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ASYMPTOTIC BEHAVIOR OF RETARDED DIFFERENTIAL EQUATIONS

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ABSTRACT. Some integral criteria for the asymptotic behdvior of oscillatory solutions of higher order retarded differential equations are given.

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1. INTRODUCTION.

Recently, Tong [1] proved the following interesting result.

Theorem. Let $f(t,u)$ be continuous on $R_+ \times R$. If there are two non-negative continuous functions $v(t)$, $p(t)$ for $t \geq 0$, and a continuous function $g(u)$ for $u \geq 0$ such that

(a) $\int_1^\infty v(t)p(t)dt < \infty$.

(b) $g(u)$ is positive and nondecreasing for $u > 0$,

(c) $|f(t,u)| \leq v(t)p(t)g(t^{-1}|u|)$ for $t \geq 1$, $u \in R$,

then the equation

$$u''+f(t,u) = 0$$

has solutions which are asymptotic to $a+bt$, where a, b are constant and $b \neq 0$.

In this note we generalize Tong's result to a more general case which improves also the results of Chen and Yeh [2] and Kusano and Singh [3]. Using this result, we establish an asymptotic behavior of oscillatory solutions of retarded differential equations.

2. MAIN RESULTS.

Consider the following retarded differential equations

(2.1) $$L_n y(t)+f(t,y(g(t))) = h(t), \quad t \geq 0, \quad n \geq 2$$

where L_n is an operator defined by

$$L_0y(t) := \frac{y(t)}{r_0(t)}, \quad L_iy(t) := \frac{1}{r_i(t)}\frac{d}{dt}L_{i-1}y(t), \quad i = 1,2,\cdots,n$$

$$r_n(t) := 1.$$

Here $r_i(t) \in C^{n-i}[R_+,R]$ with $r_i(t) > 0$ for $i = 0,1,\cdots,n-1$.

Sufficient smoothness to guarantee the existence of solutions of (2.1) on an infinite subinterval of R_+ will be assumed without mention. The following conditions are assumed to hold in this note.

(i) $f \in C[R_+\times R,R]$ and there exist two positive functions $p(t)$, $H(t)$ $\in C[R_+,R_+]$ with $H(t)$ nondecreasing and $kH(t) \le H(kt)$ for any $k > 0$ such that

$$|f(t,u)| \le p(t)H(|u|),$$

(ii) $g,\ h \in C[R_+,R]$, $g(t) \le t$, $\lim_{t\to\infty} g(t) = \infty$,

(iii) $\liminf_{t\to\infty} \frac{1}{r_0(t)} > 0$, $\limsup_{t\to\infty} \frac{w_i(t,u)}{w_{n-1}(t,u)} < \infty$, $i = 1,2,\cdots,n-2$,

where $w_i(t,u)$ is defined by

$$w_i(t,u) := \int_u^t r_1(s_1)\int_u^{s_1} r_2(s_2)\cdots\int_u^{s_{i-1}} r_i(s_i)ds_i\cdots ds_2ds_1.$$

Theorem 1. Let

$$\int^\infty w_{n-1}(t)p(t)dt < \infty \tag{2.2}$$

$$\int^\infty |h(t)|dt < \infty \tag{2.3}$$

hold. If $y(t)$ is a solution of (2.1), then $y(g(t)) = O(w_{n-1}(t,T))$ for some $T \ge 0$.

Proof. Let $y(t)$ be a solution of (2.1) on an interval $[T_0,\infty)$, $T_0 \ge 0$. It follows from (ii) and (iii) that there exist a $T \ge T_0$ and a positive constant m such that

$$g(t) \ge T_0 \quad \text{for} \quad t \ge T$$

and

$$\inf_{t\ge T} \frac{1}{r_0(t)} = \frac{1}{m}.$$

By (iii), there is a positive constant c such that

$$w_i(t,T) \le cw_{n-1}(t,T), \quad i = 1,2,\cdots,n-2.$$

Now a simple argument shows that

$$\frac{|y(g(t))|}{m} \le |L_0y(g(t))| \le \sum_{i=0}^{n-1} |L_iy(T)|w_i(g(t),T)$$

$$+\int_T^{g(t)} r_1(s_1)\int_T^{s_1} r_2(s_2)\cdots\int_T^{s_{n-2}} r_{n-1}(s_{n-1})\int_T^{s_{n-1}} |L_nY(s)|dsds_{n-1}\cdots ds_1$$

$$\leq cw_{n-1}(t,T)\sum_{i=0}^{n-1}|L_iy(T)|+w_{n-1}(t,T)\int_T^t |L_ny(s)|ds.$$

Hence

$$\frac{|y(g(t))|}{w_{n-1}(t,T)} \leq cm\sum_{i=0}^{n-1}|L_iy(T)|+m\int_T^t |h(s)|ds+m\int_T^t p(s)H(y(g(s)))ds$$

$$\leq M+m\int_T^t w_{n-1}(s,T)p(s)H\left(\frac{|y(g(s))|}{w_{n-1}(s,T)}\right)ds,$$

where

$$M := cm\sum_{i=0}^{n-1}|L_iy(T)|+m\int_T^\infty |h(s)|ds.$$

By Bihari's inequality [4] or LaSalle's inequality [5] we have

$$\frac{|y(g(t))|}{w_{n-1}(t,T)} \leq G^{-1}\left(G(M)+\int_T^t w_{n-1}(s,T)p(s)ds\right),$$

where $G(x) := \int_T^x \frac{dt}{H(t)}$ and $G^{-1}(x)$ is the inverse function of $G(x)$. This and (2.2) imply $\frac{|y(g(t))|}{w_{n-1}(t,T)}$ is bounded. This completes the proof.

Remark 1. For $n = 2$, $r_0(t) = r_1(t) = 1$ and $g(t) = t$, Theorem 1 improves Tong's result [1].

Remark 2. For $H(u) = |u|^r$, where $r \in (0,1]$, Theorem 1 improves the results of Chen and Yeh [2, Theorem 1] and Singh and Kusano [3, Theorem 1] which require the condition

$$\int^\infty r_i(t)dt = \infty, \quad \text{for } i = 1,2,\cdots,n-1.$$

Using Theorem 1, we can prove the following theorem which extends Theorem 3 of Philos [6].

Theorem 2. Let (2.2) and (2.3) hold. Assume that for some $T \geq 0$

$$(2.4)\quad \int_T^\infty r_1(s_1)\int_{s_1}^\infty r_2(s_2)\cdots\int_{s_{n-2}}^\infty r_{n-1}(s_{n-1})\int_{s_{n-1}}^\infty p(s)H(cw_{n-1}(s,T))ds\,ds_{n-1}\cdots ds_1 < \infty$$

for any constant $c > 0$, and

$$(2.5)\quad \int_T^\infty r_1(s_1)\int_{s_1}^\infty r_2(s_2)\cdots\int_{s_{n-2}}^\infty r_{n-1}(s_{n-1})\int_{s_{n-1}}^\infty |h(s)|ds\,ds_{n-1}\cdots ds_1 < \infty$$

hold. Then every oscillatory solution $y(t)$ of (2.1) satisfies

$$\lim_{t\to\infty} L_iy(t) = 0 \quad \text{for } i = 1,2,\cdots,n-1.$$

The proof of Theorem 2 is essentially the same as that of Theorem 3 in [6], so we omit the details.

Example 1. The differential equation

$$(ty'(t))'+\frac{1}{t}y(t) = \frac{2}{t^2}, \quad t \geq 1$$

has an oscillatory solution $y(t) = \frac{1}{t}+\sin(\ln t)$ but $\lim_{t\to\infty} y(t)$ does not exist. In this example, condition (2.2) and (2.4) are not satisfied, while (2.3) and (2.5) are valid.

Example 2. Consider the differential equation

$$(e^{-t}y')''+e^{-3t-\pi}y(t-\pi) = e^{-2t}[\sin t+7\cos t-e^{-2t}\sin t],$$

for $t \geq 0$. All conditions of Theorem 2 are satisfied. It has $y(t) = e^{-t}\sin t$ as an oscullatory solution which approaches zero as $t \to \infty$.

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