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SOME CONDITIONS FOR FINITENESS OF A RING

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ABSTRACT. Extending a result of Putcha and Yaqub, we prove that a non-nil ring must be finite if it has both ascending chain condition and descending chain condition on non-nil subrings. We also prove that a periodic ring with only finitely many non-central zero divisors must be either finite or commutative.

KEYS WORDS AND PHRASES. Finiteness of rings, Periodic rings, Commutativity.
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1. INTRODUCTION AND TERMINOLOGY.

Over the years several authors have given sufficient conditions for a ring R to be finite, among them the following:

(I) (Szele, [9]) R has both ascending chain condition and descending chain condition on subrings;

(II) (Ganesan, [4], [5]) R has non-trivial left zero divisors, of which there are only a finite number;

(III) (Bell, [1]) R contains no infinite zero ring and no infinite subring without non-zero nilpotent elements;

(IV) (Putcha and Yaqub, [8]) R is non-nil and has only finitely many non-nilpotent elements.

The present study, which presents some new conditions for finiteness, was motivated by the Putcha-Yaqub paper. Our first two theorems are ones suggested by that paper; the third is a new result on the old theme of commutativity and finiteness.

Throughout the paper the term zero divisor will refer to a one-sided (i.e. not necessarily two-sided) zero divisor. By a left (right) zero divisor we shall mean an element y for which there exists $x \neq 0$ such that $yx = 0$ ($xy = 0$).

If $x_1, x_2, \dots, x_k \in R$, the subring generated by the x_i will be denoted by $\langle x_1, x_2, \dots, x_k \rangle$; and for each $x \in R$, the symbols $A_\ell(x)$ and $A_r(x)$ will denote respectively the left and right annihilators of x. The symbols C and N will be used for the center of R and the set of nilpotent elements of R. The symbol Z will denote the ring of integers, and Z^+ the set of positive integers.

Finally, the ring R is called periodic if for each $x \in R$, there exist distinct $m, n \in Z^+$ for which $x^m = x^n$.

2. TWO FINITENESS THEOREMS FOR NON-NIL RINGS.

Our first theorem, which employs (IV) in its proof, is an extension of (II).

THEOREM 1. Let R be a ring, and let S be the set of non-nilpotent zero divisors of R. If S is finite and non-empty, then R is finite.

PROOF. Let $x \in S$. Applying the pigeonhole principle to the powers of x yields distinct $m,n \in Z^+$ for which $x^m = x^n$; consequently, there exists a non-zero idempotent zero divisor e, which we assume to be a right zero divisor. Write $R = eR + A_r(e)$. Since each summand consists of zero divisors of R, each has only finitely many non-nilpotent elements, hence by (IV) is either finite or nil. It is immediate that eR is finite, and to complete the proof we proceed on the assumption that $A_r(e)$ is nil. Let $0 \neq x \in A_r(e)$, with $x^s = 0 \neq x^{s-1}$. Then $(e+x)x^{s-1} = 0$, so e+x is a zero divisor. Moreover, e+x is non-nilpotent, since for any $k \geq s$, we have $(e+x)^k = e + \sum_{i=1}^{k-1} x^i e$; and the assumption that $(e+x)^k = 0$ gives, on left multiplication by e, the contradiction $e = 0$. It follows that the set $\{e+x \mid x \in A_r(e)\}$ is finite, hence $A_r(e)$ is finite and so is R.

THEOREM 2. If R is any non-nil ring having both ascending chain condition and descending chain condition on non-nil subrings, then R is finite.

PROOF. Note that by (I) and (III), any infinite ring R satisfying our hypotheses, and indeed every infinite subring of R, must contain an infinite zero ring. Moreover, for any non-nilpotent element x, the chain $\langle x\rangle \supseteq \langle x^2\rangle \supseteq \langle x^4\rangle \supseteq \dots$ becomes stationary at some point, hence there exist $n \in Z^+$ and $p(x) \in Z[X]$ for which $x^n = x^{n+1}p(x)$; and since this last condition is obviously satisfied by nilpotent elements as well, a result of Chacron ([3], [2, Theorem 1]) shows that R is periodic, hence contains non-zero idempotents. The following lemma gives crucial information about the idempotents.

LEMMA. If R satisfies the hypotheses of Theorem 2 and e is any non-zero idempotent, then $A_r(e)$ and $A_\ell(e)$ are finite.

PROOF. Assume without loss that e is a left zero divisor; note that in any periodic ring, idempotents have finite additive order. Recall our initial remark, which implies that if $A_r(e)$ is infinite, it must contain an infinite zero ring.

Let B be any zero ring contained in $A_r(e)$, and let u be an arbitrary element of B. Considering the chain $\langle e,u\rangle \supseteq \langle e,2u\rangle \supseteq \langle e,4u\rangle \supseteq \dots$ yields $k \in Z^+$ such that $2^k u \in \langle e, 2^{k+1}u\rangle$ - that is, there exist $p,q,t \in Z$ such that

$$2^k u = pe + q2^{k+1}u + t2^{k+1}ue.$$

Left-multiplying by e yields pe=0, hence $(2^k - q2^{k+1})u = t2^{k+1}ue$, and the fact that e has finite additive order shows that u does also. We now know that any subring E of R generated by e and a finite number of elements of B is finite. Choosing a maximal E, say E_1, and noting that $B \subseteq E_1$, we see that B is finite. The proof of the lemma is now complete.

Returning to the proof of Theorem 2, suppose that e is an idempotent which is a zero divisor, say a left zero divisor; and write $R = eR + A_r(e) = eRe + (eR \cap A_\ell(e)) + A_r(e)$. The last two summands are finite by the lemma, and the first is a ring satisfying our original hypotheses and having a multiplicative identity element. Of course, if all idempotents of R are regular, then R has a multiplicative identity element; therefore, we have reduced the problem to proving the theorem under the additional hypothesis that R has 1, in which case the periodicity of R implies that R has non-zero characteristic.

If there exists a non-zero idempotent $f \neq 1$, the decomposition $R = fR + (1-f)R$ shows that R is finite, since both summands are finite by the lemma. Therefore, assume that 1 is the only non-zero idempotent, and use the periodicity of R to obtain the property that every element is either nilpotent or invertible - a property that forces N to be an ideal [7]. The factor ring $\frac{R}{N}$ has ascending chain condition and descending chain condition on subrings, hence is finite by (I). Now consider N, and let B_1 be any zero ring contained in N. Among subrings of R generated by 1 and finitely many elements of B_1, choose M to be a maximal one. Note that M is finite and $B_1 \subseteq M$; hence B_1 is finite, N is finite, and R is finite.

3. A THEOREM ON PERIODIC RINGS.

The final theorem may be thought of as an extension of Herstein's result ([6], [2, Theorem 2]) that periodic rings with N C are necessarily commutative.

THEOREM 3. Let R be a periodic ring having only finitely many non-central zero divisors. Then R is either finite or commutative.

PROOF. Let n(R) denote the number of non-central zero divisors, and note that Herstein's result implies commutativity of R if $n(R) = 0$. Assume henceforth that $n(R) \geqq 1$; and consider first the case that every element of R is a left zero divisor or, more generally, the case that the set D of left zero divisors is a non-trivial additive subgroup of R. Then for $d \in D \setminus C$ and $u \in D \cap C$, $d+u \in D \setminus C$; hence $\{d+u \mid u \in D \cap C\}$ is finite. Thus, D is finite; and R is finite by (II). This argument covers the case R=N, so we assume that $R \neq N$ and therefore R contains non-zero idempotents.

If every non-zero idempotent is regular, there exists a unique non-zero idempotent, necessarily 1; and every element is invertible or nilpotent. It follows, again by [7], that N is an ideal; and since N is equal to the set D of left zero divisors, R is finite.

Assume now that we have a counterexample R with n(R) as small as possible. Then there exists $y \notin D$ and therefore an idempotent $e \notin D$. Thus R has a left identity element; and since we can repeat our previous arguments for right zero divisors, R has a right identity as well, hence R has 1. Moreover, by the argument in the previous paragraph, R has an idempotent e which is a zero divisor. If $e \notin C$, then at least one of eR and Re must be non-commutative. On the other hand, if $e \in C$, then $R = eR \oplus (1-e)R$, where $\oplus$ denotes a ring-theoretic direct sum; and since R was a counterexample, one of the summands must be non-commutative. Thus, in any event we may assume eR to be non-commutative.

Now eR must contain a non-central element d which is a left zero divisor in eR; otherwise, eR would be commutative by Herstein's result. For $u = (1-e)x \in C \cap (1-e)R$, we have $eu=ue=0$, hence u left-annihilates eR and d+u is a non-central left zero divisor in R. Thus, $C \cap (1-e)R$ is finite; and since $(1-e)R$ consists of zero divisors in R, it contains only finitely many elements not in C, hence must be finite. Now eR cannot be finite as well, since $R = eR + (1-e)R$; therefore $n(eR) = n(R)$, and every non-central zero divisor in R must be a zero-divisor in eR. It follows that $(1-e)R \subseteq C$. But then for any non-central zero divisor d and any element $u \in (1-e)R$, d+u is a non-central zero divisor, so both d and d+u are in eR and therefore $u \in eR$. But this implies $(1-e)R = \{0\}$, which is a contradiction. This completes the proof.

4. REMARKS.

In Theorem 3 the hypothesis of finitely many non-central zero divisors cannot be replaced by the assumption that R has only finitely many non-central nilpotent elements. A counterexample is the direct sum $F \oplus S$, where F is an infinite periodic field and S is a finite non-commutative nil ring.

A plausible extension of the Putcha-Yaqub result - namely, that a ring R having only a finite number of regular elements must either be finite or consist entirely of zero divisors - is also false, even for commutative rings. To see this, consider the algebra A over GF(2) having basis $\{1, e_1, e_2, \ldots, e_n, \ldots\}$, where the e_i are pairwise orthogonal idempotents. Certainly A is not finite, and it is easily shown that 1 is the unique regular element.

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REFERENCES

1. BELL, H.E. Infinite Subrings of Infinite Rings and Near-rings, Pacific J. Math. 59 (1975), 345-358.

2. BELL, H.E. On Commutativity of Periodic Rings and Near-rings, Acta Math. Acad. Sci. Hungar. 36 (1980), 293-302.

3. CHACRON, M. On a Theorem of Herstein, Canad. J. Math. 21 (1969), 1348-1353.

4. GANESAN, N. Properties of Rings with a Finite Number of Zero Divisors, Math. Ann. 157 (1964), 215-218.

5. GANESAN, N. Properties of Rings with a Finite Number of Zero Divisors II, Math. Ann. 161 (1965), 241-246.

6. HERSTEIN, I.N. A Note on Rings with Central Nilpotent Elements, Proc. Amer. Math. Soc. 5 (1954), 620.

7. OUTCALT, D. and YAQUB, A. A Commutativity Theorem for Rings, Bull. Austral. Math. Soc. 2 (1970), 95-99.

8. PUTCHA, M.S. and YAQUB, A. Rings with a Finite Set of Nonnilpotents, Internat. J. Math. & Math. Sci. 2 (1979), 121-126.

9. SZELE, T. On a Finiteness Criterion for Modules, Publ. Math. Debrecen 3 (1954), 253-256.