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COMPUTATION OF RELATIVE INTEGRAL BASES FOR ALGEBRAIC NUMBER FIELDS

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ABSTRACT. At first we are given conditions for existence of relative integral bases for extension $(K;k) = n$. Then we will construct relative integral bases for extensions $O_{K_6}(\sqrt[6]{-3})/O_{k_2}(\sqrt{-3})$, $O_{K_6}(\sqrt[6]{-3})/O_{k_3}(\sqrt[3]{-3})$, $O_{K_6}(\sqrt[6]{-3})/Z$.

KEY WORDS AND PHRASES. Integral Bases and Principal Ideal.

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1. EXISTENCE OF A RELATIVE INTEGRAL BASES.

The following criterion has been shown in [1] for existence of a Relative Integral Bases, for any finite extension K/k.

THEOREM 1.1. Let $(K;k) = n$, and let h_k be an odd integer, then O_K has a "relative integral bases" over $O_k \leftrightarrow d_{K/k}$ is a principal ideal. See also [2].

COROLLARY 1.2. If O_K = P.I.D., then $h_k = 1$ and $d_{K/k}$ = P.I. Therefore for every finite extension of k where O_k = P.I.D., a relative integral bases exists.

Let $k_1 = Q$, $k_2 = Q(\sqrt{-3})$, $k_3 = Q(\sqrt[3]{-3})$, $K_6 = Q(\sqrt[6]{-3})$. Since $h_{k_1} = h_{k_2} = h_{k_3} = 1$, so O_{K_1}, O_{K_2}, O_{K_3} are P.I.D. and then by corollary 1.2, relative integral bases for extensions K_6/k_1, K_6/k_2, K_6/k_3 exists.

Now, we will compute the relative discriminant for the extensions. Let $(K;k) = n$ and for some $\theta \in K$, $O_K = O_k(\theta)$ and θ satisfies an equation $F(\theta) = 0$ of degree n. Then $D_{K/k} = (F(\theta) = \Pi(\theta - \theta^{(t)})$, where $\theta, \theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}$ are conjugates [3].

Since extensions K_2/K_1, K_3/K_1 have discriminant divisible by 3 [3], by theorem in [3] discriminants K_6/k_2, K_6/k_3, K_6/k_1 are also divisible by 3 and 3 is completely ramified in k_1, k_2, k_3.

For extension K_6/k_2, $\theta = \sqrt[6]{-3}$ we therefore have:

$$D_{K_6/k_2} = (\theta - \theta^{(1)})(\theta - \theta^{(2)}) = (\sqrt[6]{-3} - \rho\sqrt[6]{-3})\ (\sqrt[6]{-3} - \rho^2\cdot\sqrt[6]{-3}),$$

$$D_{K_6/k_2} = (-3)^{4/3} \text{ for } \rho = \frac{-1 + \sqrt{-3}}{2}.$$ By the definition in [4],

$$d_{K_6/k_2} = N_{K_6/k_2}\ (D_{K_6/k_2}) = (-3)^4.$$

For extension K_6/k_3, $\theta = \sqrt[6]{-3}$, $D_{K_6/k_3} = (\theta - \theta^{(1)}) = (-3)^{1/6}$, then $d_{K_6/k_3} = (-3)^{1/2}$.

By theorem in [4], $D_{K_6/k_1} = D_{K_6/k_2} \cdot D_{k_2/k_1} = (-3)^{4/3} \cdot (-3)^{1/2} = (-3)^{11/6}$, then $d_{K_6/k_1} = (-3)^{11}$.

Now we will construct relative integral bases for the extensions. See also [5] for associated work.

For K_3/k_1, $O_{K_3} = (1, \sqrt[3]{-3}, \sqrt[3]{(-3)^2}) \cdot Z$, [3].

For K_2/k_1, $O_{K_2} = (1, \frac{1+\sqrt{-3}}{2}) \cdot Z$, [3].

2. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/O_2(\sqrt{-3})$.

Let $O_6 = (1,\alpha,\beta)O_2$ for α,β in O_6. By theorem in [6], disc $(1,\alpha,\beta) = d_{K_6/k_2}$,

$$\mathrm{disc}(1,\alpha,\beta) = \begin{vmatrix} 1 & \alpha & \beta \\ 1 & \rho\alpha & \rho^2\beta \\ 1 & \rho^2\alpha & \rho\beta \end{vmatrix}^2 = d_{K_6/k_2} = (-3)^4.$$

Now $\alpha^2\beta^2(3\rho^2 - 3\rho)^2 = (-3)^4$ and from here $\alpha \cdot \beta = \sqrt{-3}$.

We may take $\alpha = \sqrt[6]{-3}$ and $\beta = \sqrt[6]{(-3)^2}$, because they satisfy an $\alpha.\beta = \sqrt{-3}$ and they are in O_6.

Since $N_{6/3}(\alpha) = \sqrt[3]{-3}$ and $N_{6/3}(\beta) = \sqrt[3]{(-3)^2}$ are in O_3, we have:

$$O_6 = (1, \sqrt[6]{-3}, \sqrt[6]{(-3)^2})\, O_2.$$

3. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/O_3(\sqrt[3]{-3})$.

Let $O_6 = (1,\alpha)O_3$ for $\alpha \in O_6$. Again by theorem [6]

$$\mathrm{disc}(1,\alpha) = \begin{vmatrix} 1 & \alpha \\ 1 & -\alpha \end{vmatrix}^2 = 4\alpha^2 = d_{K_6/k_3} = \sqrt[3]{-3},$$

Note $\alpha = \frac{\sqrt[6]{-3}}{2} \notin O_6$, because $N_{6/3}(\alpha) = \frac{\sqrt[6]{-3}}{2} \; \frac{-\sqrt[6]{-3}}{2} = \frac{-\sqrt[3]{-3}}{4} \; \varepsilon \; O_3$. Hence, $(1,\alpha)$ is not a relative integral bases.

We define $\alpha = \frac{\beta + \sqrt[3]{-3}}{2}$ for $\beta \in O_3$ such that $N_{6/3}(\alpha)$ is divisible by $2.2 = 4$ and $\alpha \in O_6$. If we take $\beta = \sqrt[3]{(-3)^2} \in O_3$, it satisfies the conditions, this is because

$$\frac{\beta + \sqrt[6]{-3}}{2} \cdot \frac{\beta - \sqrt[6]{-3}}{2} = \frac{\sqrt[3]{(-3)^4} - \sqrt[6]{(-3)^2}}{4} = \sqrt[3]{-3} \in O_3, \text{ by theorem [6]},$$

Also, $\mathrm{disc}(1,\alpha) = d_{k_6/k_3}$, so that:

$$O_6 = \left(1, \frac{\sqrt[3]{(-3)^2} + \sqrt[6]{-3}}{2}\right) \cdot O_3.$$

4. RELATIVE INTEGRAL BASES FOR $O_6(\sqrt[6]{-3})/Z$.

Since $K_6 = Q(\sqrt[6]{-3})$, at first we start by:

$$O_6 = (1,\theta,\theta^2,\theta^3,\theta^4,\theta^5)\, Z$$

Let $\theta = \sqrt[6]{-3} \in O_6$. Since $\mathrm{disc}(1,\theta,\theta^2,\theta^3,\theta^4,\theta^5) = 2^2 \cdot 2^2 \cdot 2^2 \cdot d_{K_6/k_1}$, we can apply theorem [3] in order to cancel out $2^2 \cdot 2^2 \cdot 2^2$ and generate a new bases.

We will build a new bases $\alpha_i^* = \{\alpha_i : 0 \le i \le 5\}$. By the theorem [3] we check which α_i is going to be changed. $\alpha_0{}^* = \alpha_0/2 = 1/2 \notin 0_6$. Thus there is no change for the first bases element $\alpha_0 = 1$.

$\alpha_1{}^* = \dfrac{g_i\alpha_0 + \alpha_1}{2} = \dfrac{g_i\alpha_0 + \theta}{2}$ for $0 \le g_i \le 1$. For any value of g_i, $\alpha_1{}^*$ is not in 0_6.

This is because

$N_{6/3}(\alpha_1{}^*) = \dfrac{1+\sqrt[6]{-3}}{2} \cdot \dfrac{1-\sqrt[6]{-3}}{2} = \dfrac{1-\sqrt[3]{-3}}{4} \notin 0_3$ and also since $N_{6/3}(\theta/2) \notin 0_3$, so there is no change for α_1.

$\alpha_2{}^* = \dfrac{g_1\alpha_0 + g_2\alpha_1 + \alpha_2}{2}$ for $0 \le g_i \le 1$. For any value of $g_i, \alpha_2{}^* \notin 0_6$, then there will be no change for α_2.

$\alpha_3{}^* = \dfrac{g_1\alpha_0 + g_2\alpha_1 + g_3\alpha_2 + \alpha_3}{2}$ for $0 \le g_i \le 1$. In this case for g1 = g2 = g3 = 1,

$\alpha_3{}^* = \sqrt[6]{(-3)^4} \in 0_6$. This is because:

$\alpha_3{}^* = \dfrac{1+\sqrt[6]{(-3)^3}}{2} \cdot \dfrac{1-\sqrt[6]{(-3)^3}}{2} = \dfrac{1-\sqrt[6]{(-3)^6}}{4} = 1 \in 0_3$, and for other values of $g_i, \alpha_3{}^* \notin 0_6$.

$\alpha_4{}^* = \dfrac{g_1\alpha_0 + g_2\alpha_1 + g_3\alpha_2 + g_4\alpha_3{}^* + \alpha_4}{2}$. In this case for g2 = g4 = 1,

$\alpha_4{}^* = \dfrac{\sqrt[6]{-3} + \sqrt[6]{(-3)^4}}{2} \in 0_6$. This is because

$N_{6/3}(\alpha_4{}^*) = \dfrac{\sqrt[6]{-3} + \sqrt[6]{(-3)^4}}{2} \cdot \dfrac{\sqrt[6]{-3} - \sqrt[6]{(-3)^4}}{2} = \dfrac{4 \cdot \sqrt[3]{-3}}{4} \in 0_3$, and for other $g_i, \alpha_4{}^* \notin 0_6$.

$\alpha_5{}^* = \dfrac{g_1\alpha_0 + g_2\alpha_1 + g_3\alpha_2 + g_4\alpha_3{}^* + g_5\alpha_4{}^* + \alpha_5}{2}$, for $g_2 = g_5 = 1$,

$\alpha_5{}^* = \dfrac{\sqrt[6]{(-3)^2} + \sqrt[6]{(-3)^5}}{2} \in 0_6$. This is because $N_{6/3}(\alpha_5{}^*) \in 0_3$, and for other values of $g_i, \alpha_5{}^* \notin 0_6$. This last assertion is since

$\mathrm{disc}(\alpha_0,\alpha_1,\alpha_2,\alpha_3{}^*,\alpha_4{}^*,\alpha_5{}^*) = \dfrac{2^2 \cdot 2^2 \cdot 2^2}{2^2 \cdot 2^2 \cdot 2^2} \cdot d_{K6/k1}$, and each α_i, $\alpha_i{}^*$ are in 0_6, then by theorem [6].

$$0_6 = \left[1, \sqrt[6]{-3}, \sqrt[6]{(-3)^2}, \frac{1+\sqrt[6]{(-3)^3}}{2}, \frac{\sqrt[6]{-3} + \sqrt[6]{(-3)^4}}{2}, \frac{\sqrt[6]{(-3)^2} + \sqrt[6]{(-3)^5}}{2} \right] \cdot Z.$$

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