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ON A CLASS OF p-VALENT CLOSE-TO-CONVEX FUNCTIONS OF ORDER β AND TYPE α

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ABSTRACT. Let $S(A,B,p,\alpha)$ denote the class of functions $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ analytic in the unit disc $U = \{z: |z| < 1\}$ and satisfying the condition

$$\frac{zg'(z)}{g(z)} < \frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}, \quad z \in U, \quad -1 \leqq B < A \leqq 1, \quad o \leqq \alpha < p.$$

Let $C(A,B,p,\beta,\alpha)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ analytic in U, and satisfying the condition

$$\text{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta, \quad z \in U, \quad g \in S(A,B,p,\alpha).$$

In this paper we determine the coefficient estimates and distortion theorems for the class $C(A,B,p,\beta,\alpha)$.

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1. INTRODUCTION.

Let A_p (p a fixed integer greater than zero) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in $U = \{z: |z| < 1\}$. Let Ω denote the class of bounded analytic functions $w(z)$ in U satisfying the conditions $w(o) = o$ and $|w(z)| \leqq |z|$, for $z \in U$. We use P to denote the class of functions $P_1(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ which are analytic in U and have a positive real part there. Also we use $P(p,\beta)$ the class of functions that have the form $P(z) = p + \sum_{k=1}^{\infty} c_k z^k$, which are analytic in U and satisfy the conditions $P(o) = p$ and $\text{Re}\{P(z)\} > \beta$ $(o \leqq \beta < p)$ in U. The class $P(p,\beta)$ was introduced by Patil and Thakare [1]. It is well known that a function $P(z)$ is in $P(p,\beta)$ if and only if there exists a function $P_1(z) \in P$ such that

$$P(z) = (p - \beta) P_1(z) + \beta . \tag{1.1}$$

For $-1 \leqq B < A \leqq 1$ and $o \leqq \alpha < p$, denote by $S(A,B,p,\alpha)$ the class of functions $g(z) \in A_p$ which satisfy

$$\frac{zg'(z)}{g(z)} < \frac{p+[pB+(A-B)\ (p-\alpha)]\,z}{1+Bz}, \quad z \in U.$$

By definition of subordination it follows that $g(z) \in S(A,B,p,\alpha)$ has a representation of the form

$$\frac{zg'(z)}{g(z)} = \frac{p+[pB+(A-B)\ (p-\alpha)]w(z)}{1+Bw(z)}, \quad w \in \Omega. \tag{1.2}$$

Obviously $S(A,B,p,\alpha)$ is a subclass of the class $S_p(\alpha)$, $o \leqq \alpha < p$, of p-valent starlike functions of order α, investigated by Goluzina [2]. The class $S(A,B,p,\alpha)$ introduced by the author [3].

Moreover, let $C(A,B,p,\beta,\alpha)$ denote the class of functions $f(z) \in A_p$ which satisfy

$$\text{Re}\,\{\frac{zf'(z)}{g(z)}\} > \beta, \quad g \in S(A,B,p,\alpha). \tag{1.3}$$

Thus if $f(z) \in C(A,B,p,\beta,\alpha)$, then we may write

$$zf'(z) = g(z)\ P(z), \quad P(z) \in P(p,\beta). \tag{1.4}$$

We note that $C(1,-1,1,\beta,\alpha) = C(\beta,\alpha)$, is a subclass of the class of close-to-convex functions of order β and type α introduced by Libera [4].

Let $\phi(z) = \sum_{n=o}^{\infty} \alpha_n z^n$ and $\psi(z) = \sum_{n=o}^{\infty} \beta_n z^n$ be any two functions, then by $\phi(z) * \psi(z)$ we shall mean the Hadamard product or convolution of $\phi(z)$ and $\psi(z)$, that is

$$\phi(z) * \psi(z) = \sum_{n=o}^{\infty} \alpha_n \beta_n z^n .$$

We state below some lemmas that are needed in our investigation.

LEMMA 1 [1]. If $P(z) = p + \sum_{k=1}^{\infty} c_k z^k \in P(p,\beta)$, then

$$|c_n| \le 2(p-\beta) \text{ for all } n. \tag{1.5}$$

The function $P_o(z)$ defined by

$$P_o(z) = \frac{p+(p-2\beta)\ \delta_1 z}{1-\delta_1 z}, \quad |\delta_1| = 1 \tag{1.6}$$

shows that the result is sharp for each $n \ge 1$.

LEMMA 2 [1]. If $P(z) \in P(p,\beta)$, then for $|z| = r < 1$

$$\frac{p-|p-2\beta|r}{1+r} \le |P(z)| \le \frac{p+|p-2\beta|r}{1-r}. \tag{1.7}$$

Equality occurs for $P_o(z)$ defined by (1.6).

LEMMA 3 [5]. If $\psi(z)$ is regular in U, $\phi(z)$ and $h(z)$ are convex univalent in U such that $\psi(z) < \phi(z)$, then $\psi(z) * h(z) < \phi(z) * h(z)$, $z \in U$.

LEMMA 4 [3]. If $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in S(A,B,p,\alpha)$, then for $n \geqq p+1$

$$|b_n| \leq \prod_{j=o}^{n-(p+1)} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)}. \tag{1.8}$$

The function $g_o(z)$ defined by

$$g_o(z) = z^p(1+B\,\delta_2 z)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \neq o, \ |\delta_2| = 1$$

shows that the result is sharp for each $n \geqq p+1$.

2. COEFFICIENT ESTIMATES FOR THE CLASS $C(A,B,p,\beta,\alpha)$.

THEOREM 1. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C(A,B,p,\beta,\alpha)$, then for $n \geqq p+1$

$$|a_n| \leqq \frac{p}{n} \cdot \prod_{j=o}^{n-(p+1)} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)} + \frac{2(p-\beta)}{n}\left[1+ \sum_{k=o}^{n-(p+2)} \prod_{j=o}^{k} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)}\right]. \tag{2.1}$$

The result is sharp.

PROOF. Since $f \in C(A,B,p,\beta,\alpha)$, it follows that

$$zf'(z) = g(z)\,P(z), \tag{2.2}$$

where $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \in S(A,B,p,\alpha)$ and $P(z) = p + \sum_{k=1}^{\infty} c_k z^k \in P(A,B,p,\beta)$.

Hence

$$[pz^p + \sum_{k=1}^{\infty} (p+k)a_{p+k} z^{p+k}] = [z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}] \times [p + \sum_{k=1}^{\infty} c_k z^k]. \tag{2.3}$$

Equating coefficients of z^n on both sides of (2.3), we obtain

$$na_n = pb_n + c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_{n-p-1} b_{p+1} + c_{n-p}$$

This gives

$$n|a_n| \leqq p|b_n| + |c_1||b_{n-1}| + |c_2||b_{n-2}| + \dots + |c_{n-p-1}||b_{p+1}| + |c_{n-p}|. \tag{2.4}$$

Substituting the value from (1.5) and (1.8) in (2.4), we obtain

$$n|a_n| \leq p \cdot \prod_{j=o}^{n-(p+1)} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)} + 2(p-\beta) \times$$

$$\left[\prod_{j=o}^{n-(p+2)} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)} + \prod_{j=o}^{n-(p+3)} \frac{|(B-A)(p-\alpha)+Bj|}{(j+1)} + \dots + (A-B)(p-\alpha)+1 \right]$$

which, on simplification, takes the form of (2.1). The function $f_o(z)$ defined by

$$f_o'(z) = z^{p-1} \cdot \frac{p+(p-2\beta)\delta_1 z}{1-\delta_1 z} \cdot (1+B\,\delta_2 z)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \neq o,$$

$$|\delta_1| = |\delta_2| = 1, \tag{2.5}$$

shows that the bound (2.1) is sharp for each $n \geq p+1$.

Remarks on Theorem 1.

(1) Choosing $p=1$ and $\alpha=\beta=o$ in Theorem 1, we get the result due to Goel and Mehrok [6].

(2) Choosing $p=1$, $A=1$, $B=-1$ and $\alpha=\beta=o$ in Theorem 1, we get the result due to Reade [7].

3. DISTORTION THEOREMS.

LEMMA 5. If $g \in S(A,B,p,\alpha)$, then for all $|s| \le 1$, $|t| \le 1$, $(s \ne t)$

$$\frac{t^p g(sz)}{s^p g(tz)} < \left[\frac{(1+Bsz)}{(1+Btz)}\right]^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \ne o; \tag{3.1}$$

$$\frac{t^p g(sz)}{s^p g(tz)} < \exp\,[A(p-\alpha)(s-t)z], \quad B=o. \tag{3.2}$$

PROOF. The proof is similar to the one given by Ruscheweyh [8] and Goel and Mehrak [6].

We first consider the case when $B \ne o$. We have

$$\frac{zg'(z)}{g(z)} < \frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}, \quad z \in U.$$

This implies that

$$\frac{zg'(z)}{g(z)} - p < \frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz} - p = \frac{(A-B)(p-\alpha)z}{1+Bz}, \tag{3.3}$$

where $\frac{(A-B)(p-\alpha)z}{1+Bz}$ is convex, univalent in U. For $|s| \le 1$, $|t| \le 1$, $(s \ne t)$,

$$h(z) = \int_o^z \left(\frac{s}{1-su} - \frac{t}{1-tu}\right)du \tag{3.4}$$

is convex, univalent in U. (3.3) and (3.4) satisfy the conditions of lemma 3, and therefore

$$\left[\frac{zg'(z)}{g(z)} - p\right] * h(z) < \frac{(A-B)(p-\alpha)}{(1+Bz)} * h(z). \tag{3.5}$$

Now for every analytic function $q(z)$ with $q(o) = o$, we have

$$q(z) * h(z) = \int_{tz}^{sz} q(u)\,\frac{du}{u}. \tag{3.6}$$

By the application of (3.6), (3.5) can be written as

$$\int_{tz}^{sz} \left[\frac{ug'(u)}{g(u)} - p\right]\frac{du}{u} < (A-B)(p-\alpha)\int_{tz}^{sz} \frac{du}{1+Bu}$$

from which (3.1) follows.

Similarly for $B=o$, we obtain (3.2).

LEMMA 6. If $g \in S(A,B,p,\alpha)$, then for $|z| = r < 1$

$$r^p(1-Br)^{\frac{(A-B)(p-\alpha)}{B}} \le |g(z)| \le r^p(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \ne o, \tag{3.7}$$

$$r^p \exp(-A(p-\alpha)r) \le |g(z)| \le r^p \exp(A(p-\alpha)r), \quad B = o. \tag{3.8}$$

$$\left|\arg \frac{g(z)}{z^p}\right| \le \frac{(A-B)(p-\alpha)}{B} \sin^{-1}(Br), \quad B \ne o, \tag{3.9}$$

$$\left|\arg \frac{g(z)}{z^p}\right| \le A(p-\alpha)r, \quad B = 0. \tag{3.10}$$

These bounds are sharp, being attained by the function $g_o(z)$ defined by

$$g_o(z) = z^p(1+B\,\delta_2 z)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \neq o,$$

$$= z^p \exp\,(A(p-\alpha)\delta_2 z), \quad B = o, \ |\delta_2| = 1.$$

PROOF. The author [3] proved the results in lemma 6 using a different method. However, we deduce them from lemma 5.

Taking s=1, t=o in (3.1) and (3.2), we get

$$\frac{g(z)}{z^p} < (1+Bz)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \neq o, \tag{3.11}$$

$$\frac{g(z)}{z^p} < \exp(A(p-\alpha)z), \quad B = o. \tag{3.12}$$

(3.11) implies that $\frac{g(z)}{z^p} = (1+Bw(z))^{\frac{(A-B)(p-\alpha)}{B}}$, $B \neq o$.

(i) When $B > o$

$$\left|\frac{g(z)}{z^p}\right| = \left|(1+Bw(z))^{\frac{(A-B)(p-\alpha)}{B}}\right|$$

$$= \left|\exp\left[\frac{(A-B)(p-\alpha)}{B} \log\,(1+Bw(z))\right]\right|$$

$$= \exp\,\mathrm{Re}\left[\frac{(A-B)(p-\alpha)}{B} \log(1+Bw(z))\right]$$

$$= \exp\left(\frac{(A-B)(p-\alpha)}{B} \log|1+Bw(z)|\right)$$

$$\leq |1+Bw(z)|^{\frac{(A-B)(p-\alpha)}{B}}$$

$$\leq (1+Br)^{\frac{(A-B)(p-\alpha)}{B}}.$$

(ii) When $B < o$, put $B = -C$, $C > o$.

$$\left|(1+Bw(z))^{\frac{(A-B)(p-\alpha)}{B}}\right| = \left|((1-Cw(z))^{-1})^{\frac{(A-B)(p-\alpha)}{C}}\right|$$

$$= |(1-Cw(z))^{-1}|^{\frac{(A-B)(p-\alpha)}{C}}$$

$$\leq \left(\frac{1}{1-Cr}\right)^{\frac{(A-B)(p-\alpha)}{C}}$$

$$= (1+Br)^{\frac{(A-B)(p-\alpha)}{B}}.$$

Similarly (3.8) is a direct consequence of (3.12).

For $|z| = r$, from (3.11), we get

$$\left|\arg \frac{g(z)}{z^p}\right| = \frac{(A-B)(p-\alpha)}{B} |\arg\,(1+Bw(z))| \leq \frac{(A-B)(p-\alpha)}{B} \sin^{-1}(Br).$$

Similarly (3.10) is a direct consequence of (3.12).

THEOREM 2. If $f \in C(A,B,p,\beta,\alpha)$, then for $|z| = r < 1$

$$r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,(1-Br)^{\frac{(A-B)(p-\alpha)}{B}} \le |f'(z)| \le r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}, \quad B \ne 0, \tag{3.13}$$

$$r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,\exp(-A(p-\alpha)r) \le |f'(z)| \le r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,\exp(A(p-\alpha)r), \quad B = 0; \tag{3.14}$$

$$\int_0^r r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,(1-Br)^{\frac{(A-B)(p-\alpha)}{B}}\,dr \le |f(z)| \le \int_0^r r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}\,dr, \quad B \ne 0; \tag{3.15}$$

$$\int_0^r r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,\exp(-A(p-\alpha)r)\,dr \le |f(z)| \le \int_0^r r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,\exp(A(p-\alpha)r)\,dr, \quad B = 0 \tag{3.16}$$

All those inequalities are sharp.

PROOF. From (2.2), we have

$$|f'(z)| = \left|\frac{g(z)}{z}\right|\,|p(z)|\,. \tag{3.17}$$

Using (1.7), (3.7) and (3.8) in (3.17), we obtain (3.13) and (3.14).

Now

$$|f(z)| = \left|\int_0^r f'(z)\,dz\right| \le \int_0^r |f'(z)|\,dr \le \begin{cases} \displaystyle\int_0^r r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,(1+Br)^{\frac{(A-B)(p-\alpha)}{B}}\,dr, & B \ne 0; \\[2ex] \displaystyle\int_0^r r^{p-1}\,\frac{p+|p-2\beta|r}{1-r}\,\exp(A(p-\alpha)r)\,dr, & B = 0. \end{cases}$$

Let z_0, $|z_0| = r$, be chosen in such a way that $|f(z_0)| \le |f(z)|$, for all z, $|z| = r$. If $L(z_0)$ is the pre-image of the segment $\overline{[0,f(z_0)]}$ in U, then

$$|f(z_0)| = \int_{L(z_0)} |f'(z)|\,|dz| \ge \int_{L(z_0)} |f'(z)|\,dr \ge \begin{cases} \displaystyle\int_0^r r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,(1-Br)^{\frac{(A-B)(p-\alpha)}{B}}\,dr, & B \ne 0; \\[2ex] \displaystyle\int_0^r r^{p-1}\,\frac{p-|p-2\beta|r}{1+r}\,\exp(-A(p-\alpha)r)\,dr, & B = 0. \end{cases}$$

Equality signs in (3.13), (3.14), (3.15) and (3.16) are attained by the function $f_o(z)$ defined by (2.5) with $B \neq o$, and

$$f_o'(z) = z^{p-1} \frac{p+(p-2\beta)\delta_1 z}{1-\delta_1 z} \exp(A(p-\alpha)\delta_2 z), \quad B = o, \quad |\delta_1| = |\delta_2| = 1.$$

Remarks on Theorem 2.

Choosing $p=1$ and $\alpha=\beta=o$ in Theorem 2, we get the result due to Goel and Mehrok [6].

4. ARGUMENT OF $f'(z)$.

LEMMA 7. Let $p(z) \in P(p,\beta)$. Then for $|z| \leq r$,

$$\left|P(z) - \frac{p+(p-2\beta)r^2}{1-r^2}\right| \leq \frac{2(p-\beta)r}{1-r^2}. \tag{4.1}$$

PROOF. It is well known [9] that for $P_1(z) \in P$

$$\left|P_1(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}. \tag{4.2}$$

Thus the result follows from (1.1) and (4.2).

LEMMA 8. If $P(z) \in P(p,\beta)$, then

$$|\arg P(z)| \leq \sin^{-1} \frac{2(p-\beta)r}{p+(p-2\beta)r^2}, \quad |z| = r. \tag{4.3}$$

The bound is sharp.

PROOF. The proof follows from Lemma 7. To see that the result is sharp, let

$$P(z) = p\left\{\frac{1+(1-\frac{2\beta}{p})\delta_1 z}{1-\delta_1 z}\right\},$$

$$\delta_1 = \frac{r}{z}\left\{\frac{\frac{2\beta}{p} r}{1-(1-\frac{2\beta}{p})r^2} + \frac{i\sqrt{1-(1-\frac{2\beta}{p})^2 r^2}\sqrt{1-r^2}}{1-(1-\frac{2\beta}{p})r^2}\right\}. \tag{4.4}$$

THEOREM 3. If $f \in C(A,B,p,\beta,\alpha)$, then

$$\left|\arg \frac{f'(z)}{z^{p-1}}\right| \leq \frac{(A-B)(p-\alpha)}{B}\sin^{-1}(Br) + \sin^{-1}\frac{2(p-\beta)r}{p+(p-2\beta)r^2}, \quad B \neq o, \tag{4.5}$$

$$\left|\arg \frac{f'(z)}{z^{p-1}}\right| \leq A(p-\alpha)r + \sin^{-1}\frac{2(p-\beta)r}{p+(p-2\beta)r^2}, \quad B = o. \tag{4.6}$$

These inequalities are sharp.

PROOF. From (2.2), we have $\frac{f'(z)}{z^{p-1}} = \frac{g(z)}{z^p} P(z)$. Thus

$$\arg \frac{f'(z)}{z^{p-1}} = \arg \frac{g(z)}{z^p} + \arg P(z). \tag{4.7}$$

Using (3.9), (3.10) and (4.3) in (4.7), we obtain (4.5) and (4.6). Equality signs in (4.5) and (4.6) hold for the function $f_1(z)$ and $f_2(z)$ respectively, where

$$f_1'(z) = p\, z^{p-1} \frac{1 + (1-\frac{2\beta}{p})\delta_1 z}{1 - \delta_1 z} (1+B\,\delta_2\, z)^{\frac{(A-B)(p-\alpha)}{B}} ,$$

$$f_2'(z) = pz^{p-1} \frac{1 + (1-\frac{2\beta}{p})\delta_1 z}{1 - \delta_1 z} . \exp(A(p-\alpha)\,\delta_2 z) ,$$

where

$$\delta_1 = \frac{r}{z} \left\{ \frac{2\frac{\beta}{p} r}{1-(1-\frac{2\beta}{p})r^2} + \frac{i\sqrt{1-(1-\frac{2\beta}{p})^2 r^2}\,\sqrt{1-r^2}}{1-(1-\frac{2\beta}{p})r^2} \right\} ,$$

$$\delta_2 = \frac{r}{z}\,[-Br + i\sqrt{1-B^2 r^2}\,].$$

Remarks on Theorem 3.

(1) Choosing $p=1$ and $\alpha=\beta=o$ in Theorem 3, we get the result due to Goel and Mehrok [6].

(2) Choosing $p=1$, $A=1$ and $B= -1$ in Theorem 3, we get the result due to Silverman [10].

(3) Choosing $p=1$, $A=1$, $B= -1$ and $\alpha=\beta=o$ in Theorem 3, we get the result due to Ogawa [11] and Krzyz [12].

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