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ON ASYMPTOTIC NORMALITY FOR M-DEPENDENT U-STATISTICS

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ABSTRACT. Let (X_n) be a sequence of m-dependent random variables, not necessarily equally distributed. We give a Berry-Esseen estimate of the convergence to normality of a suitable normalization of a U-statistic of the (X_n). This bound holds under moment assumptions quite weaker than the existence of third moments for the kernel. Since we obtain the sharpest bound, the order of the bound can not be improved.

KEY WORDS AND PHRASES. U-Statistics, m-dependent random variables, Berry-Esseen bound.
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1. INTRODUCTION.

Let $(X_1, X_2, \ldots, X_n)$ be random variables (r.v.). A very important and common class of statistics is the class of U-statistics, of the type

$$U = \sum_{1 \le i_1 < \ldots < i_k \le n} h(X_{i_1}, \ldots, X_{i_k})$$

where h is a measurable symmetric function which is called a kernel. (For notational convenience, we do not consider the average of h but the sum of its values. This will not make any difference since we will normalize U latter.) In the case of an independent identically distributed (i.i.d.) sequence (X_i), it has been shown by various authors that the distribution function of normalized U-statistics converges to standard normal distribution function with the rate of $n^{-1/2}$, the latest and sharpest result being due to R. Helmers and W. R. Van Zwet [1]. These authors studied U-statistics of order 2, viz.,

$$U = \sum_{1 \le j < k \le n} h(X_j, X_k).$$

But as they pointed out, their results can be extended to any order and to the multi-sample case.

In this work, we relax the independent and equi-distributed assumptions about the sequence (X_i) of r.v.'s. Consider a m-dependent sequence (X_i) of r.v.'s, i.e.

for each $1 \le s < n-m$, the sequence $(X_i)_{i \le s}$ and $(X_i)_{i>s+m}$ are independent of each other, and which is not necessarily equi-distributed. Then we obtain a universal Berry-Esseen bound for the convergence of the suitably normalized U-statistic to standard normal. This bound involves the same moments as in the Helmers and Van Zwet's result, and leads the best rate of convergence for the independent case. We deal only with the one sample case of order two; but there is no doubt that the same methods could extend to the general case.

2. RESULTS.

Consider a fixed sequence of m-dependent r.v.'s, $X_1, \ldots, X_n$ and a fixed measurable function $h\colon R^2 \to R$. We want to study $U = \sum_{1 \le j \le k \le n} h(X_j, X_k)$. Let us denote by $\underline{F}_i$ the field generated by X_i. Then the sequence $\underline{F}_i$ is m-dependent, in the sense that for $i \le s < n-m$, the two σ-fields $\bigvee_{i \le s} \underline{F}_i$ and $\bigvee_{i>s+m} \underline{F}_i$ are independent (where $\bigvee_{i \in I} \underline{F}_i$ denote the σ-field generated by the $\underline{F}_i$, $i \in I$). It is conceptually more elegant, and also more convenient to look at $h(X_j, X_k)$ as a function $h_{j,k}$ which is $\underline{F}_j \vee \underline{F}_k$ measurable. So we fix a sequence $(\underline{F}_i)_{1 \le i \le n}$ of m-dependent σ-fields and consider for $1 \le j \le k \le n$ r.v. $h_{j,k}$ which are $\underline{F}_j \vee \underline{F}_k$ measurable. (Here we allow the possibility $j = k$ since the proof will be the same and since it is convenient and allows to extend the result to V-statistics.)

For $j \le k$, let

$$g_{j,k} = E(h_{j,k} | \underline{F}_k) \qquad g_{k,j} = E(h_{j,k} | \underline{F}_j).$$

Let $f_j = \sum_{1 \le j \le n} g_{j,i}$. Let $3/2 \le p < 2$. We make the following assumptions:

For any $1 \le j \le k \le n$, $E(h_{j,k}) = 0$ (2-1)

For any $1 \le j < k \le n$, $h_{j,k}$ has a moment of order p. (2-2)

For any $1 \le i \le n$, f_i has a moment of order 3. (2-3)

For $1 \le j < k \le n$, let $Y_{j,k} = h_{j,k} - g_{j,k} - g_{k,j}$. Suppose now $m = 1$. Let $\underline{G}_1 = \bigvee_{|i-j| \ge 2} \underline{F}_i$, $\underline{G}_2 = \bigvee_{|i-k| \ge 2} \underline{F}_i$. It is easily checked, and fundamental, that if $k \ge j+2$ we have $E(Y_{j,k} | \underline{G}_1) = E(Y_{j,k} | \underline{G}_2) = 0$. Moreover, if $k = j+1$ and $\underline{G} = \bigvee_{i \notin \{j-1, j, j+1, j+2\}} \underline{F}_i$, we have $E(Y_{j,j+1} | \underline{G}) = 0$. We set

$$S = \sum_{i=1}^n f_i, \ \sigma^2 = ES^2, \ L = \sigma^{-3} \sum_{i=1}^n E|f_i|^3.$$

For $a \in \{1, 3/2, p\}$, we set

$$M_a = \sigma^{-a} \sum_{j<k} E|Y_{j,k}|^a, \qquad M'_a = \sigma^{-a} \sum_{j<k} E|h_{j,k}|^a$$

$$M = (M_{3/2})^{2/3}, \qquad M' = (M'_{3/2})^{2/3}$$

$$N = \sigma^{-3/2} \sum_{0<k-j \le 3} E|Y_{j,k}|^{3/2}, \qquad N' = \sigma^{-3/2} \sum_{0<k-j \le 3} E|h_{j,k}|^{3/2}.$$

Let Φ denote the distribution of the standard normal law. The main result is as follows:

THEOREM A. Assume $m = 1$. There is a universal constant K such that

$$\sup_t |P(U\sigma^{-1} < t) - \Phi(t)| \le K\{L + M'L^{2/3} + NL^{1/3} + (\log L^{-1})^{p/2} M'_p$$
$$+ M'_p L^{5/3-p} + M'_1 L^2\}.$$

Let us investigate this theorem in the case that $h_{i,j} = h(X_i, X_j)$ and the X_i are i.i.d. If $g = (Eh(X_1, X_2)|X_2)$ and if we set $a = (Eg^2)^{-1/2}$, and for $t \in [1, 3]$ (resp. $[1, p]$) we set $b_t = E|g|^t$, $c_t = E|h_{1,2}|^t$, we get the bound

$$K\{n^{-1/2}a^3 b_3 + n^{-1/2}a^3(b_3 c_{3/2})^{2/3} + n^{-2/3}(c_{3/2})^{2/3} g_3^{1/3} a^{5/2}$$
$$+ (1/2 \log n - \log a^3 b_3)^{p/2} n^{2-3p/2} a^p c_p$$
$$+ n^{1/3-p/2} a^{5-2p} c_p b_p^{5/3-p} + n^{-1/2} a^7 c_1 b_3^2\}.$$

For $p = 3/2$, this bound is $O(n^{-1/4} \log^{3/4} n)$ and for $p = 5/3$, it is of the best possible order $O(n^{-1/2})$. A bound of the same order is obtained in the stationary case.

It is possible with our method to find many other bounds of the same type. An example of possible variation is given in section 6. More importantly, the term $M'_p L^{5/3-p}$ can be replaced by a term of the form $M'_p L^{\gamma-p}$ for any $\gamma < 2$ (but the constant K will grow very fast when γ gets close to 2). It is also possible to replace the term $M'_1 L^2$ by $M'_1 L^q$ for any $q > 0$, (but then K will grow with q). Hence it can be said that the main terms in theorem A are L, $M'L^{2/3}$ and $M'_p \log(L^{-1})^{p/2}$ if $p \le 5/3$ or $M'_p L^{5/3-p}$ if $p > 5/3$.

Theorem A can also be used to give bounds in the case $m > 1$. To do this, we proceed by blocking. More specifically if for $0 \le i \le [n/m]$, we set $\underline{G}_i = \bigvee_{j \in J_i} \underline{F}_i$, where $J_i = \{j: im < j \le \operatorname{Inf}((i+1)m, n)\}$, the fields $\underline{G}_i$ are 1-dependent. Moreover, if $h'_{j,k} = \sum_{\ell \in J_j, \ell' \in J_k} h_{\ell,\ell'}$, for $j \le k$, we have $U = \sum_{0 \le j \le k \le [n/m]} h'_{j,k}$, and it is possible to apply theorem A to this U-statistic. (It is very useful at this point that we did not assume that $h'_{j,k}$ is of the form $h(X_i, X_j)$ and to allow the case $j = k$!). The quantities involving the moments of the functions $(h'_{j,k})$ can easily be expressed in terms of the moments of the $h_{\ell,\ell'}$. However, this does not seem to be the case of the normalizing factor. This is why we do not state formally the result when $m > 1$.

3. METHODS.

We shall use three basic techniques, viz. the method of R. Helmers and W. Van Zwet [1], a method of V. Shergin [2] to deal with m-dependent r.v. and his result about the convergence to normality of a sequence of m-dependent r.v. and an estimate by the author of $|Ee^{itS}|$ where S is a sum of m-dependent r.v. [3] (which is also based on Shergin's technique).

We shall denote by $K_1, K_2, \ldots$ universal constants. No attempt has been made to find small numerical values: their choice is made crudely, to check the consistency of the construction.

We suppose $m = 1$. We shall use Esseen smoothing inequality [4].

$$\sup_t |P(U\sigma^{-1} \le t) - \phi(t)| \le K_1 \{T^{-1} + \int_{-T}^{T} |t|^{-1} |E \exp(itU\sigma^{-1}) - \exp(-t^2/2)| \, dt\}. \tag{3-1}$$

Let $\Delta = \sigma^{-1} \sum_{1 \le j < k \le n} Y_{j,k}$. We have

$$|E \exp(itU\sigma^{-1}) - \exp(-t^2/2)| \le |E \exp(it\sigma^{-1}S)(\exp(it\Delta)-1)| + |E \exp(it\sigma^{-1}S) - \exp(-t^2/2)|.$$

The second term will be taken care of by Shergin's theorem. Considering that $|e^{it} - 1 - it| \le 2|t|^p$, we have

$$|E \exp(it\sigma^{-1}S)(\exp(it\Delta)-1)| \le |t| |E(\Delta \exp(it\sigma^{-1}S))| + 2|t|^p E|\Delta|^p. \tag{3-2}$$

We shall evaluate these two terms directly. The above evaluation will be used for $t \le 10 \log L^{-1}$. For $t \ge 10 \log L^{-1}$, we have $\exp(-t^2/2) \le L^5$, and so it is enough to bound $|E \exp(itU\sigma^{-1})|$. Let I be an interval of $[1, n]$, (to be chosen appropriately). Let

$$\Delta_1 = \sigma^{-1} \sum_{\substack{1 \le j < k \le n \\ j,k \notin I}} Y_{j,k}, \qquad \Delta_2 = \Delta - \Delta_1.$$

We have, by expanding $\exp(it\Delta_2)$.

$$|E \exp(itU\sigma^{-1})| \le |E \exp(it(\sigma^{-1}S + \Delta_1)) + |t| |E \Delta_2 \exp(it\sigma^{-1}S + \Delta_1)| + 2|t|^p E|\Delta_2|^p \tag{3-3}$$

and we shall evaluate these three terms. In these evaluations, we shall several times encounter the same difficulty. Say, for example we want to estimate $E|\exp(it(\sigma^{-1}S + \Delta_1))|$. Let $S' = \sum_{i \in I} f_i$. Then S' depends on the $\underline{F}_i$ for $i \in I$. Moreover $\sigma^{-1}(S-S') + \Delta_1$ depends on the $\underline{F}_i$ for $i \notin I$.

If we knew that the $\underline{F}_i$ were independent, we would have

$$|E \exp(it(\sigma^{-1} S + \Delta_1))| \le |E \exp(it\sigma^{-1}S')|$$

for which good estimates are known (of the type $\exp(-t^2\sigma^{-2}E(S'^2))$ for t not too large). However, we must proceed in a different way. We shall use the technique mentioned above to show that modulo a small perturbation one can (roughly speaking) do as if S' and $\sigma^{-1}(S-S') + \Delta_1$ were independent and then use the estimate of theorem 4-5.

In order to prove theorem A, one can suppose $K \geq 10^{-3}$, so we can assume $L \leq 10^{-3}$. It then follows that if $H = [\log L^{-1} + 1]$, we have $HL \leq 10^{-2}$ and $H \geq 7$. Moreover for each i,

$$Ef_i^2 \leq (E|f_i|^3)^{2/3} \leq \sigma^2 L^{2/3} \leq \sigma^2/100. \tag{3-4}$$

4. LEMMAS.

This section contains some of the technical tools we need.

LEMMA 4-1 [4]. Let $X_1, \ldots, X_k$ be r.v.. Then for $r \geq 1$,

$$E|\sum_{i \leq k} X_i|^r \leq k^{r-1} \sum_{i \leq k} E|X_i|^r. \tag{4-1}$$

LEMMA 4-2 [5]. Let $X_1, \ldots, X_k$ be a martingale difference. Then for $p \leq 2$, we have

$$E|\sum_{i \leq k} X_i|^p \leq 2 \sum_{i \leq k} E|X_i|^p. \tag{4-2}$$

LEMMA 4-3 [2]. For a sequence $(X_i)_{i \leq n}$ of m-dependent r.v. of mean zero and $r \geq 2$,

$$E|\sum X_i|^r \leq (m+1)^{r-1} E(\sum X_i^2)^{r/2}. \tag{4-3}$$

THEOREM 4-4. (V. Shergin [2]). Let S be a sum of 1-dependent r.v. $(f_i)_{i \leq n}$ of zero mean and $\sigma^2 = ES^2$. Let $L = \sigma^{-3} \sum_{i \leq n} E|f_i|^3$. There exists a universal constant K_2 such that

$$\int_{|t|K_2 L \leq 1} |t|^{-1} |E \exp(itS\sigma^{-1}) - \exp(-t^{-2}/2)| dt \leq K_2 L. \tag{4-4}$$

THEOREM 4-5. [3]. Under the hypothesis of the above theorem one has for $|t|K_2 L \leq 1$ and $|t| \geq K_2$:

$$|E \exp(itS\sigma^{-1})| \leq (1+K_2|t|) \operatorname{Max}(\exp(-t^2/80), (tK_2L)^{1/4 \log L^{-1}}).$$

In particular, for $\sigma^{-1}K_2 \leq |t|$ and $\sigma|t|K_2 L \leq 1$, we have

$$|E \exp(itS)| \leq (1 + K_2\sigma|t|) \operatorname{Max}(\exp(-t^2\sigma^2/80), (t\sigma K_2 L)^{1/4 \log L^{-1}}). \tag{4-5}$$

PROPOSITION 4-6. With the above notation, for $a \in \underline{R}$, let

$$F = \{i \in [1, n]: \sigma^{-2} E|f_i|^2 \geq 400L^2\}.$$

Then

$$\sum_{i \in F} E|f_i|^2 \leq \sigma^2/20. \tag{4-6}$$

PROOF. Let q = card F. We have

$$\sigma^4 \sum_{i \in F} E|f_i|^2 \geq 400q \left(\sum_{1 \leq i \leq n} E|f_i|^3 \right)^2 \geq 400q \left(\sum_{i \in F} E|f_i|^3 \right)^2. \tag{4-7}$$

From Holder's inequality, we have

$$\sum_{i \in F} E|f_i|^2 \leq \sum_{i \in F} (E|f_i|^3)^{2/3} \leq q^{1/3} \left(\sum_{i \in F} E|f_i|^3 \right)^{2/3}.$$

So (4-7) gives

$$\sigma^4 \sum_{i\varepsilon F} E|f_i|^2 \geq 400(\sum_{i\varepsilon F} E|f_i|^2)^3.$$

The result follows.

5. BOUND FOR $E|\Delta_2|^p$.

We are going to bound at the same time $E|\Delta_2|^p$ and $E|\Delta|^p$ (notice $\Delta = \Delta_2$ if $I = [1, n]$.).

LEMMA 5-1. $E|\Delta_2|^p \leq K_3\sigma^{-p} \sum_{(j,k)\varepsilon V} E|Y_{j,k}|^p$,

where $V = \{j, k: 1 \leq j < k \leq n$, either j or k belongs to $I\}$.

PROOF. Since

$$\sigma\Delta_2 = \sum_{V_1} Y_{j,k} + \sum_{V_2} Y_{j,k} = \Delta' + \Delta''$$

$$V_1 = \{j, k: 1 \leq j < k \leq n, k \varepsilon I\}$$

$$V_2 = \{j, k: 1 \leq j < k \leq n, j \varepsilon I, k \notin I\},$$

it is enough by (4-1) to bound each of these sums. We bound the first one. The proof for the second is very similar. For $k \varepsilon I$ let $Z_k = \sum_{j\leq k-2} Y_{j,k}$.

Then

$$\Delta' = \sum_{k\varepsilon I} Z_k + \sum_{k\varepsilon I} Y_{k-1,k}.$$

For $i = 0, 1, 2$, let

$$A_i = \sum_{3\ell+i\varepsilon I} Y_{3\ell+i-1,3\ell+i}.$$

The sequence $Y_{3\ell+i-1,3\ell+i}$ is a martingale difference, since if

$$\underline{H}_\ell = \bigvee_{j\leq 3\ell+i} \underline{F}_j, \text{ then } E(Y_{3\ell+i-1,3\ell+i}|\underline{H}_{\ell-1}) = 0.$$

So (4-2) implies

$$E|A_i|^p \leq 2 \sum_{3\ell+i\varepsilon I} E|Y_{3\ell+i-1,3\ell+i}|^p.$$

Thus (4-1) implies

$$E|\sum_{k\varepsilon I} Y_{k-1,k}|^p \leq 6 \sum_{k\varepsilon I} E|Y_{k-1,k}|^p.$$

The sequence $(Z_{2k})_{2k\varepsilon I}$ is a martingale difference. Indeed, if $\underline{H}_k = \bigvee_{i\leq 2k} \underline{F}_i$, then Z_{2k} is $\underline{H}_k$ measurable and $E(Z_{2k+2}|\underline{H}_k) = 0$. A similar result holds for the sequence (Z_{2k+1}). So (4-1) and (4-2) give

$$E|\sum_{k\varepsilon I} Z_k|^p \leq 4 \sum_{k\varepsilon I} E|Z_k|^p.$$

Let us fix $k \varepsilon I$. Then, it is easily seen that both sequences $(Y_{2j,k})_{1\leq 2j\leq k-2}$ and $(Y_{2j+1,k})_{1\leq 2j+1\leq k-2}$ are martingale differences. It then follows by (4-1) and (4-2) that

$$E|Z_k|^p \leq 4 \sum_{j\leq k-2} E|Y_{j,k}|^p.$$

The result follows these estimates.

6. BOUND FOR $|E(\Delta \exp it\, \sigma^{-1}S)|$.

Let $H = [\log L^{-1} + 1]$. For $|t| \geq 10K_2$, let

$$\Psi(t) = (1+K_2|t|)\, \mathrm{Max}(\exp(-t^2/8000),\ (100K_2 tL)^{1/4(\log L^{-1}/1000)}) \qquad (6\text{-}1)$$

and for $|t| \leq 10K_2$, let $\Psi(t) = 1$.

LEMMA 6-1. There exist universal constants K_4, K_5 such that for $|tK_4L| < 1$, we have

$$|E(\Delta \exp(it\sigma^{-1}S))| \leq K_5\{t^2\Psi(t)ML^{2/3} + t\Psi(t)NL^{1/3} + M_1(40tL)^H\}.$$

We write

$$\sigma|E(\Delta \exp(it\sigma^{-1}S))| \leq \sum_{1\leq j<k\leq n} |E(Y_{j,k}\exp(it\sigma^{-1}S))|.$$

We shall evaluate each of the terms of the right-hand side.

LEMMA 6-2. If $k \leq j+2$, we have for $100K_2|t|L \leq 1$:

$$|E(Y_{j,k}\exp(it\sigma^{-1}S))|$$

$$\leq 24|t|\Psi(t)(E|Y_{j,k}|^{3/2})^{2/3}(\sigma^{-3}\sum_{i=j}^{k+1} E|f_i|^3)^{1/3} + E|Y_{j,k}|(40tL)^H.$$

PROOF. First step: We prove that one of the following cases occurs.
First Case : Their exist $s < j$ satisfying the following conditions:

$$\text{for } s \leq s' < j,\ E|\sum_{i=1}^{s'} f_i|^2 \geq \sigma^2/10 \text{ and} \qquad (6\text{-}2)$$

there are at least 2H indices $i \in\,]s, j-2[$ for which $E|f_i|^2 \leq 400\, L\sigma^2$. (6-3)

Second Case : There exists $s > k$ satisfying the following conditions:

$$E|\sum_{i=s'}^{n} f_i|^2 \geq \sigma^2/10, \text{ for } k < s' \leq s \text{ and} \qquad (6\text{-}4)$$

there are at least 2H indices $i \in\,]k+2, s[$ for which $E|f_i|^2 \leq 400\, L\sigma^2$. (6-5)

Indeed, let $S_1 = \sum_{1\leq i\leq j} f_i$, and $S_2 = S - S_1$. We have $\sigma^2 = ES_1^2 + ES_2^2 + 2Ef_jf_{j+1}$. It follows from (3-4) that $ES_1^2 + ES_2^2 \geq 98\sigma^2/100$. We prove that the first case occurs if $ES_1^2 \geq 49\sigma^2/100$ (or even $ES_1^2 \geq 24\sigma^2/100$. Let $F = \{i\epsilon I: E|f_i|^2 \geq 400\ \sigma^2L^2\}$. Let s be the largest integer such that $]s, j-2[$ contains 2H indices which do not belong to F. For $s' > s$, since $HL \leq 10^{-2}$ and $L \leq 10^{-3}$, from (4-3) and (4-6), we get

$$E|\sum_{i=s'+1}^{j} f_i|^2 \leq 2\sum_{i=s'+1}^{j} Ef_i^2 \leq 2\sum_{i\epsilon F} Ef^2 + 2(2H+3)400L^2\sigma^2$$

$$\leq \sigma^2/10 + 3\cdot 10^{-3}\sigma^2 + 16\cdot 10^{-3}\sigma^2 \leq 12\sigma^2/100.$$

It then follows that

$$E|\sum_{1\leq i\leq s'} f_i|^2 \geq ES_1^2 - E|\sum_{i=s'+1}^{j} f_j|^2 - 2\, Ef_{s'}f_{s'+1}$$

$$\geq 48\sigma^2/100 - 12\sigma^2/100 - 2\sigma^2/100 \geq \sigma^2/10.$$

Similarly, the second case occurs for $ES_2^2 \geq 49\sigma^2/100$.

Second Step. We suppose that the first case occurs, the second being similar. We can pick for $1 \leq \ell \leq H$, indices $s < p(\ell) < j-2$ such that $p(\ell) \notin F$ and $p(\ell+1) \leq p(\ell)-2$ for $1 \leq \ell \leq H-1$. Let

$$Z_0 = \sigma^{-1} \sum_{j-1 \leq i \leq k+1} f_i, \qquad Z_1 = \sigma^{-1} \sum_{i>p(1)} f_i - Z_0$$

and for $1 \leq \ell \leq H$ (resp. $1 \leq \ell \leq H-1$), let

$$Z_{2\ell} = \sigma^{-1} f_{p(\ell)} \quad (\text{resp. } Z_{2\ell+1} = \sigma^{-1} \sum_{p(\ell+1)<i<p(\ell)} f_i).$$

Let $S_\ell = \sigma^{-1}S - Z_0 - \ldots - Z_\ell$. Let us define $\gamma_\ell = \exp(itZ_\ell) - 1$. It is easy to check that

$$\begin{aligned} Y_{j,k}\exp(it\, S\sigma^{-1}) &= Y_{j,k}\exp(itZ_0)\exp(it\, S_1) \\ &+ \sum_{\ell=1}^{2H-1} Y_{j,k}\exp(itZ_0)\prod_{q=1}^{\ell}\gamma_q(itS_{\ell+1}) \\ &+ Y_{j,k}\exp(itZ_0)\prod_{q=1}^{2H}\gamma_q\exp(itS_{2H}). \end{aligned} \tag{6-6}$$

Note that for each q, $|\gamma_q| \leq 2$ and $E|\gamma_q| \leq |t|E|Z_q|$. The last term of (6-6) has an expectation bounded by

$$2^H E|Y_{j,k}| \prod_{q=1}^{H} E|\gamma_{2q}| \leq E|Y_{j,k}|(40tL)^H \tag{6-7}$$

since $E|Z_{2q}| = \sigma^{-1}E|f_{p(2q)}| \leq \sigma^{-1}(Ef^2_{p(2q)})^{1/2} \leq 20L$. For $1 \leq \ell \leq 2H-1$, $\prod_{q=1}^{\ell}\gamma_q \exp(itS_{\ell+1})$ is measurable for the σ-field generated by the $\underline{F}_i$ for $i \leq j-2$ and $i \geq k+2$, hence independent of $Y_{j,k}$. So, if $\gamma_0 = \exp(itZ_0)-1$, since $E(Y_{j,k}) = 0$, we get

$$\begin{aligned} E_\ell &= E(Y_{j,k}\exp(itZ_0)\prod_{q=1}^{\ell}\gamma_q\exp(itS_{\ell+1})) \\ &= E(Y_{j,k}\prod_{q=0}^{\ell}\gamma_q\exp(itS_{\ell+1})). \end{aligned}$$

Since $S_{\ell+1}$ is independent of $Y_{j,k}\prod_{q=0}^{\ell}\gamma_q$, we get

$$\begin{aligned} |E_\ell| &\leq E|Y_{j,k}\gamma_0| \prod_{2q\leq\ell} E|\gamma_{2q}| \cdot 2^{[\ell/2]+1}|E\exp(itS_{\ell+1})| \\ &\leq 2E|Y_{j,k}\gamma_0|(40tL)^{[\ell/2]}|E\exp(itS_{\ell+1})|. \end{aligned}$$

Since $S_{\ell+1}$ is of the form $S_{\ell+1} = \sigma^{-1}\sum_{i=1}^{s'} f_i$ where $s < s' < j$ by construction, it follows from step 1 that $\sigma'^2 = ES^2_{\ell+1} \geq 1/10$. So we have

$$L' = \sigma'^{-3}\sum_{i=1}^{s'} E|\sigma^{-1} f_i|^3 \leq 10^2\sigma'^{-1}L.$$

It follows from (4-5) that for $10^2K_2|t|L \le 1$ we have $|E\exp(itS_{\ell+1})| \le \Psi(t)$. Moreover, since $K_2 \ge 1$ and $10^2K_2|t|L \le 1$, we have $(20tL)^{[\ell/2]} \le 2^{-[\ell/2]}$. Finally,

$$E|Y_{j,k}\gamma_0| \le tE|Y_{j,k}Z_0| \le t(E|Y_{j,k}|^{3/2})^{2/3}(E|Z_0|^3)^{1/3}.$$

In the same manner, we have

$$E|Y_{j,k}\exp(itZ_0)\exp(itS_1)| \le E|Y_{j,k}\gamma_0|\Psi(t).$$

The lemma follows from these estimates and the estimate of $E|Z_0|^3$ from (4-1).

LEMMA 6-3. If $k > j+2$, we have for $100K_2|t|L \le 1$,

$$\sigma|E(Y_{j,k}\exp(it\sigma^{-1}S))|$$
$$\le 40t^2\Psi(t)(E|Y_{j,k}|^{3/2})^{2/3}(\sigma^{-3}\sum_{|i-j|\le 1}E|f_i|^3)^{1/3}(\sigma^{-3}\sum_{|i-k|\le 1}E|f_i|^3)^{1/3}$$
$$+ E|Y_{j,k}|(40tL)^H.$$

PROOF. First Step. One of the following cases occur: first case, second case: identical to the first and second case of lemma 6-2 respectively. Third case, there exists $j+2 < s_1 < s_2 < k-2$ satisfying the following conditions: for $j+1 < s' < s_1 < s_2 < s'' < k-1$, we have

$$E|\sum_{s'\le i\le s''}f_i|^2 \ge \sigma^2/10 \text{ and } \tag{6-8}$$

there are at least 2H indices $i \in]j+2, s_1[$ for which $E|f_i|^2 \le 400\sigma^2L$ and 2H indices $i \in]s_2, k-2[$ with the same property. (6-9)

The proof uses the same method as in lemma 6-2. We omit it.

Second Step. We shall treat only the first case. The second case is identical and the third uses the same idea. For $1 \le \ell \le H$, we pick indices $s \le p(\ell) < j-2$ such that $Ef^2_{p(\ell)} \le 400\sigma^2L^2$ and $p(\ell+1) \le p(\ell)-2$ for $1 \le \ell \le H-1$. Let

$$Z' = \sigma^{-1}\sum_{|i-j|\le 1}f_i,\ Z'' = \sigma^{-1}\sum_{|i-k|\le 1}f_i \text{ and } Z_0 = Z' + Z'',$$

and for $1 \le \ell \le 2H$, define Z_ℓ and γ_ℓ as in lemma 6-2. Then (6-6) holds.

Let $W_\ell = \prod_{q=1}^{\ell}\gamma_q\exp(itS_{\ell+1})$. For $1 \le \ell \le 2H-1$, the r.v. $\exp(itZ'')W_\ell$ is measurable for the σ-field $\underline{G}_1$ generated by the $\underline{F}_1$ for $|i-j| \ge 2$. (Here we use the fact that $k \ge j+3$). Since $E(Y_{j,k}|\underline{G}_1) = 0$, we have $E(Y_{j,k}\exp(itZ'')W_\ell) = 0$. Similar $E(Y_{j,k}\exp(itZ')W_\ell) = 0$ and $E(Y_{j,k}W_\ell) = 0$. It follows that if

$$\gamma_0 = (\exp(itZ') - 1)(\exp(itZ'') - 1),$$

we have

$$E_\ell = E(Y_{j,k}\exp(itZ_0)\prod_{q=1}^{\ell}\gamma_q\exp(itS_{\ell+1}))$$
$$= E(Y_{j,k}\gamma_0\prod_{q=1}^{\ell}\gamma_q\exp(itS_{\ell+1})).$$

The rest of proof is entirely similar to lemma 6-2, except that we estimate

$$E|Y_{j,k}Y_0| \le t^2E|Y_{j,k}Z'Z"| \le t^2(E|Y_{j,k}|^{3/2})^{2/3}(E|Z'Z"|^3)^{1/3}$$
$$\le t^2(E|Y_{j,k}|^{3/2})^{2/3}(E|Z'|^3)^{1/3}(E|Z"|^3)^{1/3}.$$

PROOF OF LEMMA 6-1 : Follows from lemmas 6-2 and 6-3 by the use of Holder's inequality (with $K_4 = 100K_3$).

REMARK: If, in the estimate of $E|Y_{j,k}Y_0|$, we use Holder's inequality with exponents p and $q = p/(p-1)$, we get in lemma (6-1) the bound

$$K_5(t^2\Psi(t)M_p^{1/2}Q^{2/q} + |t|\Psi(t)N_p^{1/2}Q^{1/q} + M_1(40tL)^H)$$

where $Q = \sigma^{-q}\sum_{i=1}^{n} E|f_i|^q$.

7. HOW TO CHOOSE I.

Let $10^{-2} \ge \theta \ge 3\cdot10^3HL^2$, which will be chosen later. The next lemma shows how to pick a subinterval I of {1, ..., n} which roughly speaking will play the role that interval {1, ..., $[n\theta]$} would play the i.i.d. case. Let

$$F = \{\ i \in \{1, \dots, n\} : E|f_i|^2 \ge 400\ L^2\}.$$

LEMMA 7-1. There exists an interval $I \subset \{1, \dots, n\}$ which has the following properties.

I is reunion of ten subintervals $I_1, \dots, I_{10}$, which (7-1)
extremities does not belong to F, and such that for $1 \le \ell \le 10$, each I_ℓ contains at least 2H elements which does not belong to F, and is such that $E(\sum_{i\in I_\ell} f_i)^2 \ge 2\theta\sigma^2$.

$$\sum_{i\in I} E|f_i|^3 \le 2400\theta \sum_{i=1}^{n} E|f_i|^3. \tag{7-2}$$

If A = {j, k; j < k, j or k belongs to I}, we have

$$\sigma^{-p}\sum_{(j,k)\in A} E|Y_{j,k}|^p \le 5000\theta M_p. \tag{7-3}$$

PROOF. Let s(1) = 1. We construct by introduction a sequence s(i) in the following way:

$$s(i+1) = \mathrm{Inf}\{s: s(i) < s \le n,\ s-1,\ s \notin F,\ [s(i), s[\setminus F \text{ contains}$$
$$\text{at least 2H elements, } E(\sum_{s(i)\le\ell<s} f_\ell)^2 \ge 2\theta\sigma^2\}.$$

The construction goes until we reach an integer s(h) such that either s(h) = n or no $s \in]s(h),n[$ satisfies the required conditions. In the second case we set s(h+1) = n.

We show now that $5\theta(h-1) \ge 1$. For each $1 \le i \le h$, let s'(i) be the largest index $s \ge s(i)$ such that $E(\sum_{s(i)\le\ell\le s} f_\ell)^2 \le 2\theta\sigma^2$. The definition of s(i+1) implies easily if $A_i =]s'(i), s(i+1)[\cap F$, then if $B_i =]s'(i), s(i+1)[\setminus A_i$, we have

card $B_i \le 2H +$ card A_i. For each i, we have

$$E\Big(\sum_{s(i)\le\ell<s(i+1)} f_\ell\Big)^2 \le E\Big(\sum_{s(i)\le\ell\le s'(i)} f_\ell\Big)^2$$
$$+ Ef^2_{s'(i)} + Ef^2_{s'(i)+1} + 2\sum_{\ell\in A_i} Ef^2_\ell + 2\sum_{\ell\in B_i} Ef^2_\ell .$$

It follows that

$$\sigma^2 \le 2h\theta\sigma^2 + \sum_{i=2}^{h}(Ef^2_{s(i)-1} + Ef^2_{s(i)}) + \sum_{i=1}^{h}(Ef^2_{s'(i)} + Ef^2_{s'(i)=1})$$
$$+ 2\sum_{i\in F} Ef_i^2 + 2\sum_i \sum_{\ell\in B_i} Ef^2_\ell .$$

Since $s(i)-1$, $s(i) \notin F$ it follows easily

$$\sigma^2 \le 2h\theta\sigma^2 + 4\sum_{i\in F} Ef_i^2 + 1600hL^2\sigma^2 + 800L^2\sum_{i=1}^{h} \text{card } B_i\sigma^2 .$$

But $\sum_{i=1}^{h}$ card $B_i \le 2Hh$ + card F, from (4-6) we got $400L^2$card $F \le \sigma^2/20$,

so we get finally

$$\sigma^2 \le 6\sigma^2/20 + h\sigma^2(2\theta + 1600L^2 + 1600HL^2).$$

Since $H \ge 7$, We get $7\sigma/10 \le 3\theta h\sigma^2$. So $\theta h \ge 7/30$. Since $10^{-2}\theta \le 1$, we have $h \ge 20$, so $5\theta(h-1) \ge 1$.

For $1 \le i \le h-1$, let $J_i =]s(i), s(i+1)[$. Let

$$a_i = \sigma^{-3}\sum_{\ell\in J_i} E|f_i|^3, \quad b_i = \sigma^{-p}\sum_{(j,k)\in A_i} E|Y_{j,k}|^p$$

where $A_i = \{(j,k): 1 \le j < k \le n,\ j \text{ or } k \text{ belongs to } J_i\}$.

It is easy to see that

$$\sum_{1\le i\le h-1} a_i \le L \quad\text{and}\quad \sum_{1\le i\le h-1} b_i \le 2M_p. .$$

It follows that there are at least $19(h-1)/20$ indices $1 \le i \le h-1$ for which $a_i \le 40(h-1)^{-1}L$ and $b_i \le 80(h-1)^{-1}M_p$. Since $(h-1) \ge 19$, it is possible to find ten consecutive indices $i+1, i+2, \ldots, i+10$ with this property. The lemma follows by letting $I_\ell = J_{i+\ell}$ for $i \le \ell \le 10$ and $I = \bigcup_{\ell=1}^{10} I_\ell$.

8. BOUND FOR $|E\exp(it(\sigma^{-1}S+\Delta_1))|$ and $|E\,\Delta_2\exp(it(\sigma^{-1}S+\Delta_1))|$.

We shall bound the above quantities when I is chosen as in the preceding paragraph. Let

$$\Psi(t,\theta) = (1+K_2|t|)\text{Max}(\exp(-t^2\theta^2/80),\ (2400tK_2L)^{1/4\log(L^{-1}\theta^{1/2}/2400)})$$

for $t \ge \theta^{-1}K_2$ and $\Psi(t,\theta) = 1$ otherwise. We first show that if s_1, $s_2 \in I$ are such that $[s_1, s_2]$ contains one of the intervals I_ℓ $(2 \le \ell \le 9)$ then, if

$S' = \sigma^{-1}\sum_{s_1\le i\le s_2} f_i$, we have $E|\exp(itS')| \le \Psi(t,\theta)$ whenever $2400|t|K_2L \le 1$.

Indeed, if $I_\ell = (s', s'')$, we have

$$E(S')^2 = E(\sum_{s_i \le i < s'} f_i)^2 + E(\sum_{i \in I_\ell} f_i)^2 + \sum(\sum_{s'' < i < s_2} f_i)^2$$

$$+ 2\, Ef_{s'-1}f_{s'} + 2\, Ef_{s''}f_{s''+1} \ge 2\theta\sigma^2 - 1600\sigma^2L^2 \ge \theta\sigma^2.$$

Moreover $\sigma^{-3} \sum_{s' \le i \le s''} E|f_i|^3 \le 2400\theta L$, so the result by (4-5).

LEMMA 8-1. If $|t|K_6L \le 1$, then

$$E|\Delta_2 \exp(it(\sigma^{-1}S + \Delta_1))| \le K_5M_1(\Psi(t,\theta) + (80tL)^H).$$

PROOF. Let us fix $j < k$. Then, among the intervals $I_1, \ldots, I_{10}$, it is possible to find three consecutive intervals J_1, J_2, J_3 which do not contain either j or k. So we can pick indices $p(\ell)$, $1 \le \ell \le H$ in $J_1 \setminus F$ such that $q(\ell+1) \ge p(\ell)+2$ for $1 \le \ell \le H-1$ and indices $q(\ell)$, $1 \le \ell \le H$ in $J_3 \setminus F$ such that $q(\ell+1) \le q(\ell)-2$ for $1 \le \ell \le H-1$. Let

$$Z_1 = \sigma^{-1} \sum_{i<p(1)} f_i + \sigma^{-1} \sum_{i>q(1)} f_i + \Delta_1$$

and for $1 \le \ell \le H$, let

$$Z_{2\ell} = \sigma^{-1}(f_{p(\ell)} + f_{q(\ell)})$$

and for $1 \le \ell \le H-1$,

$$Z_{2\ell-1} = \sigma^{-1} \sum_{p(\ell)<i<p(\ell+1)} f_i + \sigma^{-1} \sum_{q(\ell+1)<i<q(\ell)} f_i.$$

Let

$$S_\ell = (\sigma^{-1}S + \Delta_1) - \sum_{i \le \ell} Z_i \text{ and } \gamma_\ell = \exp(itZ_\ell) - 1.$$

We have

$$Y_{j,k}\exp(it\sigma^{-1}S) = Y_{j,k} \exp(itZ_1) \exp(itS_2)$$

$$+ Y_{j,k} \sum_{\ell=2}^{2H-1} \exp(itZ_1) \prod_{q=2}^{\ell} \gamma_q \exp(itS_{\ell+1})$$

$$+ Y_{j,k}\exp(itZ_1)\prod_{q=2}^{2H}\gamma_q\exp(itS_{2H}).$$

So, by using the same type of majorations as in section 6, and since $|\exp(it\, S_\ell)| \le \Psi(t,\theta)$ for $2 \le \ell \le 2H$ by the preceding remarks, we get

$$|E\, Y_{j,k}\exp(it\sigma^{-1}\, S)| \le 4E|Y_{j,k}|\Psi(t,\theta) + E|Y_{j,k}|(80tL)^H$$

and the lemma follows by summation (with $K_6 = 2400K_2 + K_4$).

A comparable but simpler proof yields the following.

LEMMA 8-2. If $|t|K_6L \le 1$, then

$$E|\exp(it(\sigma^{-1}S+\Delta_1))| \le K_5(\Psi(t,\theta) + (80tL^H)).$$

9. PROOF OF THEOREM A.

Since we can suppose $K_6 \ge 2400K_2$, straightforward computation from (6-1) shows that for $i = 0, 1, 2,$

$$\int_{e^{-12} \, |t| K_6 L < 1} |t|^i \Psi(t)dt \le K_7. \tag{9-1}$$

Let us denote J(T) the integral in the right hand side (3-1). Let

$$T_0 = \mathrm{Inf}\ ((80000 \cdot \log L)^{1/2},\ (e^{-40} K_6 L)^{-1}).$$

It follows from (3-2), theorem 4-4, lemmas 5-1 and 6-1, and 9-1 that (since $K_6 \ge 40$)

$$J(T_0) \le K_8 (L + (\log L^{-1})^{p/2} M_p + ML^{2/3} + NL^{1/3} + M_1 L^{10}).$$

For $T_0 \le t \le (e^{-40} K_6 L)^{-1}$ (if such t exists) we let

$$\theta = \mathrm{Max}\ (8000t^{-2} \log^{-1},\ 2400^2 L^{5/6}).$$

The first term is $\le 10^{-2}$, and it is indeed possible to assume $\theta \le 10^{-2}$ for otherwise since we can take $K \ge (24000)^2$, and theorem A will be automatically satisfied. Moreover $\theta \ge 4 \cdot 10^{-3} HL$. Hence choose I as in section 6 and use the estimates (3-3) and lemmas 5-1, 8-1 and 8-2. Notice that for this choice of θ, straightforward computation gives $\Psi(t,\theta) \le K_8 L^4$. It then follows that

$$J((\exp(-40)K_6 L)^{-1}) - J(T_0) \le K_9 (L + M_1 L^3 + M_p L^{5/6-p}).$$

If we put all these estimates together we get theorem A, with the bound as stated, but where the quantities with a "dash" are replaced by corresponding quantities without a dash. However, since for $1 \le r \le p$ we have $E|g_{j,k}|^r \le E(|h_{j,k}|^r | \underline{F}_j)$, lemma 4-1 shows that the "undashed" quantities are bounded by a universal constant times the "dashed" ones. This concludes the proof of theorem A.

In order to get the extensions of theorem A mentioned as remarks after the statement of this theorem one replaces e^{-40} by $e^{-(8q+2)}$ in the choice of T_0 and $L^{5/6}$ by L^{γ} in the choice of θ.

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