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COEFFICIENT ESTIMATES FOR SOME CLASSES OF p-VALENT FUNCTIONS

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ABSTRACT. Let A_p, where p is a positive integer, denote the class of functions $f(z) = z^p + \sum_{n=p+1} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$.

For $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, let $F_\lambda(\alpha,\beta,p)$ denote the class of functions $f(z) \in A_p$ which satisfy the condition

$$\left|\frac{H(f(z))-1}{H(f(z))+1}\right| < \lambda \quad \text{for} \quad z \in U ,$$

where
$$H(f(z)) = \frac{e^{\frac{i\alpha z f'(z)}{f(z)}} - \beta\cos\alpha - ip\sin\alpha}{(p-\beta)\cos\alpha} .$$

Also let $C_\lambda(b,p)$, where p is a positive integer, $0 < \lambda < 1$, and $b \neq 0$ is any complex number, denote the class of functions $g(z) \in A_p$ which satisfy the condition

$$\left|\frac{H(g(z))-1}{H(g(z))+1}\right| < \lambda \quad \text{for} \quad z \in U, \text{ where}$$

$$H(g(z)) = 1 + \frac{1}{pb}\left(1 + \frac{zg''(z)}{g'(z)} - p\right).$$

In this paper we obtain sharp coefficient estimates for the above mentioned classes.

KEY WORDS AND PHRASES. p-Valent, starlike, convex, spirallike functions.
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1. INTRODUCTION.

Let A_p, where p is a positive integer, denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$. We use Ω_λ, $0 < \lambda \leq 1$, to denote the class of analytic functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < \lambda$, $0 < \lambda \leq 1$.

Padmanabhan introduced the class of starlike functions of bounded order λ, $0 < \lambda \leq 1$, defined as follows [11]:

DEFINITION 1. A function $f \varepsilon A_1$ and satisfying

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1} \right| < \lambda \tag{1.1}$$

for a given λ, $0 < \lambda \leq 1$, $|z| < 1$ is said to be starlike of bounded order λ in $|z| < 1$ and this class is denoted $S(\lambda)$, the class of all such functions for a given λ.

Let $F(\alpha,\beta,p)$ ($|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$) denote the class of functions $f(z) \varepsilon A_p$ and for which there exists a $\rho = \rho(f)$ such that

$$\mathrm{Re}\ \{e^{i\alpha} \frac{zf'(z)}{f(z)}\} > \beta \cos\alpha \tag{1.2}$$

and

$$\int_0^{2\pi} \mathrm{Re}\ \{\frac{zf'(z)}{f(z)}\}\ d\Theta = 2\pi p \quad \text{for} \quad z = re^{i\Theta},\ \rho < r < 1\ . \tag{1.3}$$

Functions in $F(\alpha,\beta,p)$ are called p-valent α-spirallike functions of order β. The class $F(\alpha,\beta,p)$ was introduced by Patil and Thakare [12].

In this paper we use a method of Clunie [3] to obtain sharp bounds for the coefficients of functions $F_\lambda(\alpha,\beta,p)$ and $C_\lambda(b,p)$, where p is a positive integer, $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, and b is any complex number, where $F_\lambda(\alpha,\beta,p)$ and $C_\lambda(b,p)$ are defined as follows:

DEFINITION 2. For $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, and $0 \leq \beta < p$, let $F_\lambda(\alpha,\beta,p)$ denote the class of functions $f(z) \varepsilon A_p$ which satisfy the condition

$$\left|\frac{H(f(z))-1}{H(f(z))+1}\right| < \lambda \tag{1.4}$$

for $z \varepsilon U$, where

$$H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos\alpha - ip \sin\alpha}{(p-\beta)\cos\alpha}\ . \tag{1.5}$$

DEFINITION 3. For p is a positive integer, $0 < \lambda \leq 1$, and $b \neq 0$ is any complex number, let $C_\lambda(p,b)$ denote the class of functions $g(z) \varepsilon A_p$ which satisfy the condition

$$\left|\frac{H(g(z))-1}{H(g(z))+1}\right| < \lambda \tag{1.6}$$

for $z \varepsilon U$,

where $$H(g(z)) = 1 + \frac{1}{pb}\ (1 + \frac{zg''(z)}{g'(z)} - p). \tag{1.7}$$

We note that by giving specific values to λ, α, β, p and b, we obtain the following important subclasses studied by various authors in earlier papers:

(1) $F_1(0,0,1) = S^*$ and $C_1(1,1) = C$, are respectively the well-known classes of starlike functions and convex functions, $F_1(0,\beta,1) = S_\beta$ and $C_1(1-\beta,1) = C_\beta$, $0 \leq \beta < 1$, are respectively the classes of starlike functions of order β and convex functions of order β introduced by Robertson [14], $F_\lambda(0,0,1) = S(\lambda)$ and $C_\lambda(1,1) = C(\lambda)$, is the class of functions g for which $zg'(z) \varepsilon S(\lambda)$.

(2) $F_1(\alpha,0,1) = S^\alpha$ and $C_1(\cos\alpha\, e^{-i\alpha},1) = C^\alpha$, $|\alpha| < \frac{\pi}{2}$, are respectively the class of α-spirallike functions introduced by Špaček [18] and the class of functions g for which $zg'(z)$ is α-spirallike introduced by Robertson [15], $F_1(\alpha,\beta,1) = S^\alpha_\beta$ and $C_1[(1-\beta)\cos\alpha\, e^{-i\alpha},1] = C^\alpha_\beta$, $|\alpha| < \frac{\pi}{2}$, $0 < \beta \le 1$, are respectively the class of α-spirallike functions of order β introduced by Libera [8] and the class of functions g for which $zg'(z)$ is α-spirallike of order β by Chichra [2] and Sizuk [17].

(3) $C_1(b,1) = C(b)$ is the class of functions $g \in A_1$ satisfying

$$\mathrm{Re}\{1 + \frac{1}{b}\frac{zg''(z)}{g'(z)}\} > 0$$

introduced by Wiatrowski [19] and studied by [9] and [10].

(4) $F_1(0,0,p) = S(p)$, $C_1(1,p) = C(p)$, $F_1(0,\beta,p) = S\ \beta(p)$ and $C_1[(1-\frac{\beta}{p}), p] = C_\beta(p)$, $0 \le \beta < p$, are respectively the classes of p-valent starlike functions, p-valent convex functions, p-valent starlike functions of order β and p-valent convex functions of order β considered by Goodman [6] and the class $S_\beta(p)$ investigated by Goluzina [5].

(5) $F_1(\alpha,0,p) = S^\alpha(p)$ and $C_1(\cos\alpha e^{-i\alpha},p)$, $|\alpha| < \frac{\pi}{2}$, are respectively the class of p-valent α-spirallike functions and the class of p-valent functions $g \in A_p$ satisfying

$$\mathrm{Re}\ e^{i\alpha}(1 + \frac{zg''(z)}{g'(z)}) > 0, \quad z \in U$$

i.e., the class of p-valent functions g for which $\frac{zg'(z)}{p}$ is p-valent α-spirallike.

(6) $F_1(\alpha,\beta,p) = F(\alpha,\beta,p)$ and $C_1[(1-\frac{\beta}{p})\cos\alpha\ e^{-i\alpha}, p]$, $|\alpha| < \frac{\pi}{2}$, $0 \le \beta < p$, is the class of p-valent functions g for which $\frac{zg'(z)}{p}$ is p-valent α-spirallike of order β.

(7) $C_1(b,p)$, is the class of functions $g \in A_p$ satisfying

$$\mathrm{Re}\ \{p + \frac{1}{b}(1 + \frac{zg''(z)}{g'(z)} - p)\} > 0, \ z \in U,$$

the class $C(b,p)$ was introduced by the author [1].

(8) $F_\lambda(\alpha,\beta,1) = F_\lambda(\alpha,\beta)$, is the class of functions investigated by Gopalakrishna and Umarani [7].

(9) $C_1[(1-\frac{\beta}{p})\cos\alpha\ e^{i\alpha}, p]$, $|\alpha| < \frac{\pi}{2}$, $0 \le \beta < p$, is the class of p-valent functions $g(z)$ for which $\frac{zg'(z)}{p} \in F_\lambda(\alpha,\beta,p)$.

We state the following lemma that is needed in our investigation.

LEMMA 1[11]. Let $f(z)$ be analytic for $|z| < 1$ and let $f(0) = 0$. Then $f(z) \in S(\lambda)$ if and only if

$$f(z) = z \exp\left[-2\int_0^z \frac{\phi(t)}{1 + t\phi(t)}\,dt\right],$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \le \lambda$, $0 < \lambda \le 1$, for $|z| < 1$.

In the rest of the paper we always assume that p is a positive integer, $0 < \lambda \leq 1$, $|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < p$, and $b \neq 0$ is any complex number.

2. REPRESENTATION FORMULAS FOR THE CLASS $F_\lambda(\alpha,\beta,p)$.

LEMMA 2. $f(z) \in F_\lambda(\alpha,\beta,p)$ if and only if for $z \in U$

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos\alpha \left\{\frac{p-(p-2\beta)w(z)}{1 + w(z)}\right\} + ip \sin\alpha, \tag{2.1}$$

$w \in \Omega_\lambda$.

PROOF. If $f(z)$ is given by (2.1), then

$$H(f(z)) = \frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos \alpha - ip \sin \alpha}{(p-\beta)\cos\alpha}$$

$$= \frac{1-w(z)}{1+w(z)}$$

so that $\frac{H(f(z)) -1}{H(f(z)) +1} = -w(z)$

and so (1.4) holds. Thus $f(z) \in F_\lambda(\alpha,\beta,p)$.

Conversely, if $f(z) \in F_\lambda(\alpha,\beta,p)$, then (1.4) holds.

Defining $w(z) = \frac{1-H(f(z))}{1+H(f(z))}$ we obtain (2.1) and the proof is complete .

LEMMA 3. $f(z) \in F_\lambda(\alpha,\beta,p)$ if and only if

$$f(z) = z^p \left[\frac{f_1(z)}{z}\right]^p \tag{2.2}$$

for some $f_1 \in F_\lambda(\alpha,\frac{\beta}{p},1)$.

PROOF. Let $f(z) = z^p \left[\frac{f_1(z)}{z}\right]^p$ for $f_1(z) = z + \sum_{n=2}^{\infty} c_n z^n \in F_\lambda(\alpha,\frac{\beta}{p},1)$, $z \in U$.

By direct computation, we obtain

$$\frac{e^{i\alpha} \frac{zf'(z)}{f(z)} - \beta\cos \alpha - ip \sin \alpha}{(p-\beta)\cos \alpha} = \frac{e^{i\alpha} \frac{zf_1'(z)}{f_1(z)} - \frac{\beta}{p}\cos\alpha - i\sin\alpha}{(1- \frac{\beta}{p}) \cos \alpha}$$

and the result follows from (1.4).

In a similar way we can prove the following lemma :

LEMMA 4. $f(z) \in F_\lambda(\alpha,\beta,p)$ if and only if

$$f(z) = z^p \left[\frac{f_2(z)}{z}\right]^{(p-\beta) \cos\alpha\, e^{-i\alpha}} \tag{2.3}$$

for some $f_2 \in S(\lambda)$.

An immediate consequence of lemmas 1 and 4 is

THEOREM 1. $f(z) \in F_\lambda(\alpha,\beta,p)$ if and only if

$$f(z) = z^p \exp[-2(p-\beta)\cos\alpha e^{-i\alpha} \int_0^z \frac{\phi(t)}{1 + t\phi(t)} dt] \tag{2.4}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq \lambda$, $0 < \lambda \leq 1$, for $|z| < 1$.

3. COEFFICIENT ESTIMATES FOR THE CLASS $F_\lambda(\alpha,\beta,p)$.

LEMMA 5. If integers p and m are greater than zero, $0 \leq \beta < p$ and $|\alpha| < \frac{\pi}{2}$, then

$$\prod_{j=0}^{m-1} \frac{\lambda^2|2(p-\beta)\cos\alpha e^{-i\alpha}+j|^2}{(j+1)^2} = \frac{\cos^2\alpha}{m^2}\{4\lambda^2(p-\beta)^2$$

$$+\sum_{k=1}^{m-1}[\lambda^2(2p-2\beta+k)^2+\lambda^2k^2\tan^2\alpha-k^2\sec^2\alpha]\times$$

$$\prod_{j=0}^{k-1}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}\}. \tag{3.1}$$

PROOF. We prove the lemma by induction on m. For $m = 1$, (3.1) is easily verified directly.

Next suppose that (3.1) is true for $m = q-1$. We have

$$\frac{\cos^2\alpha}{q^2}\{4\lambda^2(p-\beta)^2+\sum_{k=1}^{q-1}[\lambda^2(2p-2\beta+k)^2+\lambda^2k^2\tan^2\alpha$$

$$-k^2\sec^2\alpha].\prod_{j=0}^{k-1}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}\}$$

$$=\frac{\cos^2\alpha}{q^2}\{4\lambda^2(p-\beta)^2+\sum_{k=1}^{q-2}[\lambda^2(2p-2\beta+k)^2$$

$$+\lambda^2k^2\tan^2\alpha-k^2\sec^2\alpha]\prod_{j=0}^{k-1}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}$$

$$+[\lambda^2(2p-2\beta+q-1)^2+\lambda^2(q-1)^2\tan^2\alpha-$$

$$(q-1)^2\sec^2\alpha]\prod_{j=0}^{q-2}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}\}$$

$$=\prod_{j=0}^{q-2}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}\times$$

$$\{\frac{\lambda^2(2p-2\beta+q-1)^2\cos^2\alpha+\lambda^2(q-1)^2\sin^2\alpha}{q^2}\}$$

$$=\prod_{j=0}^{q-1}\frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}$$

Thus (3.1) holds for $m=q$ which proves lemma 5.

THEOREM 2. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in F_\lambda(\alpha,\beta,p)$, then

$$|a_n| \le \prod_{k=0}^{n-(p+1)} \frac{\lambda|2(p-\beta)\cos\alpha\, e^{-i\alpha}+k|}{k+1} \tag{3.2}$$

for $n \ge p+1$ and these bounds are sharp for all admissible α,β and λ for each n.

PROOF. As $f \in F_\lambda(\alpha,\beta,p)$, from Lemma 2, we have

$$\{e^{i\alpha}\sec\alpha\, zf'(z)+(p-2\beta-ip\tan\alpha)f(z)\}\,w(z)$$

$$=(p+ip\tan\alpha)f(z)-e^{-i\alpha}\sec\alpha zf'(z)$$

for $z \in U$, $w \in \Omega_\lambda$. Hence we have

$$\sum_{k=0}^{\infty}[\{(p+k)\,e^{i\alpha}\sec\alpha+(p-2\beta-ip\tan\alpha)\}\,a_{p+k}z^k]\,w(z)$$

$$=\sum_{k=0}^{\infty}[p+ip\tan\alpha-(p+k)e^{i\alpha}\sec\alpha]a_{p+k}z^k \tag{3.3}$$

where $a_p = 1$ and $w(z) = \sum_{k=0}^{\infty} b_{k+1}z^{k+1}$.

Equating coefficients of z^m on both sides of (3.3), we obtain

$$\sum_{k=0}^{m-1} \{(p+k)e^{i\alpha}\sec\alpha + (p-2\beta-ip\tan\alpha)\}\, a_{p+k}\, b_{m-k}$$

$$= \{p+ip\tan\alpha - (p+m)e^{i\alpha}\sec\alpha\}a_{p+m};$$

which shows that a_{p+m} on right hand side depends only on

$$a_p,\ a_{p+1},\ldots,\ a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\sum_{k=0}^{m-1} [\{(p+k)e^{i\alpha}\sec\alpha + (p-2\beta-ip\tan\alpha)\}\, a_{p+k}z^k]\, w(z)$$

$$= \sum_{k=0}^{m} [p + ip\tan\alpha-(p+k)e^{i\alpha}\sec\alpha]\, a_{p+k}\, z^k + \sum_{k=m+1}^{\infty} A_k z^k \tag{3.4}$$

for $m = 1,2,3\ldots$ and a proper choice of A_k $(k \geq 0)$.

Denoting the right member of (3.4) by $G(z)$ and the factor multiplying $w(z)$ in the left member of (3.4) by $F(z)$, (3.4) assmes the form

$$G(z) = F(z)\, w(z) \quad \text{for} \quad z \in U.$$

Since $|w(z)| < \lambda$ for $z \in U$ this yields for $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2\, d\theta \leq \frac{\lambda^2}{2\pi} \cdot \int_0^{2\pi} |F(re^{i\theta})|^2\, d\theta,$$

hence, using the definitions of $G(z)$ and $F(z)$

$$\sum_{k=0}^{m} |p+ip\tan\alpha - (p+k)\, e^{i\alpha}\sec\alpha|^2\, |a_{p+k}|^2\, r^{2k}$$

$$+ \sum_{k=m+1}^{\infty} |A_k|^2\, r^{2k} \leq$$

$$\lambda^2\{\sum_{k=0}^{m-1} |(p+k)e^{i\alpha}\sec\alpha + (p-2\beta-ip\tan\alpha)|^2\, |a_{p+k}|^2\, r^{2k}\}. \tag{3.5}$$

Setting $r \to 1$ in (3.5), the inequality (3.5) may be written as

$$\sum_{k=0}^{m-1} \{\lambda^2\, |(p+k)e^{i\alpha}\sec\alpha + (p-2\beta-ip\tan\alpha)|^2 -$$

$$|p+ip\tan\alpha - (p+k)e^{i\alpha}\sec\alpha|^2\}\, |a_{p+k}|^2$$

$$\geq |p+ip\tan\alpha - (p+m)e^{i\alpha}\sec\alpha|^2\, |a_{p+m}|^2\,. \tag{3.6}$$

Simplification of (3.6) leads to

$$|a_{p+m}|^2 \leq \frac{\cos^2\alpha}{m^2} \cdot \sum_{k=0}^{m-1} \{\lambda^2(2p-2\beta+k)^2 +$$

$$\lambda^2 k^2\tan^2\alpha - k^2\sec^2\alpha\}\, |a_{p+k}|^2. \tag{3.7}$$

Replacing $p+m$ by n in (3.7), we are led to

$$|a_n|^2 \leq \frac{\cos^2\alpha}{(n-p)^2} \cdot \sum_{k=0}^{n-(p+1)} \{\lambda^2(2p-2\beta+k)^2 +$$

$$\lambda^2k^2\tan^2\alpha - k^2\sec^2\alpha\}\, |a_{p+k}|^2 \tag{3.8}$$

where $n \geq p + 1$.

For $n = p + 1$, (3.8) reduces to

$$|a_{p+1}|^2 \le 4(p-\beta)^2 \lambda^2 \cos^2 \alpha$$

or

$$|a_{p+1}| \le 2(p-\beta) \lambda \cos \alpha \tag{3.9}$$

which is equivalent to (3.2).

To establish (3.2) for $n > p+1$, we will apply induction argument.

Fix n, $n \ge p + 2$, and suppose (3.2) holds for $k = 1,2,\dots, n-(p+1)$. Then

$$|a_n|^2 \le \frac{\cos^2\alpha}{(n-p)^2}\{4\lambda^2(p-\beta)^2 + \sum_{k=0}^{n-(p+1)} [\lambda^2(2p-2\beta+k)^2 + \lambda^2k^2 \tan^2 \alpha - k^2 \sec^2\alpha] \times \prod_{j=0}^{k-1} \frac{\lambda^2|2(p-\beta)\cos \alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}\} \tag{3.10}$$

Thus from (3.8), (3.10) and Lemma 5 with $m = n - p$, we obtain

$$|a_n|^2 \le \prod_{j=0}^{n-(p+1)} \frac{\lambda^2|2(p-\beta)\cos\alpha\, e^{-i\alpha}+j|^2}{(j+1)^2}.$$

This completes the proof of Theorem 2.

Equality holds in (3.2) for $n \ge P + 1$ for the function $f(z) \in A_p$ defined by (2.1) with $w(z) = \lambda z$.

REMARK ON THEOREM 2. For various choices of the parameters, known results can be regained: [7], [8], [12], [13], [14], [16] [20].

In a similar way we can prove the following: Lemma 6, 7, and Theorem 3 for functions in $C_\lambda(b,p)$.

4. REPRESENTATION FORMULAS FOR THE CLASS $C_\lambda(b,p)$

LEMMA 6. $g(z) \in C_\lambda(b,p)$ if and only if for $z \in U$

(i) $$\frac{zg''(z)}{g'(z)} = \frac{(p-1)+(p-2pb-1)w(z)}{1+w(z)}, \quad w \in \Omega_\lambda. \tag{4.1}$$

(ii) $$g'(z) = pz^{p-1} [\frac{g_1(z)}{z}]^{pb} \tag{4.2}$$

for some $g_1 \in S(\lambda)$.

(iii) $$g'(z) = pz^{p-1} \exp[-2pb \int_0^z \frac{\phi(t)}{1+t\, \phi(t)} dt], \tag{4.3}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \le \lambda$, $0 < \lambda < 1$, for $|z| < 1$.

5. COEFFICIENT ESTIMATES FOR THE CLASS $C_\lambda(b,p)$.

LEMMA 7. If integers p and m are greater than zero; $b \ne 0$ and complex, then

$$\prod_{j=0}^{m-1} \frac{\lambda^2|2pb+j|^2}{(j+1)^2} = \frac{1}{m^2} \{4 p^2|b|^2 \cdot \lambda^2 + \sum_{k=1}^{m-1} (k^2(\lambda^2-1) +4p^2|b|^2 \lambda^2 + 4pk \operatorname{Re}\{b\}\lambda^2) \prod_{j=0}^{k-1} \frac{\lambda^2|2pb+j|^2}{(j+1)^2}\}. \tag{5.1}$$

THEOREM 3. If $g(z) = z^p + \sum_{n=p+1}^{\infty} d_n z^n \in C_\lambda(b,p)$, then

$$|d_n| \le \frac{p}{n} \cdot \prod_{k=0}^{n-(p+1)} \frac{\lambda|2pb+k|}{(k+1)} \tag{5.2}$$

for $n \ge p+1$. Equality holds in (5.2) for the function $g(z) \in A_p$ defined by (4.1) with $w(z) = \lambda z$.

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