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RECENT DEVELOPMENTS ON THE STIELTJES TRANSFORM OF GENERALIZED FUNCTIONS

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ABSTRACT. This paper is concerned with recent developments on the Stieltjes transform of generalized functions. Sections 1 and 2 give a very brief introduction to the subject and the Stieltjes transform of ordinary functions with an emphasis to the inversion theorems. The Stieltjes transform of generalized functions is described in section 3 with a special attention to the inversion theorems of this transform. Sections 4 and 5 deal with the adjoint and kernel methods used for the development of the Stieltjes transform of generalized functions. The real and complex inversion theorems are discussed in sections 6 and 7. The Poisson transform of generalized functions, the iteration of the Laplace transform and the iterated Stieltjes trans-from are included in sections 8, 9 and 10. The Stieltjes transforms of different orders and the fractional order integration and further generalizations of the Stieltjes transform are discussed in sections 11 and 12. Sections 13, 14 and 15 are devoted to Abelian theorems, initial-value and final-value results. Some applications of the Stieltjes transforms are discussed in section 16. The final section deals with some open questions and unsolved problems. Many important and recent references are listed at the end.

KEY WORDS AND PHRASES. Stieltjes transforms of ordinary and generalized functions, Real and complex inversion theorems, Kernel and adjoint methods, Poisson transforms, Laplace transforms, fractional order integration, Weyl fractional integral, hyper-geometric transform, Abelian theorems, Final-value and initial-value results.
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1. INTRODUCTION.

The development and extension to generalized functions of the Fourier transform became a remarkably powerful tool in the theory of partial differential equations. Following these developments, the theory and applications of the integral transforms of generalized functions have been an active research area for the last thirty years. This period has produced major advances in the extension to generalized functions of many different integral transformations. Since the various types of such transformations are numerous, it is hardly possible to review all such transformations in one expository and survey article. So the purpose of this paper is to discuss and present recent developments of the Stieltjes transform of generalized functions and its applications. Our primary objective, therefore, is not concerned with proofs of many results and theorems but rather to provide researchers with the fundamental concepts, the underlying principles and various results and theorems.

2. STIELTJES TRANSFORM OR ORDINARY FUNCTIONS.

The Stieltjes transform F(x) of an ordinary function f(t) with $(1+t)^{-1} f(t) \in L(0,\infty)$ is defined by

$$F(x) = S[f(t)] = \int_0^\infty (x+t)^{-1} f(t)dt \quad (x > 0). \tag{2.1}$$

This transform (2.1) arises as an iteration of the Laplace transform, i.e., if

$$F(x) = \int_0^\infty e^{-xy} \psi(y)dy \tag{2.2}$$

where

$$\psi(y) = \int_0^\infty e^{-yt} f(t)dt, \tag{2.3}$$

then F(x) is given by (2.1). For example

(i) $S[(t+a)^{-1}] = \frac{1}{a-x} \log |\frac{a}{x}|$ provided $|\arg a| < \pi$ and $|\arg x| < \pi$.

(ii) $S[t^{-\frac{1}{2}} e^{-at}] = \pi x^{-\frac{1}{2}} e^{ax} \operatorname{erfc}(\sqrt{ax})$ where erfc(y) is the usual complementary function defined in a book by Myint-U and Debnath (1987).

Real and complex inversion formulae and several other results for the Stieltjes transform are given by Widder (1946). Widder distinguished between "complex" inversion formulae, which make use of values of f at points off the real-axis, and "real" inversion formulae, which use only values of f on the positive real axis.

Let p be any complex number except zero and the negative integers. Then for all s in the region Ω , where Ω is the s-plane cut from the origin along the negative real axis, the Stieltjes transform in its general form is defined by

$$F(s):= S_s[F(t)] := \int_0^\infty \frac{f(t)}{(s+t)^p}\, dt, \tag{2.4}$$

For real and positive values of p Pollard (1942) defined the operator

$$L^p_{k,t}[F(t)]: = \frac{(-1)^{k-1}(2k-1)!\, \Gamma(p)}{k!\,(k-2)!\,\Gamma(2k+p+1)} [t^{2k+p-2} F^{(k-1)}(t)]^{(k)} \tag{2.5}$$

and gave the real inversion formula

$$\lim_{k\to\infty} L^p_{k,t}[F(t)] = f(t), \tag{2.6}$$

If $L_{n,t}$ denotes the differential operator

$$L_{n,t} \equiv -tD \prod_{k=1}^{n} [1 - \frac{(tD)^{2k}}{4k^2}],\ D \equiv \frac{d}{dt} \tag{2.7ab}$$

and if

$$F(x) = \int_0^\infty \frac{f(t)dt}{x^2+t^2} \tag{2.8}$$

converges for some non-zero x, then if $f(t) \in L(R^{-1},R)$ for every positive $R > 1$,

$$L_{n,x}[F] \to f(t) \quad \text{almost everywhere, as } n \to \infty. \tag{2.9}$$

Love and Byrne (1980, p. 301) gave the following real inversion theorem:

If p is any complex number except zero or a negative integer, a and b are any integers, f is a function locally integrable on $(0,\infty)$, F is defined by

$$F(s) := \int_{\to 0}^{\to\infty} \frac{\Gamma(p)}{(s+t)^p} f(t)dt, \tag{2.10}$$

x is positive and $f(x\pm)$ exist in the sense that

$$f(x\pm) = \lim_{\eta\to 0+} \frac{1}{\eta} \int_0^\eta f(x\pm t)dt, \tag{2.11}$$

then

$$\frac{D^{n+b}\, x^{2n+a+b+p-1}\, (-D)^{n+a}\, F(x)}{\Gamma(n+a+p)\ \Gamma(n+b)} \to \tfrac{1}{2}\, [f(x+) + f(x-)] \tag{2.12}$$

as $n \to \infty$ through positive integral values.

Result (2.12) reduces to Widder's inversion formula (1946, p. 348) for $p = 1$, $a = b = -1$.

The next real inversion formula is due to Love and Byrne (1982, p. 281).

If p is any complex number except zero and the negative integers, a and b are any integers, f is a complex valued function locally integrable in $(0,\infty)$, F is defined by the convergent improper integral (2.10), x is positive and $f(x\pm)$ exist in the sence of (2.11), then as $n \to \infty$ through integral values

$$\frac{2^{2n+a+b+p-1}}{\Gamma(2n+a+b+p)}\ D^{n+b}\, x^{\frac{1}{2}(2n+a+b+p)}\, (-D)^{n+a}\, F(x) \to \tfrac{1}{2}\{f(x+0) + f(x-0)\}. \tag{2.13}$$

Several other real inversion formulae for the Stieltjes transform involving Stieltjes integrals can be found in papers by Love and Byrne (1980, 1982).

Summer (1949) defined the complex inversion operator $M_{\eta t}$ by

$$M_{\eta t}[F]\ (t) := -\frac{1}{2\pi i} \int_{C_{\eta t}} (z+t)^{p-1}\, F'(z)dz \tag{2.14}$$

where $C_{\eta t}$ is the contour which starts at the point $-t-i\eta$, proceeds along the straight line $\operatorname{Im}(z) = -\eta$ to the points $-i\eta$, then along the semicircle $|z| = \eta$, $\operatorname{Re} z \geq 0$, to the point $i\eta$, and finally along the line $\operatorname{Im}(z) = \eta$ to the point $-t+i\eta$.

Summer showed that if $f \in L(0 \leq u \leq R)$ for all positive R and is such that (2.4) converges, then

$$\lim_{\eta\to 0+} M_{\eta t}(F)(t) = \tfrac{1}{2}[f(t+) + f(t-)] \tag{2.15}$$

for any positive t at which both $f(t+)$ and $f(t-)$ exist.

Byrne and Love (1974) gave the following complex inversion formulae for (2.4):

If Re $p > 1$, f is locally integrable in $[0,\infty)$, improper Lebesgue integral (2.4) converges, and $\lambda > 0$; then, for each positive x for which the Lebesgue limits $f(x \pm 0)$ exist

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\} = \lim_{\eta\to 0+} \frac{p-1}{2\pi i} \int_{-x}^{\lambda} (x+t)^{p-2} \{F(t-i\eta)-F(t+i\eta)\}dt. \tag{2.16}$$

If Re $p > 1$, $\frac{f(t)}{1+t^2} \in L(0,\infty)$ and the improper Lebesgue integral (2.4) converges, then for each positive x for which the Lebesgue limits $f(x \pm 0)$ exist,

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\} = \lim_{\eta\to 0+} \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} \{F(t-i\eta)-F(t+i\eta)\}dt. \tag{2.17}$$

Several other complex inversion formulas for the generalized Stieltjes transform of ordinary functions have been obtained by Byrne and Love (1974).

Zemanian (1968) extended (2.1) as a special case of his theory of general convolution transform with positive real values of the Stieltjes transform variable x. Replacing x with $z = x+iy$, we can extend the definition of (2.1) and the integral (2.1) can be written as $\langle f,\psi(t,z)\rangle$ where $\psi(t,z) = (z+t)^{-1}$.

Finally, we state the following operational properties:

(a) $S[f(at)] = F(ax)$, $S[t^{-1}f(\frac{a}{t})] = x^{-1}\ F(\frac{a}{x})$

(b) $S[f'(t)] = -[x^{-1}\ f(0) + F'(x)]$.

3. THE STIELTJES TRANSFORM OF GENERALIZED FUNCTIONS.

To introduce the Stieltjes transform of generalized functions, Zemanian (1968, §4.2) defined the testing function space $M_{a,b}$ which is the space of all smooth complex-valued function $\theta(x)$ on I $(0 < x < \infty)$. According to him, for each $x > 0$, $\psi(t,x)$ belongs to $M_{a,b}$ provided $a \leqq 1$ and $b \geqq 0$. So the Stieltjes transform can be defined for the elements of $M'_{a,b}$ (which are generalized functions but they are not distributions in the sense of Zemanian (1968, p. 39)). The space $M'_{a,b}$ is the dual of $M_{a,b}$ and it is a linear space to which we assign the usual (weak) topology.

Pandey (1972) further extended the theory of the Stieltjes transform of generalized functions which belong to the dual $S'_\alpha(I)$ of the testing function space $S_\alpha(I)$. The $S_\alpha(I)$ is the space of all infinitely differentiable complex-valued function $\phi(x)$ defined over I where

$$\gamma_k(\phi) = \sup_{0<x<\infty} (1+x)^{\alpha}\ |(xD)^k\phi(x)| < \infty,\ D \equiv \frac{d}{dx} \tag{3.1}$$

for any fixed k (k=0,1,2,3,...) and $\alpha(\leqq 1)$ is a given real number. Clearly, $S_\alpha(I)$ is a vector(linear) space over the field of complex numbers. The topology over $S_\alpha(I)$ is generated by the semi-norms $\{\gamma_k(\phi)\}_{k=0}^{\infty}$. It can be shown that $S_\alpha(I)$ is a locally convex Hausdorff topological vector space. Obviously, $D(I) \subset S_\alpha(I)$ where $D(I)$ is the testing function space of infinitely differentiable functions with compact supports in I. The dual $D'(I)$ of $D(I)$ represents the space of distributions defined over the testing function space $D(I)$. The topology of $D(I)$ is such that it makes the dual space $D'(I)$ of Schwartz distributions. Furthermore, the topology of $D(I)$ is stronger than that induced on $D(I)$ by $S_\alpha(I)$, and as such the restriction of any element of $S_\alpha(I)$ to $D(I)$ is in $D'(I)$. Finally, it can be shown that $S\ (I)$ is a sequentially complete space.

Another testing function space $S_{\underline{\alpha}}(I)$ is the space of all infinitely differentiable complex-valued function $\psi(x)$ defined over I provided

$$\rho_k(\psi) = \sup_{0<x<\infty} \left|(1+x)^\alpha (xD)^k \psi(x)\right| < \infty \tag{3.2}$$

for all $k=0,1,2,3,\ldots$ where α is a fixed real number. This space has properties similar to those of $S_\alpha(I)$.

Pandey (1972) introduced the Stieltjes transform $F(z)$ of generalized function $f(t) \in S'_\alpha(I)$ by

$$F(z) = \langle f(t), (z+t)^{-1} \rangle \tag{3.3}$$

for z lying in the complex plane with a cut along the negative real axis.

He proved several results including

(i) $$F^{(k)}(z) = \langle f(t), \frac{(-1)^k k!}{(z+t)^{k+1}} \rangle \tag{3.4}$$

where $k=1,2,3,\ldots$ and with a cut along the negative real axis of the complex z-plane.

(ii) complex inversion formula: For an arbitrary element $\phi(x)$ in $D(I)$,

$$\langle \frac{1}{2\pi i} [F(-\xi-i\eta)-F(-\xi+i\eta)], \phi(\xi)\rangle \to \langle f,g\rangle \text{ as } \eta \to 0+ , \tag{3.5}$$

(iii) For a fixed $\alpha \leqq 1$ and $x > 0$, and for an arbitrary element $\phi(x)$ of $D(I)$,

$$\langle L_{k,x}[F(x)], \phi(x)\rangle \to \langle f,\phi\rangle \quad \text{as} \quad k \to \infty, \tag{3.6}$$

where $F(x)$ is the Stieltjes transform of $f(t)$ defined by (3.3) and

$$L_{k,x}\, \psi(x) = \frac{(-x)^{k-1}}{k!\,(k-2)!} \frac{d^{2k-1}}{dx^{2k-1}} [x^k \psi(x)], \tag{3.7}$$

and $\psi(x) \in S'_\alpha(I)$ and the differentiation in (3.7) is assumed to be in the distributional sense.

(iv) A transform of generalized functions related to the Poisson integral: For a fixed $\alpha \leqq 1$ and for $\phi(x)$ in $D(I)$,

$$\langle L_{n,x} F(x), \phi(x)\rangle \to \langle f(t), \phi(t)\rangle \quad \text{as} \quad n \to \infty, \tag{3.8}$$

where $F(x) = \langle f(t), t(x^2+t^2)^{-1}\rangle$ and the operator $L_{n,x}$ is given by (2.7ab).

For proofs, the reader is referred to Pandey (1972).

Pathak (1976) and Erdelyi (1977) extended (2.4) to generalized functions. The former author used kernel method which states that the kernel of the transform is embedded in a testing functin space, whereupon a numerical-valued generalized transform is defined by the action on the kernel of an element of the dual space. This method allows him to determine the real inversion formula (2.6) and the complex invertion formula (2.14).

On the other hand, Erdelyi's (1977) work was devoted to an extension of the Stieltjes transform to generalized function both by kernel and adjoint methods. Tiwari (1976) has extended (2.10) to generalized functions and established complex inversion formulas (2.16) and (2.17) in the distributional sense. In what follows we shall discuss extensions of real and complex inversion formulas for (2.10) for real and complex values of p by both kernel and adjoint methods.

4. THE ADJOINT METHOD.

Following Erdeyi (1977) we introduce the test function spaces $M_{a,b}$ and M(a,b) for real numbers a and b. The space $M_{a,b}$ is defined by

$$M_{a,b} = \{\phi \in \dot{C}^{\infty}(0,\infty): \ \mu_{a,b,k}(\phi) < \infty \ ; \ k=0,1,2,3,\ldots \} \tag{4.1}$$

where

$$\mu_{a,b,k}(\phi) = \sup_{0<t<\infty} \{t^{1-a+k}(1+t)^{a-b}|\phi^{(k)}(t)|\} \tag{4.2}$$

which are semi-norms and $\mu_{a,b,0}$ and is a norm. The topology over $M_{a,b}$ is generated by $\{\mu_{a,b,k}\}_{k=0}^{\infty}$, and the space $M_{a,b}$ is a complete multinormal space (Zemanian (1968), §4.2).

For $c \geqq a$, $d \leqq b$ we have $M_{a,b} \supset M_{c,d}$, and the topology of $M_{c,d}$ is stronger than that induced on it by $M_{a,b}$. However, if $c > a$ or $d < b$, then $M_{c,d}$ is not dense in $M_{a,b}$ (Zemanian (1968). The elements of the dual space $M'_{a,b}$ are generalized functions.

Then the space M(a,b) is defined as the countable union space in the sense of Zemanian (1968, §4.2) of all spaces M_{a_n,b_n} where $\{a_n\}_1^{\infty}$, $\{b_n\}_1^{\infty}$ are real sequences with $a_n \downarrow a$, $b_n \uparrow b$ as $n \to \infty$ and $-\infty \leqq a < \infty$, $-\infty < b \leqq \infty$. Thus

$$M(a,b) = \bigcup_{n=1}^{\infty} M_{a_n,b_n}$$

is the space of all smooth functions belonging to $M_{c,d}$ for $c > a$ and $d < b$. If $c \geqq a$ and $d \leqq b$, then $M(c,d) \subset M(a,b)$ and convergence in M(c,d) is stronger than that in M(a,b). Also, D(I), the Schwartz test function space of smooth function of compact support is dense in M(a,b) for any a and b. M(c,d) is dense in M(a,b). The dual of M(a,b) is denoted by M'(a,b).

Let $a \leqq 1$, $b \leqq 1-p$ where $p \in \mathbb{R}$. Then for $z \in \mathbb{C} - (-\infty,0)$, $(z+t)^{-p} \in M_{a,b}$ and a numerical-valued Stieltjes transform of $f \in M'_{a,b}$ is defined by

$$F(z) := \hat{f}(z) := \langle f(t), (z+t)^{-1} \rangle . \tag{4.3}$$

It can be shown that F is an analytic function of z in the topology of $M_{a,b}$ and

$$F^{(n)}(z) = (-1)^n (p)_n \langle f(t), (z+t)^{-p-n} \rangle . \tag{4.4}$$

Using the boundedness property of generalized functions it can be further shown that

$$F^{(n)}(z) = 0\,(|z|^{\min(0,1-a-p-n)}\,(1+|z|)^{\max((a,1-p-n)-\min(b,1))} \tag{4.5}$$

uniformly in any sector $0 < |z| < \infty$, $|\arg z| \leqq \pi-\delta$, $\delta > 0$.

Restricting z to positive real values, we see from (4.4) that

$$F \in M_{\min(1,2-a-p),\ \max(1-p,2-b-p)} \tag{4.6}$$

and also $F \in M'_{(\max(0,a+p-1),\ \min(p,b+p-1))}$. (4.7)

Let $\alpha > 0$, $\beta < p$ then it has been shown by Erdélyi that the Stieltjes transform (2.4) maps $M_{\alpha,\beta}$ continuously in $M_{a,b}$ provided that

$$\begin{aligned} &a \leq 1,\ a \leq 1+\alpha-p \quad \text{and} \quad a < 1 \quad \text{if} \quad \alpha = p, \\ &b \geq 1-p,\ b \geq 1+\beta-p \quad \text{and} \quad b > 1-p \quad \text{if} \quad \beta = 0, \end{aligned} \tag{4.8ab}$$

If $\alpha \geq 0$, $\beta \leq p$, then the Stieltjes transform maps also $M(\alpha,\beta)$ continuously into $M_{1+\min(0,\alpha-p),1-p+\max(0,\beta)}$.

Now, the Stieltjes transform $\hat{f}$ of the generalized function $f \in M'_{a,b}$ with $a \leq 1$, $b \geq 1-p$ is defined by

$$< \hat{f},\phi > = < f,\hat{\phi} >, \tag{4.9}$$

where ϕ $M_{\alpha,\beta}$ with $\alpha > 0$, $\alpha \geq a+p-1$ and $\alpha > p$ if $a = 1$; $\beta < p$, $\beta \leq b+p-1$ and $\beta < 0$ if $b = 1-p$ and $\hat{\phi} \in M_{a,b}$. Since each element of $M(\max(0,a+p-1), \min(p,b+p-1))$ lies in some $M_{\alpha,\beta}$ with (4.9) satisfied, we see that the Stieltjes transformation maps $M'_{a,b}$ with $a \leq 1$, $b \geq 1-p$ continuously into $M'(\max(0,a+p-1), \min(p,b+p-1)$. Furthermore, the Stieltjes transformation maps also $M'(a,b)$ with $a < 1$, $b > 1-p$ continuously into $M'(\max(0, a+p-1), \min(p,b+p-1))$.

Let

$$L_{m,n,p,x} : = \frac{(-1)^n\ \Gamma(p)}{m!\ \Gamma(n+p-1)} \left(\frac{d}{dx}\right)^m x^{m+n+p-1} \left(\frac{d}{dx}\right)^n \tag{4.10}$$

and let the generalized Stieltjes transform of $f \in M'_{a,b}$ be defined by (4.3) for $z = x \geq 0$. Then using (4.9) Erdélyi (1977) proved the following inversion formulae

$$< L_{m+r,n+r;p,x}\ \hat{f}(x),\ \phi(x) > \to < f,\phi > \tag{4.11}$$

as $r \to \infty$ for each $\phi \in M_{a,b}$ provided $\alpha > 1-p$, $\alpha \geq a$, and $\alpha > 1$ with $a = 1$, $a \leq 1$, $b \geq 1-p$, $\beta < 1$, $\beta \leq b$, and $\beta < 1-p$ if $b = 1-p$; and

$$< (L_{m+r,n+r,p,x}\ f\hat{)}\ (x),\ \phi(x) > \to < f,\phi > \tag{4.12}$$

as $r \to \infty$ for each $\phi \in M_{a,b}$ provided $\alpha > 0$, $\alpha \geq a$, $\alpha > p$ if $a = p$, $a \leq p$, $b \geq 0$, $\beta < p$, $\beta \leq b$, $\beta < 0$ if $b = 0$.

Erdelyi (1977) makes a further investigation for (4.3) in $M'(a,b)$ and combines it with his fractional calculus (1975) to deal with a hypergeometric integral equation which has the form

$$\int_0^\infty t^{-b}\ {}_2F_1(a,b,c;\ -\frac{x}{t})\ \phi(t)dt = \psi(x) \tag{4.13}$$

where ${}_2F_1$ is the standard Gauss hypergeometric function.

Love (1975) investigated this equation classically and shows how the operator on the left is expressible in terms of fractional integrals and the Stieltjes transform of ordinary functions. It is not difficult to estend Love's study to generalized functions using Erdelyi's theory.

5. THE KERNEL METHOD.

In what follows extension of (2.4) will be given by the kernel method. And inversion formulas (2.6) and (2.15)-(2.17) will be established in the distributional sense. The analysis is based upon the works of Pathak (1976) and Tiwari (1976).

It has been shown by Tiwari (1976) that for a complex s except negative integers or zero, $(s+x)^{-p}$ belongs $S_\alpha(I)$ where $\alpha \leqq \mathrm{Re}(p)$. Hence, for $\alpha < \mathrm{Re}(p)$, the Stieltjes transform, F(s) of a generalized function $f(x) \in S'_\alpha(I)$ can be defined by

$$F(s) := S_s[f(x)] := \langle f(x), (s+x)^{-p} \rangle \tag{5.1}$$

where s belongs to the complex plane cut along the negative real axis including the origin.

It can be shwon that F(s) is an analytic function of s and that

$$F^{(m)}(s) = \left(\frac{d}{ds}\right)^m F(s) = \quad f(x), \frac{(-1)^m (p)_m}{(s+x)^m} \quad , \tag{5.2}$$

where $(p)_m = p(p+1)(p+2) \ldots (p+m-1)$.

The function $F^{(m)}(x)$ for real x, where F(s) is the Stieltjes transform of $f \in S'(I)$, satisfies the following asymptotic properties:

$$F^{(m)}(x) = \left\{ \begin{array}{lll} O(x^{-m}) & \text{as } x \to \infty \ , & \text{if } \alpha < \mathrm{Re}\,(p) \\ O(x^{-k}) & \text{as } x \to \infty \ , & \text{if } \alpha = \mathrm{Re}\,(p) \\ O(x^{-k-\mathrm{Re}\ p}) & \text{as } x \to 0+ \ , & \text{if } \alpha \leqq \mathrm{Re}\,(p). \end{array} \right\} \tag{5.3}$$

6. COMPLEX INVERSION THEOREMS.

The following theorems, 6.1 and 6.2 are due to Tiwari (1976) and provide inversion of (5.1) for complex values of p. Their proofs are too technical; the interested reader may refer to Tiwari (1976, 1979).

THEOREM 6.1. For fixed $\alpha < 1$ and $\mathrm{Re}\, p > 1$, let $f \in S'(I)$ and let F(s) be the Stieltjes transform of f(t) as defined by (5.1). Then

$$\lim_{\eta\to 0} \left\langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2}[F(t-i\eta) - F(t+i\eta)]dt, \phi(x) \right\rangle$$

$$= \langle f,\phi \rangle \text{ for all } \phi \in D(I). \tag{6.1}$$

THEOREM 6.2. Let $\mathrm{Re}\, p > 1 > \alpha$ and f(t) $S'_\alpha(I)$. If F(s) is the Stieltjes transform of f(t) defined by (5.1), then for $\lambda > 0$ and each $\phi \in D(I)$,

$$\lim_{\eta\to 0+} \left\langle \frac{p-1}{2\pi i} \int_{-x}^{\lambda} [F(t-i\eta) - F(t+i\eta)]\,(x+t)^{p-2}\,dt, \phi(x)\right\rangle = \langle f,\phi \rangle . \tag{6.2}$$

The next theorem is due to Pathak (1976) and provides inversion of (5.1) for real values of $p > 0$.

THEOREM 6.3. Let $f \in \underline{S}'_{\alpha}(I)$ where $\alpha \leqq p$, $p > 0$ and let $F(s)$ be defined by (5.1). Then, for each $\phi \in D(I)$, we have

$$< -\frac{1}{2\pi i} \int_{C_{\eta t}} (z+t)^{p-1} F'(z)dz, \phi(t)> \to < f,\phi > \quad \text{as } \eta \to 0+ \tag{6.3}$$

where $C_{\eta t}$ is the same contour as in (2.14).

Outline of the Proof: The theorem is proved by justifying the following steps:

$$< -\frac{1}{2\pi i} \int_{C_{\eta t}} (z+t)^{p-1} F'(z)dz, \phi(t) > \tag{6.4}$$

$$= < \phi(t), \frac{1}{2\pi i} \int_{C_{\eta t}} (z+t)^{p-1} < f(x), \frac{p}{(z+x)^{p+1}} > dz > \tag{6.5}$$

$$= < \phi(t), < f(x), \frac{1}{2\pi i} \int_{C_{\eta t}} \frac{p}{(z+x)^{p+1}} (z+t)^{p-1} dz > > \tag{6.6}$$

$$= < f(x), < \phi(t), \frac{1}{2\pi i} \int_{C_{\eta t}} \frac{p(z+t)^{p-1}}{(z+x)^{p+1}} dz > > \tag{6.7}$$

$$\to < f(x), \phi(x) > \quad \text{as } \eta \to 0+ . \tag{6.8}$$

Since the integrand in (6.4) is analytic and are valued on Ω [Summer (1949), p. 178], the integral on $C_{\eta t}$, is an analytic function of t. Consequently, (6.4) has a meaning for $\phi \in D(I)$ and is, in fact, an ordinary integration of t. That (6.4) equals (6.5) is obvious in view of (5.2). The equality of (6.5) to (6.6) and also that of (6.6) to (6.7) can be proved by the technique of the Riemann sums.

To show that (6.7) $\to$ (6.8) as $\eta \to 0+$ we need the following lemma:

LEMMA 6.1. For $p > 0$, let

$$G(\eta;t,x) = \frac{\eta^p}{2\pi i} \frac{1}{t-x} \left[\frac{1}{\{\eta-i(t-x)\}^p} - \frac{1}{\{\eta+i(t-x)\}^p} \right].$$

Then

$$\lim_{\eta\to 0+} \int^{\infty} G(\eta;t,x)dx = 1.$$

LEMMA 6.2. For $\alpha \leqq p$ and $m = 0,1,2,\ldots,$

$$\sup_{0<x<\infty} \left|(1+x)^{\alpha} x^m \frac{d^m}{dx^m} \left[\int^{\infty} G(\eta;t,x) \phi(t)dt-\phi(x)\right]\right| \to 0 \text{ as } \eta \to 0+ ,$$

where $\phi \in D(I)$.

7. REAL INVERSION THEOREMS.

The following theorem provides an extension of the real inversion formula (1.6) to generlaized functions belonging to $\underline{S}'_{\alpha}$. Details of its proof can be found in the papers by Pathak (1976) and Erdelyi (1977).

THEOREM 7.1. For a fixed $\alpha \leqq p$, $p > 0$ and $x > 0$ let $F(x)$, the generalized Stieltjes transform of $f \in \underline{S}'_\alpha(I)$ be defined by (5.1). Then, for each $\phi \in D(I)$,

$$< L^p_{k,x} F(x), \phi(x) > \to < f, \phi > \quad \text{as } k \to \infty, \tag{7.1}$$

where the differentiation is supposed to be in the distributional sense.

<u>Outline of the Proof</u>: By a direct computation it follows taht

$$L^p_{k,x} F(x) = x^{p-1} P(x \frac{d}{dx}) F(x)$$

where $P(x)$ is a polynomial in x of finite degree depending upon k. For $\phi(x)$ $D(I)$, we have

$$< L^p_{k,x} F(x), \phi(x) = < x^{p-1} P(x \frac{d}{dx}) F(x), \phi(x) >$$

$$= < P(x \frac{d}{dx}) F(x), x^{p-1} \phi(x) > \tag{7.2}$$

$$= < F(x), P(-x \frac{d}{dx} - 1) x^{p-1} \phi(x) > \tag{7.3}$$

$$= < < f(t), \frac{1}{(x+t)^p} >, \zeta(x) > \tag{7.4}$$

$$= < f(t), < \frac{1}{(x+t)^p}, \zeta(x) >>, \tag{7.5}$$

where $\zeta(x) = P(-x \frac{d}{dx} - 1) x^{p-1} \phi(x)$.

The step (7.3) follows from (7.2) on integration by parts. The equality (7.4) and (7.5) can be proved by the technique of the Riemann sums. The proof is completed by showing that

$$< \zeta(x), \frac{1}{(x+t)^p} > \to \phi(t) \text{ in } \underline{S}_\alpha(I), \text{ as } k \to \infty . \tag{7.6}$$

8. THE POISSON TRANSFORM OF GENERALIZED FUNCTIONS.

For fixed $\alpha \leqq 1$, let $f \in \underline{S}'_\alpha(I)$. Then the Poisson transform of $f \in \underline{S}'_\alpha(I)$ is defined by

$$F(x) := < f(t), \frac{t}{x^2 + t^2} > . \tag{8.1}$$

The following theorem provides inversion of the Poisson transform by means of the differential operator

$$L_{n,x} = - \theta \prod_{k=1}^{n} (1 - \frac{\theta^{2k}}{4k}), \quad \theta = x \frac{d}{dx} . \tag{8.2}$$

THEOREM 8.1. If the Poisson transform of $f \in \underline{S}'_\alpha(I)$ be defined by (8.1), then, for each $\phi \in D(I)$

$$< L_{n,x} F(x), \phi(x) > \to < f, \phi > \quad \text{as } n \to \infty. \tag{8.3}$$

For a proof see Pandey (1972).

9. ITERATION OF THE LAPLACE TRANSFORM.

In this section, we show that the Stieltjes transform of a generalized function f in $\underline{S}'_\alpha(I)$ for fixed $\alpha < 1$ is obtained by iterating its Laplace transform. It can be easily seen that e^{-st} with Re $s > 0$ is an element of $\underline{S}_\alpha(I)$ for a fixed $\alpha < 1$ and consequently, its Laplace transform can be defined by

$$F(s) := \langle f(t), e^{-st} \rangle \tag{9.1}$$

which exists for $s > 0$.

Using boundedness property of generalized functions (Zemanian (1968)) it can be easily seen that

$$\begin{aligned} F(s) &= O(s^k) \quad \text{as} \quad s \to \infty \\ &= O(s^{-\alpha}) \quad \text{as} \quad s \to 0+, \end{aligned} \tag{9.2ab}$$

where k is a non-negative integer depending upon f.

THEOREM 9.1. Let α be a fixed number less than 1 and suppose that $s > 0$. Assume that $f \in \underline{S}'_\alpha(I)$. Then for fixed $y > 0$ we have

$$\langle \langle f(t), e^{-st} \rangle , e^{-sy} \rangle = \langle f(t), \frac{1}{t+y} \rangle . \tag{9.3}$$

PROOF. In view of the asymptotic orders of F(s) both sides of equations in (9.3) are meaningful. In order to prove (9.3) we need to show that

$$\int_0^\infty \langle f(t), e^{-st} \rangle e^{-sy} ds = \langle f(t), \int_0^N e^{-s(t+y)} ds \rangle . \tag{9.4}$$

By using the technique of the Riemann sums, we can easily show that

$$\int_0^N \langle f(t), e^{-st} \rangle e^{-sy} ds = \langle f(t), \int_0^N e^{-s(t+y)} ds \rangle . \tag{9.5}$$

Since $\int_N^\infty e^{-s(t+y)} dx \to 0$ in $\underline{S}_\alpha(I)$, as $N \to \infty$, (9.6)

we can easily justify taking limits $N \to \infty$ in (9.5) and obtain (9.4).

10. ITERATED STIELTJES TRANSFORM.

The iterated Stieltjes transform of a generalized function f is defined by

$$\tilde{\tilde{f}}(x) = \int_0^\infty \frac{du}{x+u} \int_0^\infty \frac{f(t)}{u+t} dt, \quad x > 0. \tag{10.1}$$

If it is permissable to change the order of integration in the above integral, we find

$$\tilde{\tilde{f}}(x) = \int_{0+}^\infty \frac{\log(x/t)}{x-t} f(t)dt, \tag{10.2}$$

where $\log(x/t)/(x-t)$ is defined by its limiting value $\frac{1}{x}$ at $t = x$. (10.2) is referred to as the S_2 - transform of the function f(t). The inversion formula for (10.2) due to Boas and Widder (1939, p. 30) is given by

$$\lim_{n\to\infty} H_{n,x} [\tilde{\tilde{f}}(x)] = f(x), \tag{10.3}$$

for almost all x 0, where, n=1,2,3,...,

$$H_{n,x}[\phi(x)] = \{\frac{1}{n!(n-2)}\}^2 \quad [x^{2n-1}\{x^{2n-1} \phi^{(n-1)}(x)\}^{2n-1}]^{(n)} \tag{10.4}$$

Taking p = 1 in (5.1) we see that the Stieltjes transform of f $\underline{S'_\alpha}(I)$ is defined by

$$G(s) = \langle f(t), \frac{1}{s+t} \rangle . \tag{10.5}$$

If G(u) is the Stieltjes transform of f for u > 0, it seems natural to define the iterated Stieltjes transform of f by

$$F(x) = \langle G(u), \frac{1}{x+u} \rangle \quad , \quad x > 0. \tag{10.6}$$

In order that the above definition be meaningful, G(u) must belong to the space $\underline{S'_\alpha}(I)$ as a regular generalized function. Since, in view of the estimate (5.3), $G(u) = 0(\frac{1}{u})$, as $u \to 0+$, $\int_0^\infty \frac{G(u)}{x+u}\, du$ does not exist in a neighborhood of zero. Therefore, the iterated Stieltjes transform of $f \in \underline{S'_\alpha}(I)$ may not be defined through (10.6). But the generalized S_2-transform of $f \in \underline{S'_\alpha}(I)$ can be defined by

$$F(x) = \langle f(t), K(t,x) \rangle \; , \quad x > 0, \tag{10.7}$$

where

$$K(t,x) = \begin{cases} \frac{\log(x/t)}{x-t} \; , & t \neq x \\ 1/x \; , & t = x \end{cases}. \tag{10.8ab}$$

Note that for each fixed x > 0, K(t,x) $\underline{S_\alpha}(I)$; hence (10.7) is meaningful . Moreover, it can be proved that F(x) is infinitely differentiable and for n=1,2,3,...

$$F^{(n)}(x) = \langle f(t), \frac{\partial^n}{\partial x^n} K(t,x) \rangle \; , \quad \text{for each} \quad x > 0 \tag{10.9}$$

Dube (1975) has proved the following inversion theorem:

THEOREM 10.1. Let f $\underline{S'_\alpha}(I)$, $0 < \alpha < 1$, and let F(x) be the generalized S_2-transform of f defined by (10.7). Then, for each $\phi \in D(I)$,

$$\langle H_{n,x} F(x), \phi(x) \rangle \to \langle f, \phi \rangle \quad \text{as} \quad n \to \infty, \tag{10.10}$$

where $H_{n,x}$ is the operator defined by (10.4) and the differentiation is assumed to be in the distributional sense.

11. STIELTJES TRANSFORMS OF DIFFERENT ORDERS AND INTEGRATION OF FRACTIONAL ORDERS.

The Stieltjes transformations of different orders can be connected by integration of fractional order.

For $\phi \in M_{\alpha,\beta}$ with $\alpha > 0$,

$$I^\lambda \phi(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-y)^{\lambda-1} \phi(y)dy \tag{11.1}$$

defines the Riemann-Liouville integral of order λ if Re $\lambda > 0$ (Debnath, 1978). It is easily seen taht $I^\lambda \phi \in C^\infty (0,\infty)$ and that

$$I^{\lambda}\phi = (I^{\lambda+\ell}\phi)^{(\ell)}, \quad \ell = 0,1,2,... \tag{11.2}$$

when $\operatorname{Re}\lambda > 0$. (11.2) is used to extend the definition of I^{λ} to all complex λ. It follows that $I^{\lambda}\phi$ is an entire function of λ.

Erdélyi (1976) has shown that I^{λ} maps $M_{\alpha,\beta}$ with $\alpha > 0$ continuously into $M_{a,b}$ provided $a \leqq \alpha + \operatorname{Re}\lambda$;

$$\left.\begin{array}{ll} b \geqq \max(0,\beta) + \operatorname{Re}\lambda & \text{if } \beta \neq 0, \\ b > \operatorname{Re}\lambda & \text{if } \beta = 0. \end{array}\right\} \tag{11.3ab}$$

Further, I^{λ} maps $M(\alpha,\beta)$ with $\alpha \geqq 0$ continuously into $M_{\alpha+\operatorname{Re}\lambda,\, \max(0,\beta)+\operatorname{Re}\lambda}$.

Let f be an ordinary function on $(0,\infty)$ such that $t^{a-1}(1+t)^{b-a} f(t) \in L(0,\infty)$ for some a,b satisfying (11.3ab) and let $\operatorname{Re}\lambda > 0$. Then

$$K^{\lambda}f(x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (y-x)^{\lambda-1} f(y)dy \tag{11.4}$$

defines the so-called Weyl fractional integral of order of f. By a simple application of Fubini's theorem, we have

$$\int_0^{\infty} (K^{\lambda}f)\phi\, dx = \int_0^{\infty} f(I^{\lambda}\phi)dx, \tag{11.5}$$

This result motivates the definition

$$\langle K^{\lambda} f, \phi \rangle := \langle f, I^{\lambda}\phi \rangle, \text{ for all } \phi \in M_{\alpha,\beta}, \tag{11.6}$$

where $f \in M'_{a,b}$, $\alpha > 0$ and a,b satisfy (11.3ab). This K^{λ} thus defined maps $M'_{a,b}$ continuously into $M'_{\alpha,\beta}$. Given a,b with $b > \operatorname{Re}\lambda$, we may take any α,β satisfying $\alpha > 0$, $\alpha + \operatorname{Re}\lambda \geqq a$;

$$\beta + \operatorname{Re}\lambda \leqq b \text{ if } b > \operatorname{Re}\lambda; \; \beta < 0 \text{ if } b = \operatorname{Re}\lambda. \tag{11.7}$$

Moreover, if in (11.6) $\phi \in M(\max(0, a-\operatorname{Re}\lambda), b-\operatorname{Re}\lambda)$, then it can be seen that K^{λ} maps $M'_{a,b}$ with $b \geqq \operatorname{Re}\lambda$ into $M'(\max(0,a-\operatorname{Re}\lambda), b-\operatorname{Re}\lambda)$.

If ϕ $M_{\alpha,\beta}$ with $\beta + \operatorname{Re}\lambda < 1$, them (11.4) defines $K^{\lambda}\phi$ for $\operatorname{Re}\lambda > 0$, and this definition can be extended for all complex values of λ by

$$K^{\lambda}\phi = (-1)^{\ell} K^{\lambda+\ell} \phi^{(\ell)}, \tag{11.8}$$

where ℓ is a non-negative integer with $\operatorname{Re}\lambda + \ell > 0$.

Erdélyi (1976) has shown that K^{λ} maps $M_{\alpha,\beta}$ with $\beta + \operatorname{Re}\lambda < 1$ continuously into $M_{a,b}$ provided that

$$a \leqq \min(1,\alpha+\operatorname{Re}\lambda) \text{ and } a < 1 \text{ if } \alpha+\operatorname{Re}\lambda = 1 : b \geqq \beta+\operatorname{Re}\lambda. \tag{11.9}$$

Also, K^{λ} maps $M(\alpha,\beta)$ with $\beta+\operatorname{Re}\lambda \leqq 1$ continuously into $M_{\min(1,\alpha+\operatorname{Re}\lambda),\, \beta+\operatorname{Re}\lambda}$. Let $f \in M'_{a,b}$ where a,b satisfy (11.9). Then

$$\langle I^{\lambda} f, \phi \rangle = \langle f, K^{\lambda}\phi \rangle, \quad \phi \in M_{\alpha,\beta}, \tag{11.10}$$

represents a continuous linear mapping I^{λ} of $M'_{a,b}$ into $M'_{\alpha,\beta}$ if $\beta+\mathrm{Re}\,\lambda < 1$. Given a,b with $a \leqq 1$, we may take α and β satisfying

$$\alpha+\mathrm{Re}\ \lambda \geqq a,\ \alpha+\mathrm{Re}\ \lambda > 1 \quad \text{if} \quad a=1;\ \beta+\mathrm{Re}\,\lambda < 1,\ \beta+\mathrm{Re}\,\lambda < b. \tag{11.11}$$

Also, I^{λ} maps $M'_{a,b}$ with $a \leqq 1$ into $M'(a-\mathrm{Re}\,\lambda\ ,\ \min(1,b)-\mathrm{Re}\,\lambda)$.

The following addition formula also holds:

$$K^{\lambda}(K^{\mu}\phi) = K^{\lambda+\mu}\phi \text{ for all } \phi \in M_{\alpha,\beta}\ ;\ \beta+\mathrm{Re}\mu < 1,\ \beta+\mathrm{Re}(\lambda+\mu) < 1. \tag{11.12}$$

Notice that K^{λ} and K^{μ} commute, if in addition to (11.12), we have $\beta+\mathrm{Re}\,\lambda < 1$. Furthermore, $K^{-\lambda}K^{\lambda}\phi = \phi$ for $\phi \quad M_{\alpha,\beta}$ if $\beta < 1$ and $\beta+\mathrm{Re}\,\lambda < 1$ but K^{λ} and $K^{-\lambda}$ are (two-sided) inverses on $M_{\alpha,\beta}$ if $\beta+|\mathrm{Re}\,\lambda| < 1$.

Let us write

$$\psi_{p,s}(t) = (s+t)^{-p}.$$

Then applying (11.4) and integrating we get

$$\Gamma(p)\ K^{\lambda} \quad \psi_{p,s} = \Gamma(p-\lambda)\ \psi_{p-\lambda,s}, \tag{11.13}$$

for $\lambda > 0$. This result is extended for all $\lambda < p$ by means of (11.8).

If $\hat{\phi}_p$ denote the integral

$$\hat{\phi}_p(s) = \int_0^{\infty} \psi_{p,s}(t)\ \phi(t)dt = \int_0^{\infty} \frac{\phi(t)}{(s+t)^p}\ dt$$

then the formula (11.13) suggests that

$$\Gamma(p)\ \hat{\phi}_p/\ \Gamma(q) = K^{q-p}\hat{\phi}_q, \tag{11.14}$$

for $\phi \in M_{\alpha,\beta}$ provided $\alpha > 0$, $\beta < \min(p,q)$, $p > 0$.

A second transformation formula arising from (11.13) is

$$\Gamma(p)\ \hat{\phi}_p/\Gamma(q) = (I^{q-p}\,\phi\,)\hat{}_q \tag{11.15}$$

for $\phi \in M_{\alpha,\beta}$ provided that $\alpha > \max(0,\ p-q)$, $\beta < p$, $p > 0$.

If $\hat{f}_p$ denote the generalized Stieltjes transform of the generalized function f then we have

$$\Gamma(p) < \hat{f}_p,\quad \phi > /\Gamma(q) = \langle (I^{q-p}\ f)\hat{}_q,\quad \phi \rangle \tag{11.16}$$

for $f \quad M'_{a,b}$, $\phi \quad M_{\alpha,\beta}$ under the conditions

$$p > 0,\ a \leqq 1,\ \alpha \leqq 1-p+q,\ b \geqq 1-p,$$

$$\alpha > 0,\ \alpha \geqq a+p-1 \quad \text{and} \quad \alpha > p \quad \text{of} \quad a = 1,\ \alpha > q \quad \text{if} \quad a = 1-p+q$$

$$\beta < \min(p,q),\ \beta \leqq b+p-1 \quad \text{and} \quad \beta < 0 \quad \text{if} \quad b = 1-p.$$

Note that the conditions in regard to α and β are always satisfied if $\phi \in M(\max(0,a+p-1),\ \min(p,q,b+p-1))$.

Furthermore,

$$\Gamma(p) \langle \hat{f}_p, \phi \rangle / \Gamma(q) = \langle K^{q-p} \hat{f}_q, \phi \rangle \tag{11.17}$$

for $f \in M'_{a,b}$, $\phi \in M_{\alpha,\beta}$ under the conditions

$p > 0$, $a \leqq 1$, $b \geqq 1-\min(p,q)$

$\alpha > \max(0,p-q)$, $\alpha \geqq a+p-1$ and $\alpha > p$ if $a=1$

$\beta < p$, $\beta \leqq b+p-1$ and $\beta < 0$ if $b = 1-p$, $\beta < p-q$ if $b = 1-q$.

Note that the conditions on α and β are always satisfied if $M(\max(0,p-q, a+p-1), \min(p,b+p-1))$.

A proof of (8.14)-(8.17) can be found in (Erdélyi, 1977; pp. 243-244)).

12. FURTHER GENERALIZATIONS OF THE STIELTJES TRANSFORM.

The hypergeometric transform (generalized Stieltjes transform) of ordinary functions defined by

$$g(x) = \frac{\Gamma(b)}{\Gamma(c)} \int_0^\infty t^{-b} F(a,b;c;-\frac{x}{t}) f(t)dt, \tag{12.1}$$

where $F(a,b;c;x)$, the Gauss-hypergeometric function, was investigated by Love (1975). Note that (12.1) reduces to (2.4) for $a = c$ and $b = p$.

We write

$$\phi_x = \frac{\Gamma(b)}{\Gamma(c)} t^{-b} F(a,b;c;-\frac{x}{t}).$$

Then $\phi_x \in M_{\rho,\sigma}$ with $\rho \leqq 1+\min(0,a-b)$ and $\rho < 1$ if $a = b$, $\sigma \geqq 1-b$. We define the hypergeometric transform of $f \in M'_{\rho,\sigma}$ by

$$g(x) = \langle f, \phi_x \rangle . \tag{12.2}$$

Then $g(x)$ is infinitely differentiable and

$$g^{(n)}(x) = \langle f, \frac{d^n \phi_x}{dx^n} \rangle$$

and g $M_{\alpha,\beta}$ for any α,β satisfying $\alpha \leqq \min(1,2-p-b)$, $\beta \geqq \max(1-a,1-b,2-\sigma-b)$ if $a \neq b$, while in case $a = b$ the last inequality must be replaced by $\beta \geqq 1-a$, $\beta \geqq 2-\sigma-a$.

Now, we discuss connection of the hypergeometric transform with the Stieltjes transform (Erdélyi, 1977).

We define $\psi_{a,c,x} \in M_{1+a-c,1-c}$ by $\psi_{a,c,x}(t) = t^{a-c}(x+t)^{-a}$. Then, for $0 < b < c$, $K_t^{c-b} \psi_{z,c,x} = \phi_x$, so that (12.2) can be written as

$$\begin{aligned} g(x) &= \langle f, K_t^{c-b} \psi_{a,c,x} \rangle = \langle I^{c-b} f, \psi_{a,c,x} \rangle , \quad b > 0 \\ &= \langle t^{a-c}(I^{c-b}f)(t), (x+t)^{-a} \rangle , \quad b > 0. \end{aligned} \tag{12.3}$$

Having expressed g as a generalized Stieltjes transform, we can apply inversion formulae of Sections 7 and 8 to invert (12.3). Applying (4.11) we have

$$\langle I^{b-c}\, t^{c-a}\, L_{r+n,s+n,a,x}\, g(x), \phi(x)\rangle \to \langle f(t), \phi(t)\rangle \tag{12.4}$$

as $n \to \infty$ under the conditions:

$b > 0$, $c > 0$, $\rho < 1$, $\rho \leq 1+a-b$

$\rho \leq 1-b+c$, $\sigma \geq 1-b$, $f \quad M'_{\rho,\sigma}$

$\phi \quad M(\max(1-b,\rho), \min(1,1+a-b,1-b+c,\sigma))$.

Using the properties of hypergeometric functions Erdélyi (1977) gave the following alternative inversion formula for (12.2).

$$\langle L_{r+n,s+n,a,x}\, x^{1-a}\, I_x^{a-c}\, x^{c-1}\, g(x), \phi(x)\rangle \to \frac{\Gamma(b)}{\Gamma(c)} \langle f,\phi\rangle \tag{12.5}$$

as $n \to \infty$ under the conditions

$b > 0$, $c > 0$, $\rho \leq 1$, $\rho \leq 1+a-b$, $\rho \leq 1+a-c$, $\rho < 1$ if $a=b$, $\sigma = 1-b$

$f \in M'_{\rho,\sigma}$, $\phi \in M(\max(1-b,\rho), \min(1,1+a-b,1+a-c,\sigma))$.

Notice that (12.4) and (12.5) determine the restriction of $f \in M'_{\rho,\sigma}$ to $M(\rho,\sigma)$.

A generalization of the Stieltjes transform analogous to (12.1) introduced by Joshi (1977) is

$$F(x) = \frac{\Gamma(\beta+\eta+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^\infty \frac{y^\beta}{x^{\beta+1}} \times$$

$$\times\ {}_2F_1\left(\beta+\eta+1,\beta+1;\alpha+\beta+\eta+1; -\frac{y}{x}\right) f(y)\,dy \tag{12.6}$$

where $\beta \geq 0$, $\eta > 0$, and the corresponding complex inversion formula is

$$f(y) = \frac{1}{2\pi i} \lim_{\omega\to\infty} \int_{\sigma-i\omega}^{\sigma+i\omega} \frac{\Gamma(b+s-\beta-1)}{\Gamma(a+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma(s)}\, y^{-s} M(s)\,ds \tag{12.7}$$

where

$$M(s) = \int_0^\infty x^{s-1} F(x)\,dx,\ a = \beta+\eta+1,\ b = a+\alpha,\ s = \sigma+i\omega,\ \beta \geq 0,\ \eta > 0.$$

(12.6) reduces to (2.4) for $\alpha = 0$.

Tiwari (1980 extended (12.6) to a class of generalized functions and established the formula (12.7) in the distributional sense. For this extension Tiwari defined a test function space $J_{c,d,\alpha}$ as follows:

A complex-valued C^∞-function ψ on $(0,\infty)$ belongs to $J_{c,d,\alpha}$ if

$$i_{c,d,k}(\psi) := \sup_{0<t<\infty} |\mu_{c,d}(t)\ (t\,D_t)^k(\sqrt{t}\,\psi(t))| \leq C_k A^k k^{k\alpha} \tag{12.8}$$

where the constants A and C_k depend on ψ. For $k = 0$, $k^{k\alpha}$ is taken as 1. Here

$$\mu_{c,d}(t) = \begin{cases} t^c & 1 \leq t < \infty \\ t^d & 0 < t < 1. \end{cases} \tag{12.9}$$

The topology over $J_{c,d,k}$ is that generalized by the countable multinorm $\{i_{c,d,k}\}_{k=0}^{\infty}$. With this topology $J_{c,d,k}$ is a countably multinormed space. It is complete under the usual definition of convergence. A countable union space is $J_{c,d} = \bigcup_{\alpha_i=1}^{\infty} J_{c,d,\alpha_i}$.

The function $\frac{\Gamma(a)}{\Gamma(b)} \frac{\Gamma(\beta+1)}{x} (\frac{T}{x})\ {}_2F_1(a, +1;b; -\frac{t}{x})$ belongs to $J_{c,d}$ for $x > 0$, $c < \frac{1}{2}$ and $d > -\beta - \frac{1}{2}$.

The distributional generalized Stieltjes transform of $f \in J'_{c,d}$ for $c < \frac{1}{2}$ and $d > -\beta - \frac{1}{2}$ is defined by

$$F(x) = \left\langle f(t), \frac{\Gamma(a)}{\Gamma(b)} \frac{\Gamma(\beta+1)}{x} \left(\frac{t}{x}\right)^{\beta} {}_2F_1(a,\beta+1;b; -\frac{t}{x}) \right\rangle . \tag{12.10}$$

Tiwari (1980) proved the following inversion formula for (12.10)

$$\lim_{r\to\infty} \left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \frac{\Gamma(b+s-\beta-1)\ M(s)}{\Gamma(a+s-\beta-1)\ \Gamma(\beta+1-s)\ \Gamma(s)}\ y^{-s}\ ds, \psi(y)\right\rangle \to \langle f,\psi \rangle \tag{12.11}$$

for each $\psi \in D(I)$.

13. ABELIAN THEOREMS.

An initial (final) value Abelian theorem concerning transforms of functions is a result in which known behavior of the function as its domain variable approaches zero (approaches ∞) is used to infer the behavior of the transform as its domain variable approaches zero (approaches ∞). Such theorems for the Stieltjes transform (2.4) with s and p real were given by Widder (1946) and Misra (1972). Corresponding distributional results were also obtained by Misra (1972). Abelain theorems for the Stieltjes trnasform of functions were also given by Carmichael (1976) and Carmichael and Hayashi (1981). Carmicheal and Hayashi considered the case when s and p are both complex and a generalizing assumption is placed on the function f(t). In what follows we shall wrote results from Lavoine and Misra (1979).

Let D'_+ be the space of Schwartz distributions having support in $[0,\infty)$. For $\alpha > 0$, $D'_+(\alpha)$ is the space of all distributions $T_x \in D'_+$ which satisfy the following: There exist $a > 0$, $k \in N$ and a function f(x) having support in $[a,\infty)$ such that $T_x = D^k f(x)$ in the sense of distributions and $(f(x)\ x^{-k-\alpha})$ is bounded.

E'_+ denotes the space of distributions of compact support in $[0,\infty)$ and T'_+ is the space of tempered distributions, with support in $[0,\infty)$. Note that $E'_+ \subset E'$ and $T'_+ \subset T'$.

From the above definition it follows that for $T \in D'_+(\alpha)$ there exist $a > 0$ and a non-negative integer k such that

$$T_t = V_t + D_t^k f(t) \tag{13.1}$$

where $V \in E'_+$ and $\text{supp}(V) \subseteq [0,a]$ where f(t) is a function having support in $[a,\infty)$ such that $f(t)t^{-k-\alpha})$ is bounded.

Another space with which we shall be concerned with which was introduced by Misra (1972) for obtaining Abelian theorems for the distributional Stieltjes transform.

Let K_α be a continuous function on $(0,\infty)$ defined by

$$K_\alpha(t) = \begin{cases} t^\alpha, & 1 \leqq t < \infty \quad , \quad \alpha \leqq 1 \\ 1 \;, & 0 < t < 1 \quad . \end{cases} \tag{13.2}$$

A complex-valued C^∞ function ϕ on $(0,\infty)$ is said to belong to the space I_α if

$$\gamma_k(\phi) = \sup_{0<t<\infty} |K_\alpha(t)\; t^{p+k}\; D_t^k\; \phi(t)| < \infty \;. \tag{13.3}$$

The concept of convergence and completeness on I_α are introduced in the usual way. It turns out that I_α is a locally convex, sequentially complete, Hausdorff topological vector space. The dual of I_α is denoted by I'_α. For additional properties, the reader is referred to Misra (1972).

The space of distributions $J'(p)$ (p being any complex number except a negative integer) consists of distributions T having support in $[0,\infty)$ and admitting the decomposition (13.1) where $V \in E'_+$ and $\operatorname{supp} V \in [0,a]$ and $f(t)$ is a function having support in $[a,\infty)$ for some finite $a \geq 0$ such that $f(t)t^{-p-k-1}$ is summable.

We note that this definition of $J(p)$ is equivalent to

$$T_x = D^n\; f_1(x), \tag{13.4}$$

where $n \in N_0 = N \cup \{0\}$, if $f_1(x) = 0$ for $x < 0$, and if

$$\int_0^\infty |f_1(x)| \quad (x+\beta)^{-p-n-1}\; dx \tag{13.5}$$

exists for $\beta > 0$.

Obviously $J'(p) \subset T'_+$ for every $r > -1$ and $J'(p_1) \subset J'(p_2)$ for $-1 < p_1 < p_2$.

The Stieltjes transformations of $T_x \in J'(p)$ (or I'_α) is defined by

$$F(s) = S_s[T_x](s) = \langle T_x, (x+s)^{-p-1} \rangle \tag{13.6}$$

for $s \in \Delta := C\setminus(-\infty,0]$. It can be easily seen that the right-hand side makes sense for $T \in J'(p)$ (or I'_α). If $T = D^n\; f_1$ for some $n \in N_0$ and locally integrable f_1 on $\mathbb{R}$ with support in $[0,\infty)$ such that (13.5) holds, it follows by the Leibnitz formula that (13.6) can be written as

$$S_s[T_x]\;(s) = (p+1)_n \int_0^\infty \frac{f_1(x)\,dx}{(x+s)^{p+n+1}} \;. \tag{13.7}$$

It can be shown that $S_s[T](s)$ is a homolomorphic function of $s \quad \Delta$. Note that for the sake of convenience in putting restrictions, the Stieltjes transform of index $p+1$ is taken in (13.6) whereas in our previous analysis index was taken as p. Unless stated otherwise, we shall take (13.6) as the definition of the distributional Stieltjes transform in the following sections.

In the next section initial value theorems are obtained for Stieltjes transfroms of f belonging to I'_α, $S'_\alpha(I)$ and $J'(p)$.

14. INITIAL VALUE RESULTS.

We say that $T \in D'_+(\alpha)$ is asymptotically equivalent to At_+^η as $t \to 0+$, A being a complex number, if there exists $a > 0$ and $V \in D'_+(\alpha)$ such that

$$T_t = At_+^\eta + V_t, \quad t \in [0,a],$$

and

$$\lim_{s\to o} \langle V_t, \; s^{-\eta-1} (1+(t/s)^{-p-1}) \rangle = 0, \quad s = \sigma + i\omega, \quad \sigma > 0.$$

The following theorems contain initial-value results:

THEOREM 14.1. Let $T \in D'_+(\alpha)$, and let T be asymptotically equivalent to At_+^η as $t \to 0+$. Let $\eta > -1$ and $p-1 > \sup(\alpha,\eta)$. Then

$$\lim_{s\to o} \frac{s^{p-\eta-1}\, \Gamma(p+1)\, S_s(T)}{\Gamma(p-\eta)\, \Gamma(\eta+1)} = A, \quad s = \sigma + i\omega, \quad \sigma > 0.$$

THEOREM 14.2. Let $T \in D'_+(\alpha)$ such that

$$\lim_{s\to o} \langle V_t, \; s^{-1} (1+(t/s))^{-p-1} \rangle = 0, \quad s = \sigma + i\omega, \quad \sigma > 0,$$

where $V \in E'_+$ is as in the decomposition (13.1). Let $p > \sup(\alpha,0)$. Then

$$\lim_{s\to o} s^p \; S_s(T) = 0, \quad s = \sigma + i\omega, \; \sigma > 0.$$

The following theorem is a generalization of a result due to Misra (1972) in which we need the concept of limit of distribution due to Lojasiewicz (1957).

Let T be a distribution defined in the neighborhood of the point x_o. Then we say that T has a value c at x_o if the distributional limit of $T(x_o + \lambda x)$ as $\lambda \to 0+$ exists in a neighborhood of zero and is a constant function c.

THEOREM 14.3. Let $T \in I'_\alpha$, $\alpha \leq 1$, and let $(T_t/t^\eta) \to A$ as $t \to 0+$ in the above sense, where A is a complex number. Let $p > \eta+1 > 0$. Then

$$\lim_{s\to o} \frac{s^{p-\eta}\, \Gamma(p+1)\, S_s(T)}{\Gamma(p-\eta)\, \Gamma(\eta+1)} = A, \quad s = \sigma + i\omega, \quad \sigma > 0.$$

Recently Misra (1987) pointed out that before applying Lojasiewicz's definition of limit of a distribution to any generalized function it is necessary to examine its validity in the space of generalized functions under consideration. He has obtained Abelian theorems for the Stieltjes transform of generalized functions belonging to some general space $I'_{\lambda,\nu}$ and has examined the validity of Lojasieqicz's definition.

The following theorem uses the idea of semi-regular distribution instead of notion of limit of a distribution in the sense of Lojasiewicz. By a semi-regular distribution we mean a distribution which is defined by a function over a subset of its support.

Let $T \in I'_\alpha$, $\alpha \leq 1$, and let $a > 0$ be fixed. We decompose T into $T = V + U$, where V has support in $[0,a]$ and U has support in $[a-\delta,\infty)$, $0 < \delta < a$. Then $V \in E'_+$ is a finite sum of distributional derivatives of continuous functions having support in an arbitrary neighborhood of $[0,a]$. In the following theorem we assume that V is a semi-regular distribution, so that $V_t = g(t)$ where $g(t)$ is a continuous function having support in $[0,a]$. Evidently, the Stieltjes transform, $S_s(g(t))$, $s = \sigma + i\omega$, $\sigma > 0$, exist for $p > -1$. Moreover, $S_s(U)$ exists for the same values of s and p since $U \in I'_\alpha$ and $T \in I'_\alpha$.

We define

$$Q_k = \{s: \ s = \sigma + i\omega \ , \ \sigma > 0, \ |\omega| \leq k\sigma\} \ ,$$

where $k \geq 0$ is an arbitrary but fixed real number. Then we have the following result:

THEOREM 14.4. Let $T \in I'_\alpha$, $\alpha \leq 1$. Let $p > \eta > -1$ and $a > 0$ be fixed. Let $T = V + U$, where $V_t = g(t)$ is a continuous function defined in the preceding paragraph. We assume further that $g(t)/t^\eta$ is bounded on $0 \leq t < \infty$ for all $y > 0$ and $(g(t)/t^\eta) \to A$ as $t \to 0+$, where A is a complex number. Then

$$\lim_{\substack{s\to 0\\ s\in Q_k}} \frac{s^{p-\eta}\,\Gamma(p+1)\,S_s(T)}{\Gamma(p-\eta)\,\Gamma(\eta+1)} = A, \tag{14.1}$$

where $k \geq 0$ is an arbitrary but fixed real number.

The following initial value theorem is due to Tiwari (1976) in which p is considered to be complex.

THEOREM 14.5. Let $\operatorname{Re} p > \alpha > \eta+1$ and $\alpha < 1$. Assume that s is a non-zero complex number not lying on the negative real axis. If for $f(t)$ S'_α there exists a constant c such that

$$\lim_{t\to 0+} \frac{f(t)}{t^\eta} = c \ \text{ in the sense of Lojasiewicz.} \tag{14.2}$$

Then

$$\lim_{\substack{|s|\to o\\ s\in Q_k}} \frac{s^{p-\eta-1}\,\Gamma(p)\,F(s)}{\Gamma(p-\eta-1)\,\Gamma(\eta+1)} = c, \tag{14.3}$$

where $F(s)$ is defined by (5.1).

In the following we obtain Abelian theorems when the assumption on the limit of the distribution is more general. We recall some of the definitions from (Lavoine and Misra (1974)).

Equivalence of distributions at the Origin

We define

$$x^\nu \log^j x_+ = \begin{cases} x^\nu \log^j x & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

and

$$x^{-n-1} \log^j x_+ = \begin{cases} x^{-n-1} \log^j x & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

where $j, n \in N$. When $\nu < -1$ and $n \in N$, the distributions represented by $x^{\nu} \log^{j} x_{+}$ and $x^{-n-1} \log^{j} x_{+}$ are understood to be in the sense of finite parts of Hadamard. In the sense of Lojasiewicz (1957), a distribution T, which has support in $[0,\infty)$ is said to satisfy

$$T_x \sim A\, F_p\, x^{\nu} \log^{j} x_{+} \quad \text{as} \quad x \to 0+,$$

where $j \in N$ and $\nu \neq -1, -2$, if there exists a number $\xi > 0$ and a distribution R having support in $[0,\xi]$ such that

$$T_x = A\, F_p\, x^{\nu} \log^{j} x_{+} + R_x \quad (x \in [0,\xi]),$$

and

$$\lambda^{-\nu-1} \log^{-j} \lambda \; [\langle Rx, \phi(x/\lambda)\rangle] \to 0 \quad \text{as} \quad \lambda \to 0+$$

for every function ϕ that is infinitely differentiable on a neighborhood of $[0,\infty)$.

When $n \in N$ and $j \in N$, we define

$$T_x \sim A\, F_p\, x^{-n-1} \log^{j} x_{+} \quad \text{as} \quad x \to 0+$$

if there exists a distribution Q having support in $[0,\xi]$ such that

$$T_x = A\, F_p\, x^{-n-1} \log^{j} x_{+} + Q_x \quad (x \in [0,\xi]),$$

and

$$x^{n} \log^{-j-1} \lambda \; [\langle Q_x, \phi(x/\lambda)\rangle] \to 0 \quad \text{as} \quad \lambda \to 0+$$

for every function ϕ which is infinitely differentiable on a neighborhood of $[0,\infty)$.

The proofs of the following initial value theorems can be found in the paper by Lavoine and Misra (1979).

THEOREM 14.6. Let $T \in J'(p)$, $j \in N$, $\nu \neq -1,-2,\ldots$, $\mathrm{Re}(p-\nu) > 0$. If

$$T_x \sim A\, F_p\, x^{\nu} \log^{j} x_{+} \quad \text{as} \quad x \to 0+$$

then

$$S_s[T] \sim AB(\nu+1, p-\nu) s^{\nu-p} \log^{j} s \quad \text{as} \quad s \to 0+ \tag{14.4}$$

in the usual sense of functions.

THEOREM 14.7. Let $T \in J'(p)$, $j \in N$, $n \in N$ with $\mathrm{Re}(p+n+1) > 0$. If

$$T_x \sim A\, F_p\, x^{-n-1} \log^{j} x_{+} \quad \text{as} \quad x \to 0+$$

then

$$S_s[T] \sim A\, \frac{(-1)^{n}\, \Gamma(p+n+1)}{(j+1)\; n!\; \Gamma(p+1)}\, s^{-n-p-1} \log^{j+1} s \quad \text{as} \quad s \to 0+ \tag{14.5}$$

in the usual sense of functions.

The following result is a consequence of Theorem 14.6.

THEOREM 14.8. Let V be a distribution having compact support with the origin being the lower bound of this support. We take $V_x = D^{m}\, h(x)$ where $m \in N$ and $h(x)$ is a locally summable function. If, in the usual sense,

$$h(x) \sim A\, x^{\nu} \log^{j} x \quad \text{as} \quad x \to 0+,$$

we have

$$S_s[V] \sim A \,\frac{\Gamma(\nu+1)\,\Gamma(p+m-\nu)}{\Gamma(p+1)}\, s^{\nu-p-n} \log^{j} s \quad \text{as} \quad s \to 0+ \tag{14.6}$$

provided that $\mathrm{Re}(p+m-\nu) > 0$.

15. FINAL-VALUE RESULTS.

Carmichael and Milton (1979) have provided a final-value Abelian theorem which generalizes earlier results of Mishra (1972, Theorem 4.2) and Lavoine and Misra (1974, Theorem III). We need the following definitions:

DEFINITION 15.1. The value of a distribution T at infinity is defined to be $A\,x^{\nu}$ ($\nu+1 \notin N$) if there exists a number $N > 1$, a positive integer k and a function $h(x)$ satisfying $D^k\, h(x) = f$ over (N,∞) and $x^{-k-\nu}\, h(x) \to A/(\nu+1)_k$, as $x \to \infty$. We designate this limiting value by $T \sim A\,x^{\nu}$.

THEOREM 15.1. If $T \in D'_+$ and if $T_t \sim At^{\eta}$, $t \to \infty$, with $\eta > -1$, then for $p > \eta$,

$$\lim_{\substack{s\to\infty \\ s\, Q_k}} \frac{s^{p-\eta}\,\Gamma(p+1)\, S_s(T)}{\Gamma(p-\eta)\,\Gamma(\eta+1)} = A, \tag{15.1}$$

where $k \geq 0$ is arbitrary.

We state another definition of limit of a distribution due to Lavoine and Misra (1974).

DEFINITION 15.2. The limit of a distribution T_x as $x \to \infty$ is a constant C if there exist $N > 1$, a non-negative integer k and a function $g(x)$ satisfying

$$T_x = D^k\, g(x) \quad \text{over } [N,\infty) \quad \text{and} \quad k!\, x^{-k}\, h(x) \to C \quad \text{as} \quad x \to \infty. \tag{15.2}$$

Using this definition of limit at infinity Carmichael and Milton (1979) proved the following theorem:

THEOREM 15.2. Let $T \in D'_+$ and $\lim_{\infty} T_t = 0$. Let $p > -1$. Then

$$\lim_{s\to\infty} s^p\, S_s(T) = 0, \quad -1 < p < 0, \tag{15.3a}$$

and

$$\lim_{\substack{s\to\infty \\ s\, Q_k}} s^p\, S_s(T) = 0, \quad p \geq 0. \tag{15.3b}$$

THEOREM 15.3. Let $T \in E'_+$ and $p+1 > p'$. Then $\lim_{s\to\infty} s^{p'}\, S_s(T) = 0$, $s = \sigma + i\omega$, $\sigma > 0$.

The following theorem contains orders of the derivatives of the distributional Stieltjes transform and generalizes a result of Misra (1972, Corollary, p. 592).

THEOREM 15.4. Let $T \in I'_{\alpha}$, $\alpha \leq 1$ and let $s = \sigma + i\omega$, $\sigma > 0$. Then for $k = 1,2,3,\ldots$

$$|D_s^k (S_s(T))| = o(|s|^{-k}), \quad |s| \to \infty,$$

$$|D_s^k (S_s(T))| = O(|s|^{-k-1}), \quad |s| \to 0, \tag{15.4ab}$$

where $p > -1$ in the definition of $S_s(T)$.

A final-value Abelian theorem for the Stieltjes transform when p is complex was given by Tiwari (1976).

THEOREM 15.5. Let $\text{Re } p > \alpha > \eta+1 > 0$, and

$$\lim_{t\to\infty} \frac{f(t)}{t^\eta} = C \text{ in the sense of definition 15.1.}$$

Then

$$\lim_{\substack{|s|\to\infty \\ s\in\Omega_k}} \frac{s^{p-\eta-1}\,\Gamma(p)\,F(s)}{\Gamma(p-\eta-1)\,\Gamma(\eta+1)} = C, \tag{15.5}$$

where F(s) is defined by (5.1).

The following final-value Abelian theorem is due to Lavoine and Misra (1979).

THEOREM 15.6. Let T be a distribution in $J'(p)$ admitting the decomposition

$$T_x = V_x + g(x) \tag{15.6}$$

where V is a distribution having compact support and g(x) is a locally summable function such that

$$g(x) \sim A\, x^\nu \log^j x \tag{15.7}$$

in the usual sense as $x \to \infty$. If $-1 < \text{Re } \nu < \text{Re } p$, then

$$S_s[T] \sim A\, B(\nu+1, p-\nu) s^{\nu-p} \log^j s \quad \text{as} \quad s \to \infty. \tag{15.8}$$

Using the notion of quasi-asymptotic behavior of distributions introduced by Drozzinov and Zavjalov (1977), Takači [1983] obtained a final-value Abelian theorem for the distributional Stieltjes transformation.

DEFINITION 15.3. A distribution $f \in T'_+$ has quasi asymptotic behavior (q.a.b.) at infinity of order η if there exists the limit

$$\lim_{t\to\infty} t^{-\eta}\, T(tx) =: \gamma(x) \quad \text{(in the sense of } T') \tag{15.9}$$

provided that $\gamma \not\equiv 0$.

Then we write $T(x) \sim \gamma(x)$ in T' when $x \to +\infty$.

We can prove that γ is a homogeneous distribution of order η (hence $\gamma \in T'$) and $\text{supp } \gamma \subset [0,\infty)$, so there exists a constant $C \neq 0$ such that $\gamma(x) = C\, f_{\eta+1}(x)$. Here

$$f_{\eta+1}(x) = H(x)\, \frac{x^\eta}{\Gamma(\eta+1)} \quad \text{for} \quad \eta > 0,$$

and (15.10ab)

$$f_{\eta+1}(x) = D^n\, f_{\eta+n+1}(x) \quad \text{for} \quad \eta \leq 0 \quad \text{and} \quad \eta+n > 0,$$

where H is the characteristic function of $(0,\infty)$.

The following structural theorem is due to Drozzinov and Zavjalov (1977):

THEOREM 15.7. The distribution $f \in T'_+$ has q.a.b. at infinity of order η (i.e. $T(x) \sim C f_{\eta+1}(x)$ in T' when $x \to +\infty$) if and only if there exist a natural number n, $n+\eta > 0$, and a continuous function F on $\mathbb{R}$ such that

$$F(x) \sim C \frac{x^{\eta+n}}{\Gamma(\eta+n+1)} \quad \text{when} \quad x \to +\infty \tag{15.11}$$

in the ordinary sense with the property $T = D^n F$.

Using the concept of q.a.b. the following final-value theorem for the Stieltjes trnasformation (13.6) was given by Takači (1983).

THEOREM 15.8. If $T \in T'_+$ has quasi asymptotic behavior of order η, then its Stieltjes transformation of order $p > -1$, $p > \eta$, has the asymptotic behavior

$$S_s[T] \sim C \frac{\Gamma(p-\eta)}{\Gamma(p+1)s^{p-\eta}} \tag{15.12}$$

for some $C \neq 0$ when $s \to \infty$ through values in the domain $Q_K = \{s \in C : s = u+iv,\ u > 0,\ |v| < K|u|\}$, $K \geq 0$.

This theorem is a generalization of Theorem 15.1 since here we do not need the condition $\eta > -1$.

16. SOME APPLICATIONS.

We cite here three main applications of the Stieltjes transform which is used (a) to obtain new inversion formulas of the Laplace transform, (b) to discuss moment problems in the semi-infinite interval and (c) to study statistical properties of many-particle spectra of a wide class of new Gaussian ensembles.

(a) We consider the Laplace transform of f(t) defined by (2.3) and the Stieltjes transform F(x) of f(t) is obtained by an iteration of the Laplace transform so that

$$\begin{aligned} F(x) &= \int_0^\infty e^{-xy}\, dy \int_0^\infty e^{-yt} f(t)\, dt \\ &= \int_0^\infty (x+t)^{-1} f(t)dt = S[f(t)] \end{aligned} \tag{16.1}$$

THEOREM 16.1. If f(t) $L(0,\infty)$ and if $\psi(y)$ is the Laplace transform of f(t), then the inverse Laplace transform is given by

$$f(x) = \lim_{k\to\infty} L_{k,x}[F(x)] \quad \text{for almost al} \quad x > 0, \tag{16.2}$$

where

$$L_{k,x}[F(x)] = \frac{(-1)^k x^{k-1}}{k!\ (k-2)!} \int_0^\infty e^{-xy}\, y^{2k-1}\ \psi^{(k)}(y)dy \tag{16.3}$$

$$= \frac{(-1)^{k-1}}{k!\ (k-2)!} [x^{2k-1} F^{(k-1)}(x)]^{(k)} . \tag{16.4}$$

where $L_{k,x}[F(x)]$ is defined by (3.7) for $k = 2,3,\ldots$.

It is noted that result (16.2) depends on the values of all the derivatives of $\psi(y)$ in the interval $(0,\infty)$.

We derive another result by the application of the operator $L_{k,x}$ to the Laplace integral directly. It turns out that

$$L_{k,x}[F(x)] = L_{k,x}\left[\int_0^\infty e^{-xy}\,\psi(y)dy\right]$$

$$= \int_0^\infty L_{k,x}[e^{-xy}]\,\psi(y)dy \tag{16.5}$$

After some computation with fixed y, result (16.5) gives

$$L_{k,x}[F(x)] = \int_0^\infty e^{-xy}\,P_{2k-1}(xy)\,\psi(y)dy$$

where

$$P_{2k-1}(t) = \frac{(-1)^{k-1}\,(2k-1)!}{k!\,(k-2)!}\sum_{n=0}^{k}\binom{k}{n}\frac{(-t)^{2k-n-1}}{(2k-n-1)!}$$

Thus, for almost all x

$$f(x) = \lim_{k\to\infty} L_{k,x}[F(x)] = \lim_{k\to\infty}\int_0^\infty e^{-xy}\,P_{2k-1}(xy)\psi(y)dy \tag{16.6}$$

This inversion formula is very useful because it depends only on the value of $\psi(y)$ in $(0,\infty)$ and not on any of its derivatives.

(b) Historically, Stieltjes introduced the theory of moments and formulated the so-called Stieltjes problem of moments in the general form. The problem is to determine a bounded non-decreasing function $\psi(x)$ in $[0,\infty)$ such that its moments M_n have a set of prescribed values $M_n = \mu_n$, $n = 0,1,2,\ldots$, where

$$M_n = \int_0^\infty x^n\,d\psi(x). \tag{16.7}$$

The Stieltjes transform occurs naturally in connection with the Stieltjes moment problem and hence is related to certain continued fractions. Since a good deal of literature is available on various version of the moment problems, we do not intend to discuss it here. However, the reader is referred to Shobat and Tamarkin (1943).

(c) In connection with statistical properties of many-particle spectra of a wide class of new Gaussian ensembles in which the matrix elements have Gaussian distributions with arbitrary specified centroids and variances, Pandey (1981) calculated the average density and two-point correlation function in terms of the Stieltjes transform. In fact he obtained the Stieltjes transform $f(z)$ of the density $\rho(x,H_{cv})$ in the form

$$f(z;H_{cv}) = \int_{-\infty}^{\infty}\frac{\rho(x;H_{cv})}{z-x}\,dx = \left\langle\frac{1}{z-H_{cv}}\right\rangle \tag{16.8}$$

where H_{cv} represent the centroid- and the variance-modified ensembles of matrices and the subscripts c and v stand for centroid and variance modifications respectively of the standard Hamiltonian ensemble H. More explicitly.

$$H_{cv} = K + \sum_{i,j} v_{ij} P_i H P_j = K + H_v$$

$$H_c = K + \alpha H, \quad H_v = \sum_{i,j} v_{ij} P_i H P_j$$

where K is a Hermitian centroid matrix ($\bar{H}_{cv} = K$), v^2 is defined in terms of its elements $(v^2)_{ij} = (v_{ij})^2 \geq 0$ and v^2 is a real symmetric variance matrix; P_i is the projection operator for the ith basis state. The centroid-modified ensemble H_c and the variance-modified ensemble H_v are particular cases of H_{cv}, defined by $v_{ij} = \alpha = \alpha^*$ and $K = 0$ respectively.

Pandey also calculated the average density $\bar{\rho}(x)$ in terms of its Stieltjes transform $\bar{f}(z)$ where the inversion formula is given by

$$\rho(x) = -\frac{1}{\pi} \operatorname{Im} f(x+i0) = (2\pi i)^{-1}[F(x-i0) - f(x+i0)], \tag{16.9}$$

and f(z) is defined in (16.8).

For two-point function S^ρ, we shall consider the two-point Stieltjes transform S^j. We have

$$S^\rho(x_1,x_2) = \overline{\rho(x_1)\rho(x_2)} - \bar{\rho}(x_1)\bar{\rho}(x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} S^F(x_1,x_2)$$

$$= -\frac{1}{4\pi^2}[S^f(x_1+i0,\ x_2+i0) + S^f(x_1-i0,\ x_2-i0)$$
$$- S^f(x_1+i0,\ x_2-i0) - S^f(x_1-i0,\ x_2+i0)\ , \tag{16.10}$$

$$S^f(z_1,z_2) = \overline{f(z_1)f(z_2)} - \bar{f}(z_1)\,\bar{f}(z_2) = \int_{-\infty}^{\infty}\int \frac{S^\rho(x_1,x_2)dx_1dx_2}{(z-x_1)\ (z-x_2)} \tag{16.11}$$

$$= \frac{\partial^2}{\partial z_1 \partial z_2} S^g(z_1,z_2) \tag{16.12a}$$

In the above, we have also introduced the two-point function S^F and its Stieltjes transform S^g for the distribution function $F(x) = \int_{-\infty}^{x} \rho(t)dt$ and corresponding transform

$$g(z) = \int_{-\infty}^{\infty} (z-x)^{-1} F(x)\ dx \tag{16.12b}$$

The inversion formula relating S^F and S^g follows from that relating S^ρ and S^f.

Pandey's analysis includes more general quantities $\bar{f}_L$ and $\overline{\delta f_{L_1}\ \delta f_{L_2}}$ for arbitrary matrices L, L_1 and L_2 where

$$f_L(z;H_{cv}) = \left\langle L \frac{1}{z-H_{cv}} \right\rangle = \left\langle LG \frac{1}{1-H_vG} \right\rangle = \sum_{p=0}^{\infty} \left\langle LG(H_vG)^p \right\rangle\ , \tag{16.13}$$

This reduces to (16.8) when L = I, the d-dimensional unit matrix.

It is observed that the Stieltjes transform is analytic everywhere in the complex plane except at poles on the real line, whose positions are given by the eigenvalues of the matrix. Although the series in (16.13) is not convergent for all z, the sum of the convergent series can be extended, except for a branch cut on the on the real axis, to all z by analytic continuation.

Pandey has shown the following result

$$\bar{f}_L(z;H_{cv}) \to \langle LG \rangle + \sum_{i,j} v_{ij}^2 \, \bar{f}_{LGP_i}(z;H_{cv}) \, \bar{f}_{P_j}(z;H_{cv}) \tag{16.14a}$$

$$= \left\langle LG \frac{1}{1 - \sum_{i,j} v_{ij}^2 \, \bar{f}_{P_j}(z,H_{cv}) P_i G} \right\rangle \tag{16.14b}$$

Except for H itself, the centroid-modified ensemble ($v_{ij} = \alpha$) follows as a special case of (16b). Thus

$$\bar{f}_L(z;H_c) = \left\langle L \frac{1}{z-K-\alpha^2 \bar{f}(z,H_c)} \right\rangle \tag{16.15}$$

$$\bar{f}(z;H_c) = \left\langle \frac{1}{z-K-\alpha^2 \bar{f}(z;H_c)} \right\rangle \tag{16.16}$$

These are the basic results which are valid for all β . Three basic types of Hermitian matrices are characterized by $\beta = 1,2,4$ [see Pandey (1981)]. Equation (16.16) is well known for $\beta = 1$ and $\beta = 2$ and is obtained by Pandey (1981) for all β.

We next consider the H-ensembles so that $K = 0$ and $\alpha = 1$. Then (16.16) gives

$$\bar{f}(z;H) = \frac{1}{z-\bar{f}(z;H)} = \tfrac{1}{2}[z \pm \sqrt{z^2-4}\,] \tag{16.17}$$

This result combines with the inversion formula (16.9) gives the Wigner semicircle result

$$\begin{aligned} \bar{\rho}(x;H) &= (2\pi)^{-1} \sqrt{4-x^2} \quad , && |x| \leqq 2 \\ &= 0, && |x| \geqq 2 \end{aligned} \tag{16.18}$$

This is true for all β. The semicircular density is found for a wide class of random matrices of zero-centered independent elements.

Pandey also discussed the two-point Stieltjes transform with application to more complicated ensembles. In conclusion, we point out that the Stieltjes transform method is found to be suitable for the study of the average and fluctuation properties of transition strengths in complicated physical systems.

17. SOME OPEN QUESTIONS AND UNSOLVED PROBLEMS.

1. The real inversion formulae (2.9) and (2.10) have not been extended to generalized functions. Such results seem to be very useful and deserve attention.

2. It would be interesting to extend the S_2-transform by adjoint method. Is it possible to attach a meaning to (10.5)?

3. As Misra (1987) has pointed out the validity of Lojasiewicz's definition of limit of a distribution needs verification in theorems 14.3 and 14.5.

4. The Abelian theorems for the Stieltjes transform of functions were obtained by Carmichael and Hayashi (1981) when the parameters p and η and the variable s are complex and the assumption on the growth is

$$\lim_{\substack{t\to 0+ \\ (\text{or } t\to\infty)}} \frac{f(t)}{t^{\eta}\,(\ln t)^{j}} = A, \qquad j = 1,2,3,\ldots \,. \tag{17.1}$$

However, in the distributional case, Lavoine and Misra (1972) obtained Abelian theorems under the above general assumptions but restricting s to real and positive. The problem, when s is also complex, has not yet been investigated.

5. In papers by Drozzinov and Zavjalov (1977 and 1979) both Abelian and Tauberian theorems for the Fourier-Laplace transformation are given. Such Tauberian theorems for the distributional Stieltjes transformation have not yet been established.

6. Consider the following two boundary value problems:

$$P_1: \quad \begin{cases} \dfrac{\partial^2}{\partial t^2}\, v(x,t) = P(x,D)\, v(x,t), & t > 0 \\ v(x,o) = \phi(x), \quad v_t(x,0) = 0 \end{cases}$$

and

$$P_2: \quad \begin{cases} \dfrac{\partial^2}{\partial y^2}\, w(x,y) + P(x,D)\, w(x,y) = 0, & y > 0 \\ w(x,0) = \phi(x) \end{cases}$$

where $x = (x_1, x_2,\ldots,x_n)$ and $D = (D_1, D_2,\ldots,D_n)$ where $D_i\, f(x) = \dfrac{\partial}{\partial x^i}\, f(x)$. For $D^{\alpha} = D_1^{\alpha_1}\, D_2^{\alpha_2} \cdots D_n^{\alpha_n}$,

$$P(x,D) = \sum_{0\le|\alpha|\le m} a_{\alpha}(x)\, D^{\alpha}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Dettman (1969) has shown that, under certain conditions if $v(x,t)$ is a solution of P_1, then the solution $w(x,t)$ of P_2 is related through the Stieltjes transform:

$$w(x,t) = \frac{2t}{\pi} \int_0^{\infty} \frac{v(x,\eta)}{(t^2+\eta^2)}\, d\eta \quad .$$

If the boundary conditions in P_1 and P_2 are distributional then such a relationship could be given by using distributional Stieltjes transform!

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