\sim i} w_{ij}$. be the weight matrix of the vertex i.

The Laplacian matrix of a graph G is defined as $L(G) = (l_{ij})$, where

$$l_{i,j} = \begin{cases} w_i; & \text{if } i = j, \\ -w_{ij}; & \text{if } i \sim j, \\ 0; & \text{otherwise.} \end{cases} \quad (1)$$

The zero denotes the $p \times p$ zero matrix. Hence $L(G)$ is square matrix of order np. Let λ_1 denote the largest eigenvalue of $L(G)$. In this paper we also use to avoid the confusion that $\rho_1(w_{ij})$ is the spectral radius of w_{ij} matrix. If V is the disjoint union of two nonempty sets V_1 and V_2 such that every vertex i in V_1 has the same $\rho_1(w_i)$ and every vertex j in V_2 has the same $\rho_1(w_j)$, then G is called a weight-semiregular graph. If $\rho_1(w_i) = \rho_1(w_j)$ in weight semiregular graph, then G is called a weighted-regular graph.

Upper and lower bounds for the largest Laplacian eigenvalue for unweighted graphs have been investigated to a great extent in the literature. Also there are some studies about the bounds for the largest Laplacian eigenvalue of weighted graphs [1–3]. The main result of this paper, contained in Section 2, gives two upper bounds on the largest Laplacian for weighted graphs, where the edge weights are positive definite matrices. These upper bounds are attained by the same methods in [1–3]. We also compare the upper bounds with the known upper bounds in [1–3]. We also characterize graphs which achieve the upper bound. The results clearly generalize some known results for weighted and unweighted graphs.

2. The Known Upper Bounds for the Largest Laplacian Eigenvalue of Weighted Graphs

In this section, we present the upper bounds for the largest Laplacian eigenvalue of weighted graphs and very useful lemmas to prove theorems.

Theorem 1 (Horn and Johnson [4]). *Let $A \in M_n$ be Hermitian, and let the eigenvalues of A be ordered such that $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$. Then,*

$$\lambda_n x^T x \leq x^T A x \leq \lambda_1 x^T x \tag{2}$$

$$\begin{aligned} \lambda_{\max} &= \lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x^T x = 1} x^T A x \\ \lambda_{\min} &= \lambda_n = \min_{x \neq 0} \frac{x^T A x}{x^T x} = \min_{x^T x = 1} x^T A x \end{aligned} \tag{3}$$

for all $x \in \mathbb{C}^n$.

Lemma 2 (Horn and Johnson [4]). *Let B be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$; then for any $\overline{x} \in R^n$ $(\overline{x} \neq \overline{0})$, $\overline{y} \in R^n$ $(\overline{y} \neq \overline{0})$,*

$$\left| \overline{x}^T B \overline{y} \right| \leq \lambda_1 \sqrt{\overline{x}^T \overline{x}} \sqrt{\overline{y}^T \overline{y}}. \tag{4}$$

Equality holds if and only if $\overline{x}$ is an eigenvector of B corresponding to λ_1 and $\overline{y} = \alpha \overline{x}$ for some $\alpha \in R$.

Lemma 3 (see [1]). *Let G be a $(\rho_1(w_i), \rho_1(w_j))$-semiregular bipartite graph of order n such that the first l vertices of the same largest eigenvalue $\rho_1(w_i)$ and the remaining m vertices of the same largest eigenvalue $\rho_1(w_j)$. Also let $\overline{x}$ be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j, where $w_i = \sum_{k \in N_i} w_{ik}$ for all i. Then $\rho_1(w_i) + \rho_1(w_j)$ is the largest eigenvalue of $L(G)$ and the corresponding eigenvector is*

$$\Big(\underbrace{\rho_1(w_i)\overline{x}^T, \ldots, \rho_1(w_i)\overline{x}^T}_{l}, \underbrace{-\rho_1(w_j)\overline{x}^T, \ldots, -\rho_1(w_j)\overline{x}^T}_{m} \Big). \tag{5}$$

Theorem 4 (see [1]). *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leq \max_{i \sim j} \left\{ \rho_1 \left(\sum_{k:k\sim i} w_{ik} \right) + \sum_{k:k\sim j} \rho_1 \left(w_{jk} \right) \right\}, \tag{6}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (6) if and only if

(i) *G is a weight-semiregular bipartite graph,*

(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j.*

Theorem 5 (see [2]). *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leq \max_{i \sim j} \left\{ \sqrt{ \begin{array}{l} \sum_{k:k\sim i} \rho_1 (w_{ik}) \left(\sum_{r:r\sim i} \rho_1 (w_{ir}) + \sum_{s:s\sim k} \rho_1 (w_{ks}) \right) \\ + \sum_{k:k\sim j} \rho_1 \left(w_{jk} \right) \left(\sum_{r:r\sim j} \rho_1 \left(w_{jr} \right) + \sum_{s:s\sim k} \rho_1 \left(w_{ks} \right) \right) \end{array} } \right\}, \tag{7}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (7) if and only if

(i) *G is a bipartite semiregular graph;*

(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j.*

Corollary 6 (see [2]). *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leq \max_i \left\{ \sqrt{2 w_i \left(w_i + \overline{w}_i \right)} \right\}, \tag{8}$$

where $\overline{w}_i = (\sum_{k:k\sim i} w_{ik} w_k)/w_i$ and w_i is the weight of vertex i. Moreover equality holds if and only if G is a bipartite regular graph.

Corollary 7 (see [2]). *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leq \max_{i \sim j} \left\{ \sqrt{w_i \left(w_i + \overline{w}_i \right) + w_j \left(w_j + \overline{w}_j \right)} \right\}, \tag{9}$$

where $\overline{w}_i = (\sum_{k:k\sim i} w_{ik} w_k)/w_i$ and w_i is the weight of vertex i. Moreover equality holds if and only if G is a bipartite semiregular graph.

Theorem 8 (see [2]). *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leq \max_{i \sim j} \left\{ \frac{\rho_1 \left(w_i \right) + \rho_1 \left(w_j \right) + \sqrt{\left(\rho_1 \left(w_i \right) - \rho_1 \left(w_j \right) \right)^2 + 4 \overline{\gamma}_i \overline{\gamma}_j}}{2} \right\}, \tag{10}$$

where $\overline{\gamma}_i = (\sum_{k:k\sim i} \rho_1(w_{ik}) \rho_1(w_k))/\rho_1(w_i)$ and w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (10) if and only if

(i) *G is a weighted-regular graph or G is a weight-semiregular bipartite graph;*

(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j.*

Corollary 9 (see [2]). *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leq \max_i \left\{ w_i + \overline{w}_i \right\}, \tag{11}$$

where $\overline{w}_i = (\sum_{k:k\sim i} w_{ik} w_k)/w_i$ and w_i is the weight of vertex i. Moreover equality holds if and only if G is a bipartite semiregular graph or G is a bipartite regular graph.

Theorem 10 (see [3]). *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \le \max_i \left\{ \sqrt{ \rho_1^2(w_i) + \sum_{k:k\sim i} \rho_1^2(w_{ik}) + \sum_{k:k\sim i} \rho_1(w_i w_{ik} + w_{ik} w_k) + \sum_{1\le i,t\le n} \sum_{s\in N_i\cap N_t} \rho_1(w_{is} w_{st}) } \right\}, \tag{12}$$

where w_{ik} is the positive definite weight matrix of order p of the edge ik and $N_i \cap N_k$ is the set of common neighbours of i and k. Moreover equality holds in (12) *if and only if*

(i) *G is a weight-semiregular bipartite graph;*

(ii) *w_{ik} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ik})$ for all i, k.*

Corollary 11 (see [3]). *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \le \max_i \left\{ \sqrt{ w_i^2 + \sum_{k:k\sim i} w_{ik}^2 + \sum_{k:k\sim i} (w_i w_{ik} + w_k w_{ik}) + \sum_{1\le i,t\le n} \sum_{s\in N_i\cap N_t} w_{is} w_{st} } \right\}. \tag{13}$$

Moreover equality holds if and only if G is a bipartite semiregular graph.

3. Two Upper Bounds on the Largest Laplacian Eigenvalue of Weighted Graphs

In this section we present two upper bounds for the largest eigenvalue of weighted graphs and compare the bounds with some examples.

Theorem 12. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \le \max_{i\sim j} \left\{ \frac{ \rho_1(w_i) + \rho_1(w_j) + \sqrt{ (\rho_1(w_i) - \rho_1(w_j))^2 + 4 \left(\sum_{k:k\sim i} \rho_1(w_{ik}) \right) \left(\sum_{k:k\sim j} \rho_1(w_{jk}) \right) } }{2} \right\}, \tag{14}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (14) *if and only if*

(i) *G is a weighted-regular graph or G is a weight-semiregular bipartite graph;*

(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j.*

Proof. Let $\overline{X} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)^T$ be an eigenvector corresponding to the largest eigenvalue λ_1 of $L(G)$. We assume that $\overline{x}_i$ is the vector component of $\overline{X}$ such that

$$\overline{x}_i^T \overline{x}_i = \max_{k\in V} \left\{ \overline{x}_k^T \overline{x}_k \right\}. \tag{15}$$

Since $\overline{X}$ is nonzero, so is $\overline{x}_i$. Let

$$\overline{x}_j^T \overline{x}_j = \max_{k:k\sim i} \left\{ \overline{x}_k^T \overline{x}_k \right\} \tag{16}$$

be. The (i, j)th element of $L(G)$ is

$$\begin{cases} w_i; & \text{if } i = j \\ -w_{i,j}; & \text{if } i \sim j \\ 0; & \text{otherwise.} \end{cases} \tag{17}$$

We have

$$L(G)\,\overline{X} = \lambda_1 \overline{X}. \tag{18}$$

From the ith equation of (18), we have

$$\left(\lambda_1 I_{p,p} - w_i\right) \overline{x}_i = -\sum_{k:k\sim i} w_{ik} \overline{x}_k, \tag{19}$$

that is,

$$\overline{x}_i^T \left(\lambda_1 I_{p,p} - w_i\right) \overline{x}_i = -\sum_{k:k\sim i} \overline{x}_i^T w_{ik} \overline{x}_k \tag{20}$$

$$\le \sum_{k:k\sim i} \left| \overline{x}_i^T w_{ik} \overline{x}_k \right| \tag{21}$$

$$\le \sum_{k:k\sim i} \rho_1(w_{ik}) \sqrt{\overline{x}_i^T \overline{x}_i} \sqrt{\overline{x}_k^T \overline{x}_k} \quad \text{by (4)} \tag{22}$$

$$\le \sum_{k:k\sim i} \rho_1(w_{ik}) \sqrt{\overline{x}_i^T \overline{x}_i} \sqrt{\overline{x}_j^T \overline{x}_j} \quad \text{by (16)}. \tag{23}$$

From (23) we have

$$\begin{aligned} (\lambda_1 - \rho_1(w_i)) \overline{x}_i^T \overline{x}_i &\le \overline{x}_i^T \left(\lambda_1 I_{p,p} - w_i\right) \overline{x}_i \quad \text{by (2)} \\ &\le \sum_{k:k\sim i} \rho_1(w_{ik}) \sqrt{\overline{x}_i^T \overline{x}_i} \sqrt{\overline{x}_j^T \overline{x}_j}, \end{aligned} \tag{24}$$

that is,

$$(\lambda_1 - \rho_1(w_i))\,\overline{x}_i^T\overline{x}_i \le \sum_{k:k\sim i}\rho_1(w_{ik})\sqrt{\overline{x}_i^T\overline{x}_i}\sqrt{\overline{x}_j^T\overline{x}_j}. \tag{25}$$

From the jth equation of (18), we get

$$\left(\lambda_1 I_{p,p} - w_j\right)\overline{x}_j = -\sum_{k:k\sim j} w_{jk}\overline{x}_k, \tag{26}$$

that is,

$$\overline{x}_j^T\left(\lambda_1 I_{p,p} - w_i\right)\overline{x}_j = -\sum_{k:k\sim j}\overline{x}_j^T w_{jk}\overline{x}_k \tag{27}$$

$$\le \sum_{k:k\sim j}\left|\overline{x}_j^T w_{jk}\overline{x}_k\right| \tag{28}$$

$$\le \sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\sqrt{\overline{x}_j^T\overline{x}_j}\sqrt{\overline{x}_k^T\overline{x}_k} \quad \text{by (4)} \tag{29}$$

$$\le \sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\sqrt{\overline{x}_i^T\overline{x}_i}\sqrt{\overline{x}_j^T\overline{x}_j} \quad \text{by (15)}. \tag{30}$$

Similarly, from (30) we get

$$\begin{aligned}\left(\lambda_1 - \rho_1\left(w_j\right)\right)\overline{x}_j^T\overline{x}_j &\le \overline{x}_j^T\left(\lambda_1 I_{p,p} - w_j\right)\overline{x}_j \quad \text{by (2)}\\ &\le \sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\sqrt{\overline{x}_i^T\overline{x}_i}\sqrt{\overline{x}_j^T\overline{x}_j},\end{aligned} \tag{31}$$

that is,

$$(\lambda_1 - \rho_1(wj))\,\overline{x}_j^T\overline{x}_j \le \sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\sqrt{\overline{x}_i^T\overline{x}_i}\sqrt{\overline{x}_k^T\overline{x}_k}. \tag{32}$$

So, from (25) and (32) we have

$$\begin{aligned}&\left(\lambda_1 - \rho_1\left(w_j\right)\right)(\lambda_1 - \rho_1(w_i))\,\overline{x}_j^T\overline{x}_j\overline{x}_i^T\overline{x}_i\\ &\quad\le \left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\cdot\left(\sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\right)\overline{x}_j^T\overline{x}_j\overline{x}_i^T\overline{x}_i.\end{aligned} \tag{33}$$

Hence we get

$$\begin{aligned}&\left(\lambda_1 - \rho_1\left(w_j\right)\right)(\lambda_1 - \rho_1(w_i))\\ &\quad\le \left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\cdot\left(\sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\right),\end{aligned} \tag{34}$$

that is,

$$\begin{aligned}&\lambda_1^2 - \lambda_1\left(\rho_1\left(w_j\right) + \rho_1(w_i)\right) + \rho_1(w_i)\,\rho_1\left(w_j\right)\\ &\quad - \left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\cdot\left(\sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\right) \le 0,\end{aligned} \tag{35}$$

that is,

$$\begin{aligned}\lambda_1 &\le \frac{\rho_1(w_i) + \rho_1\left(w_j\right)}{2}\\ &\quad + \frac{\sqrt{\left(\rho_1(w_i) - \rho_1\left(w_j\right)\right)^2 + 4\left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\left(\sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\right)}}{2}\\ &\le \max_{i\sim j}\left\{\frac{\rho_1(w_i)+\rho_1\left(w_j\right)}{2}\right.\\ &\quad\left. + \frac{\sqrt{\left(\rho_1(w_i)-\rho_1\left(w_j\right)\right)^2+4\left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\left(\sum_{k:k\sim j}\rho_1\left(w_{jk}\right)\right)}}{2}\right\}.\end{aligned} \tag{36}$$

This completes the proof of (14).

Now suppose that equality holds in (14). Then all inequalities in the previous argument must be equalities.

From equality in (23), we get

$$\overline{x}_i^T\overline{x}_i = \overline{x}_k^T\overline{x}_k \quad \text{for all } k, k\sim i. \tag{37}$$

Since $\overline{x}_i \ne 0$, we get that $\overline{x}_k \ne 0$ for all k, $k \sim i$. From equality in (22) and Lemma 2, we get that $\overline{x}_i$ is an eigenvector of w_{ik} for the largest eigenvalue $\rho_1(w_{ik})$. Hence we say that $\overline{x}_k = a\overline{x}_i$ for some a, for any k, $k \sim i$.

On the other hand, from (37) we get

$$a^2\overline{x}_i^T\overline{x}_i = \overline{x}_i^T\overline{x}_i, \tag{38}$$

that is,

$$a^2 = 1 \quad \text{as } \overline{x}_i^T\overline{x}_i > 0. \tag{39}$$

From equality in (21), we have

$$-\sum_{k:k\sim i}\overline{x}_i^T w_{ik}\overline{x}_k = \sum_{k:k\sim i}\left|\overline{x}_i^T w_{ik}\overline{x}_k\right|. \tag{40}$$

Since $\overline{x}_k = a\overline{x}_i$, from (40) we get

$$\begin{aligned}-\sum_{k:k\sim i}a\overline{x}_i^T w_{ik}\overline{x}_i &= \sum_{k:k\sim i}|a|\left|\overline{x}_i^T w_{ik}\overline{x}_k\right|\\ &= \sum_{k:k\sim i}|a|\,\overline{x}_i^T w_{ik}\overline{x}_k \quad \text{as } \overline{x}_i^T w_{ik}\overline{x}_k > 0.\end{aligned} \tag{41}$$

Hence we get

$$a = -1 \tag{42}$$

from equalities in (41). Therefore we have

$$\overline{x}_k = -\overline{x}_i \quad \text{for all } k, k\sim i. \tag{43}$$

Similarly from equality in (29), we get that $\overline{x}_j$ is an eigenvector of w_{jk} for the largest eigenvalue $\rho_1(w_{jk})$. Hence we say that $\overline{x}_k = b\overline{x}_j$ for some b, for any k, $k \sim j$. From equality in (16) we have

$$\overline{x}_j^T\overline{x}_j = \overline{x}_k^T\overline{x}_k \quad \text{for } k \sim i, \tag{44}$$

that is,

$$b^2\overline{x}_j^T\overline{x}_j = \overline{x}_j^T\overline{x}_j, \tag{45}$$

that is,

$$b^2 = 1 \quad \text{as } \overline{x}_j^T\overline{x}_j > 0. \tag{46}$$

Applying the same methods as previously, we get

$$b = -1. \tag{47}$$

Therefore we have

$$\overline{x}_k = -\overline{x}_j \quad \text{for all } k, k \sim j. \tag{48}$$

For $i \sim j$

$$\overline{x}_i = -\overline{x}_j. \tag{49}$$

Hence we take that $U = \{k : \overline{x}_k = \overline{x}_i\}$ and $W = \{k : \overline{x}_k = -\overline{x}_i\}$ from (43), (48), and (49). So, $N_j \subset U$ and $N_i \subset W$. Also, $U \neq W \neq \emptyset$ since $\overline{x}_i \neq 0$. Further, for any vertex $s \in N_{N_i}$ there exists a vertex $r \in N_i$ such that $r \sim j\ell r \sim s$, where N_{N_i} is the neighbor of neighbor set of vertex i. Therefore $\overline{x}_r = -\overline{x}_i$ and $\overline{x}_s = \overline{x}_i$. So $N_{N_i} \subset U$. By similar argument we can present that $N_{N_j} \subset W$. Continuing the procedure, it is easy to see, since G is connected, that $V = U \cup W$ and that the subgraphs induced by U and W, respectively, are empty graphs. Hence G is bipartite. Moreover, $\overline{x}_i$ is a common eigenvector of w_{ik} and w_i for the largest eigenvalue $\rho_1(w_{ik})$ and $\rho_1(w_i)$.

For $i, k \in U$

$$\lambda_1\overline{x}_i = w_i\overline{x}_i + \sum_{k:k\sim i} w_{ik}\overline{x}_i = w_k\overline{x}_i + \sum_{k:k\sim i} w_{ik}\overline{x}_i, \tag{50}$$

that is,

$$w_i\overline{x}_i = w_k\overline{x}_i. \tag{51}$$

Since $\overline{x}_i$ is an eigenvector of w_i corresponding to the largest eigenvalue of $\rho_1(w_i)$ for all i, we get

$$\rho_1(w_i)\overline{x}_i = \rho_1(w_k)\overline{x}_i, \tag{52}$$

that is,

$$(\rho_1(w_i) - \rho_1(w_k))\overline{x}_i = 0, \tag{53}$$

that is,

$$\rho_1(w_i) = \rho_1(w_k) \quad \text{as } \overline{x}_i \neq 0. \tag{54}$$

Therefore we get that $\rho_1(w_i)$ is constant for all $i \in U$. Similarly we can show that $\rho_1(w_j)$ is constant for all $j \in W$.

Hence G is a bipartite semiregular graph.

Conversely, suppose that conditions (i)-(ii) of the theorem hold for the graph G. Let G be $(\rho_1(w_i), \rho_1(w_j))$-semiregular bipartite graph. Let x be a common eigenvector of w_{ik} corresponding to the largest eigenvalue $\rho_1(w_{ik})$ for all i, k. Then we have

$$\begin{aligned}\rho_1(w_i) &= \sum_{k:k\sim i}\rho_1(w_{ik}),\\ \rho_1(w_j) &= \sum_{k:k\sim j}\rho_1(w_{jk}).\end{aligned} \tag{55}$$

By Lemma 3, we get

$$\lambda_1 = \rho_1(w_i) + \rho_1(w_j), \tag{56}$$

that is,

$$\lambda_1 = \frac{\rho_1(w_i) + \rho_1(w_j)}{2} + \frac{\sqrt{(\rho_1(w_i) - \rho_1(w_j))^2 + 4\left(\sum_{k:k\sim i}\rho_1(w_{ik})\right)\left(\sum_{k:k\sim j}\rho_1(w_{jk})\right)}}{2}. \tag{57}$$

□

Corollary 13 (see [1]). *Let G be a simple connected weighted graph where each edge weight $w_{i,j}$ is a positive number. Then*

$$\lambda_1 \leq \max_{i\sim j}\{w_i + w_j\}. \tag{58}$$

Moreover equality holds in (58) *if and only if G is bipartite semiregular graph.*

Proof. We have $\rho_1(w_i) = w_i$ and $\rho_1(w_{ij}) = w_{ij}$ for all i, j. From Theorem 12, we get the required result. □

Corollary 14 (see [5]). *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leq \max_{i\sim j}\{d_i + d_j\}, \tag{59}$$

where d_i is the degree of vertex i. Moreover equality holds in (59) *if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.*

Proof. For unweighted graph, $w_{i,j} = 1$ for $i \sim j$. Therefore $w_i = d_i$. Using Corollary 6, we get the required results. □

Theorem 15. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leq \max_{i\sim j}\left\{\sqrt{\left(\rho_1(w_i) + \sum_{k:k\sim i}\rho_1(w_{ik})\right)\left(\rho_1(w_j) + \sum_{k:k\sim j}\rho_1(w_{jk})\right)}\right\}, \tag{60}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (60) *if and only if*

(i) *G is a weighted-regular bipartite graph;*

(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{ij})$ for all i, j.*

Proof. Let $\overline{X} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ be an eigenvector corresponding to the largest eigenvalue λ_1 of $L(G)$. We assume that $\overline{x}_i$ is the vector component of $\overline{X}$ such that

$$\overline{x}_i^T \overline{x}_i = \max_{k \in V} \left\{ \overline{x}_k^T \overline{x}_k \right\}. \tag{61}$$

Since $\overline{X}$ is nonzero, so is $\overline{x}_i$. Let

$$\overline{x}_j^T \overline{x}_j = \max_{k:k \sim j} \left\{ \overline{x}_k^T \overline{x}_k \right\} \tag{62}$$

be. We have

$$L(G)\,\overline{X} = \lambda_1 \overline{X}. \tag{63}$$

From the ith equation of (43), we have

$$\lambda_1 \overline{x}_i = w_i \overline{x}_i - \sum_{k:k\sim i} w_{ik} \overline{x}_k, \tag{64}$$

that is,

$$\begin{aligned} \lambda_1 \overline{x}_i^T \overline{x}_i &= \left| \overline{x}_i^T w_i \overline{x}_i - \sum_{k:k\sim i} \overline{x}_i^T w_{ik} \overline{x}_k \right| \\ &\le \left| \overline{x}_i^T w_i \overline{x}_i \right| + \sum_{k:k\sim i} \left| \overline{x}_i^T w_{ik} \overline{x}_k \right| \\ &\le \rho_1(w_i)\, \overline{x}_i^T \overline{x}_i + \sum_{k:k\sim i} \rho_1(w_{ik}) \sqrt{\overline{x}_i^T \overline{x}_i} \sqrt{\overline{x}_k^T \overline{x}_k} \quad \text{by (2)} \\ &\le \rho_1(w_i)\, \overline{x}_i^T \overline{x}_i + \sum_{k:k\sim i} \rho_1(w_{ik})\, \overline{x}_i^T \overline{x}_i \quad \text{by (40)}. \end{aligned} \tag{65}$$

Hence we get

$$\lambda_1 \le \rho_1(w_i) + \sum_{k:k\sim i} \rho_1(w_{ik}). \tag{66}$$

By the same method, from the jth equation of (43), we have

$$\lambda_1 \overline{x}_j = w_j \overline{x}_j - \sum_{k:k\sim j} w_{jk} \overline{x}_k, \tag{67}$$

that is,

$$\begin{aligned} \lambda_1 \overline{x}_j^T \overline{x}_j &= \left| \overline{x}_j^T w_j \overline{x}_j - \sum_{k:k\sim j} \overline{x}_j^T w_{jk} \overline{x}_k \right| \\ &\le \left| \overline{x}_j^T w_j \overline{x}_j \right| + \sum_{k:k\sim j} \left| \overline{x}_j^T w_{jk} \overline{x}_k \right| \\ &\le \rho_1(w_j)\, \overline{x}_j^T \overline{x}_j + \sum_{k:k\sim j} \rho_1(w_{jk}) \sqrt{\overline{x}_j^T \overline{x}_j} \sqrt{\overline{x}_k^T \overline{x}_k} \quad \text{by (2)} \\ &\le \rho_1(w_j)\, \overline{x}_j^T \overline{x}_j + \sum_{k:k\sim j} \rho_1(w_{jk})\, \overline{x}_j^T \overline{x}_j \quad \text{by (41)}. \end{aligned} \tag{68}$$

Hence we get

$$\lambda_1 \le \rho_1(w_j) + \sum_{k:k\sim j} \rho_1(w_{jk}). \tag{69}$$

From (49) and (58), we have

$$\lambda_1^2 \le \left(\rho_1(w_i) + \sum_{k:k\sim i} \rho_1(w_{ik}) \right) \left(\rho_1(w_j) + \sum_{k:k\sim j} \rho_1(w_{jk}) \right), \tag{70}$$

that is,

$$\lambda_1 \le \max_{i\sim j} \left\{ \sqrt{\left(\rho_1(w_i) + \sum_{k:k\sim i} \rho_1(w_{ik}) \right) \left(\rho_1(w_j) + \sum_{k:k\sim j} \rho_1(w_{jk}) \right)} \right\}. \tag{71}$$

This completes the proof of (60).

Now we show the case of equality in (60). By similar method in Theorem 12. In the part of equalit, the necessary condition can show easily. So we will show the sufficient condition.

Suppose that conditions (i)-(ii) of Theorem hold for the graph G. We must prove that

$$\lambda_1 = \max_{i\sim j} \left\{ \sqrt{\left(\rho_1(w_i) + \sum_{k:k\sim i} \rho_1(w_{ik}) \right) \left(\rho_1(w_j) + \sum_{k:k\sim j} \rho_1(w_{jk}) \right)} \right\}. \tag{72}$$

Let G be regular bipartite graph. Therefore we have $\rho_1(w_i) = \alpha$ for $i \in U$ and $\rho_1(w_j) = \alpha$ for $j \in W$ such that $V = U \cup W$. Let $\overline{x}$ be a common eigenvector of w_{ik} corresponding to the largest eigenvalue $\rho_1(w_{ik})$ for all i, k. Hence we have

$$\rho_1(w_i) = \sum_{k:k\sim i} \rho_1(w_{ik}). \tag{73}$$

From (71) we get that

$$\lambda_1 \le 2\alpha. \tag{74}$$

On the other hand, the following equation can be easily verified:

$$(2\alpha)\begin{pmatrix} \overline{x} \\ \overline{x} \\ \vdots \\ \overline{x} \\ -\overline{x} \\ -\overline{x} \\ \vdots \\ -\overline{x} \end{pmatrix} = \begin{pmatrix} w_1 & \cdot & 0 & -w_{1,k+1} & \cdot & -w_{1,n} \\ 0 & \cdot & 0 & -w_{2,k+1} & \cdot & -w_{2,n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ 0 & \cdot & w_k & -w_{k,k+1} & \cdot & -w_{k,n} \\ -w_{k+1,1} & \cdot & -w_{k+1,k} & w_{k+1} & \cdot & 0 \\ -w_{k+1,2} & \cdot & -w_{k+2,k} & 0 & \cdot & 0 \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ -w_{n,1} & \cdot & -w_{n,k} & 0 & \cdot & w_n \end{pmatrix} \times \begin{pmatrix} \overline{x} \\ \overline{x} \\ \vdots \\ \overline{x} \\ -\overline{x} \\ -\overline{x} \\ \vdots \\ -\overline{x} \end{pmatrix}. \tag{75}$$

Thus 2α is an eigenvalue of $L(G)$. Since λ_1 is the largest eigenvalue of $L(G)$, we get

$$2\alpha \leq \lambda_1. \tag{76}$$

So from (74) and (76) we obtain

$$\lambda_1 = \max_{i\sim j}\left\{\sqrt{\left(\rho_1(w_i) + \sum_{k:k\sim i}\rho_1(w_{ik})\right)\left(\rho_1(w_j) + \sum_{k:k\sim j}\rho_1(w_{jk})\right)}\right\}. \tag{77}$$

□

Corollary 16. *Let G be a simple connected weighted graph where each edge weight $w_{i,j}$ is a positive number. Then*

$$\lambda_1 \leq \max_{i\sim j}\left\{2\sqrt{w_i w_j}\right\}. \tag{78}$$

Moreover equality holds in (78) *if and only if G is bipartite semiregular graph.*

Proof. We have $\rho_1(w_i) = w_i$ and $\rho_1(w_{ij}) = w_{ij}$ for all i, j. From Theorem 15 we get the required result. □

Corollary 17. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leq \max_{i\sim j}\left\{2\sqrt{d_i d_j}\right\}, \tag{79}$$

where d_i is the degree of vertex i. Moreover equality holds in (79) *if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.*

Proof. For unweighted graph, $w_{i,j} = 1$ for $i \sim j$. Therefore $w_i = d_i$. Using Corollary 16, we get the required results. □

Example 18. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be a weighted graph where $V_1 = \{1, 2, 3, 4\}$, $E_1 = \{\{1, 3\}, \{2, 4\}, \{3, 4\}\}$ and each weight is the positive definite matrix of order three. Let $V_2 = \{1, 2, 3, 4, 5, 6, 7\}$, $E_2 = \left\{\begin{smallmatrix}\{1,4\},\{2,4\},\{3,4\},\\ \{4,5\},\{5,6\},\{5,7\}\end{smallmatrix}\right\}$ such that each weight is the positive definite matrix of order two. Assume that the following Laplacian matrices of G_1 and G_2 are as follows:

$$L(G_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 & 0 & 0 & -5 & -3 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & -3 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 1 & -2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 7 & 2 & -2 & 6 & 2 & -2 \\ 0 & -5 & -3 & 0 & 0 & 0 & 2 & 11 & 1 & 2 & 6 & -2 \\ 0 & -3 & -3 & 0 & 0 & 0 & -2 & 1 & 13 & -2 & -2 & 10 \\ 0 & 0 & 0 & 2 & -1 & 0 & 6 & 2 & -2 & 8 & 1 & -2 \\ 0 & 0 & 0 & -1 & 2 & -1 & 2 & 6 & -2 & -3 & 8 & -3 \\ 0 & 0 & 0 & 0 & -1 & 2 & -2 & -2 & 10 & -2 & -2 & 12 \end{bmatrix},$$

$$L(G_2)=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & -1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 4 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & -3 & -1 & -4 & 4 & 14 & -1 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & 3 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & 3 & 18 & -1 & -6 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -6 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -7 & 0 & 0 & 1 & 7 \end{bmatrix}. \quad (80)$$

The largest eigenvalues of $L(G_1)$ and $L(G_2)$ are $\lambda_1 = 25,66$, $\lambda_2 = 26.16$ rounded two decimal places and the previously mentioned bounds give the following results:

	(6)	(7)	(10)	(12)	(14)	(60)
G_1	32.90	32.88	27.90	29.55	30.88	30.93
G_2	34.12	29.86	27.11	27.22	34.05	33.90

(81)

For G_1, we see that the upper bounds in (14) and (60) are better than upper bounds in (6) and (7). But they are not better than upper bounds in (10) and (12) from (81).

For also G_2, we see that upper bounds in (14) and (60) are only better than the upper bound in (6).

Consequently, we cannot exactly compare all the bounds for weighted graphs, where the weights are positive definite matrices. Modifications according to each weight of edges, especially for matrices can be shown.

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