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MATRIX TRANSFORMATIONS AND CONVEX SPACES

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ABSTRACT. In a Hausdorff topological linear space we examine relations between r-convexity and a condition on matrix transformations between null sequences. In particular, for metrizable spaces the condition implies r-convexity. For locally bounded spaces the condition implies sequential completeness.

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1. INTRODUCTION.

If $0 < r \leq 1$, a non-empty subset U of a complex linear space is called absolutely r-convex if $x, y \in U$ and $|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu y \in U$. Equivalently, $x_1, \ldots, x_n \in U$ and

$$\sum_{k=1}^{n} |\lambda_k|^r \leq 1 \text{ imply } \sum_{k=1}^{n} \lambda_k x_k \in U.$$

By X we denote a Hausdorff topological linear space with origin Θ, and by the term neighbourhood we shall mean neighbourhood of the origin in X.

Following Landsberg [1] we say that a space X is r-convex if every neighbourhood contains an absolutely r-convex neighbourhood. Thus a 1-convex space is, in the usual sense, a locally convex space - see for example, Robertson and Robertson [2].

Our aim in this note is to examine relations between the r-convexity of a space and the following condition on matrix transformations:

$$A \in (c_o(X), c_o(X)) \text{ whenever } (|a_{nk}|^r) \in (c_o, c_o). \tag{1.1}$$

In (1.1) we denote by $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk}. By $(|a_{nk}|^r) \in (c_o, c_o)$ we mean that $(|a_{nk}|^r)$ maps the space c_o of null sequences of complex numbers into itself. Well-known necessary and sufficient conditions for this are:

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^r < \infty \text{ and } a_{nk} \to 0 \ (n \to \infty, \text{ each } k). \tag{1.2}$$

By $c_o(X)$ we denote the set of null sequences in X. Thus $(x_k) \in c_o(X)$ means that $x_k \to \Theta$ $(k \to \infty)$. By the statement $A \in (c_o(X), c_o(X))$ we mean that for each

$x \in c_o(X)$ and each natural number n we have

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

convergent in X, amd also $A_n(x) \to \Theta$ $(n \to \infty)$.

2. THE MAIN RESULTS.

THEOREM 1. If X is r-convex and sequentially complete then (1.1) holds.

PROOF. Since X is r-convex its topology is determined by the collection {p} of all continuous r-seminorms on X. Recall that a real-valued function p on X is an r-seminorm if it satisfies the conditions:

$$p(x+y) \le p(x) + p(y) \text{ and } p(\lambda x) = |\lambda|^r p(x)$$

for all x, $y \in X$ and all complex numbers λ.

Take any continuous r-seminorm p on X and let $(x_k) \in c_o(X)$. Then $p(x_k) \to 0$ as $k \to \infty$, and for natural numbers a,b with $a \le b$ we have

$$p\left(\sum_{k=a}^{b} a_{nk} x_k\right) \le \sum_{k=a}^{b} |a_{nk}|^r p(x_k). \tag{2.1}$$

Hence, if $(|a_{nk}|^r) \in (c_o, c_o)$, then (1.2) holds, and so $\sum_{k=1}^{\infty} a_{nk} x_k$ is a Cauchy series in X, whence convergent. Taking a = 1 and letting $b \to \infty$ in (2.1) a further application of (1.2) shows that $A_n(x) \to \Theta$ $(n \to \infty)$. This completes the proof of Theorem 1.

In Theorem 1 we may observe that neither r-convexity alone, nor sequential completeness alone, implies (1.1). For example, consider the r-convex space of all finite sequences, regarded as a subspace of ℓ_r. Let us define $x_k = k^{-1} e_k$, where e_k denotes the k-th unit vector. Then $x_k \to \Theta$ $(k \to \infty)$. Now define $A = (a_{nk})$ by $a_{1k} = 2^{-k}$ and $a_{nk} = 0$ for $n > 1$. Then $(|a_{nk}|^r) \in (c_o, c_o)$, but

$$\sum_{k=1}^{\infty} a_{1k} x_k$$

does not converge to any finite sequence.

Next, consider the sequentially complete space ℓ_s where $s = r/(r+1)$. Define $x_k = k^{-1} e_k$ and let

$$\begin{aligned} a_{nk} &= n^{-1/r} && \text{for } 1 \le k \le n, \\ &= 0 && \text{for } k > n. \end{aligned}$$

Then $(|a_{nk}|^r) \in (c_o, c_o)$ and $x_k \to \Theta$ in ℓ_s but the ℓ_s norm of

$$\sum_{k=1}^{n} a_{nk} x_k$$

is equal to

$$n^{-s/r} \sum_{k=1}^{n} k^{-s}$$

which does not tend to 0 as $n \to \infty$.

For metrizable spaces the next result is a partial converse to Theorem 1.

THEOREM 2. If X is metrizable and (1.1) holds then X is r-convex.

PROOF. Since X is metrizable we may determine a countable base $\{U_1, U_2, \ldots\}$ of balanced neighbourhoods such that $U_{n+1} \subset 2^{-1}U_n$ for $n = 1, 2, \ldots$.

Now for any balanced neighbourhood U we define the Minkowski gauge p_U by

$$p_U(x) = \inf\{\lambda > 0 : x \in \lambda V\}$$

for each $x \in X$. Also, for each natural number i we shall write $p_i = p_{U_i}$. Then, since $\{U_1, U_2, \ldots\}$ is a base, it follows that if (x_j) is a sequence such that $p_i(x_j) \to 0$ $(j \to \infty)$ for each i, then $x_j \to \Theta$ $(j \to \infty)$.

Let us suppose, if possible, that X is not r-convex. Then there exists a neighbourhood V such that for each natural number n the absolutely r-convex hull of U_n is not contained in V. Hence, for each n, there exist $x(n,1)$, $x(n,2)$, ..., $x(n,m(n))$ in U_n and there exist complex numbers $\lambda(n,1)$, $\lambda(n,2)$,..., $\lambda(n,m(n)))$ with

$$\sum_{k=1}^{m(n)} |\lambda(n,k)|^r \leq 1 \tag{2.2}$$

such that

$$\sum_{k=1}^{m(n)} \lambda(n,k)\; x(n,k) \notin V. \tag{2.3}$$

Now define an infinite matrix $A = (a_{nk})$ as follows:

$a_{1k} = \lambda(1,k)$ for $1 \leq k \leq m(1)$ and $a_{1k} = 0$ otherwise;
$a_{2k} = \lambda(2,k)$ for $m(1) < k \leq m(2)$ and $a_{2k} = 0$ otherwise;
$a_{3k} = \lambda(3,k)$ for $m(2) < k \leq m(3)$ and $a_{3k} = 0$ otherwise, ...

Then it follows from (2.2) that $(|a_{nk}|^r) \in (c_o, c_o)$, whence $A \in (c_o(X), c_o(X))$ by (1.1).

Next, we define a sequence $x = (x_j)$ by

$$x = (x(1,1),\ x(1,2),\ \ldots,\ x(1,m(1)),\ x(2,1),\ \ldots,\ x(2,m(2)),\ \ldots).$$

Thus, it is clear that

$$A_n(x) = \sum_{k=1}^{m(n)} \lambda(n,k)x(n,k).$$

Let us choose any natural number i. Since $U_{n+1} \subset 2^{-1}U_n$ for $n = 1,2,\ldots$ it follows that

$$p_i(x(n,k)) \leq 2^{i-n} \tag{2.4}$$

for each $n \geq i$ and for $1 \leq k \leq m(n)$. But (2.4) implies that $p_i(x_j) \to 0$ $(j \to \infty)$, whence $x_j \to \Theta$ $(j \to \infty)$, and since $A \in (c_o(X), c_o(X))$ it follows that $A_n(x) \to \Theta$ $(n \to \infty)$, which is contrary to (2.3). Hence X must be r-convex, which completes the proof.

In our final theorem below we shall show that, for the special class of metrizable spaces known as locally bounded spaces, condition (1.1) implies sequential completeness of the space. We recall that a Hausdorff topological linear space X is called locally bounded if X contains a bounded neighbourhood B, that is a neighbourhood B such that for every neighbourhood V there exists $\lambda > 0$ with $B \subset \lambda V$.

THEOREM 3. Let X be locally bounded and suppose that (1.1) holds. Then X is sequentially complete.

PROOF. Since local boundedness implies metrizability, it follows from Theorem 2 above that X is r-convex, whence, being locally bounded and r-convex, X must be r-normable (see for example Köthe [3], page 160). Let $||.||$ be a suitable r-norm and suppose that (y_k) is any sequence in X such that

$$\sum_{k=1}^{\infty} ||y_k|| < \infty.$$

By the general convergence principle for series we may construct a positive real sequence (q_k) with $q_k \to \infty$ $(k \to \infty)$ such that

$$\sum_{k=1}^{\infty} q_k ||y_k|| < \infty. \tag{2.5}$$

Now define a sequence (x_k) by

$$x_k = \frac{1}{\sqrt{q_k}} \cdot \frac{y_k}{||y_k||^{1/r}} \text{ if } ||y_k|| > 0,$$

and $x_k = \theta$ otherwise. Then $||x_k|| \le q_k^{-r/2}$ for all $k \ge 1$, whence $x_k \to \theta$ $(k \to \infty)$. Also, we have

$$y_k = x_k \sqrt{q_k} \cdot ||y_k||^{1/r}$$

and by (2.5) we see that

$$\sum_{k=1}^{\infty} (\sqrt{q_k})^r ||y_k|| < \infty. \tag{2.6}$$

Let us define a matrix $A = (a_{nk})$ by

$$a_{1k} = \sqrt{q_k} \cdot ||y_k||^{1/r} ; \quad a_{nk} = 0 \text{ for } n \ge 2.$$

Then (2.6) implies that $(|a_{nk}|^r) \in (c_o, c_o)$, and so (1.1) implies that

$$\sum_{k=1}^{\infty} a_{1k} x_k = \sum_{k=1}^{\infty} y_k$$

converges.

To summarize : whenever (y_k) is such that $||y_1|| + ||y_2|| + \dots < \infty$ it follows that $y_1 + y_2 + \dots$ converges. It is readily seen that this implies the sequential completeness of X, which proves the theorem.

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