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ANOTHER NOTE ON KEMPISTY'S GENERALIZED CONTINUITY

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ABSTRACT. Under a fairly mild completeness condition on spaces Y and Z we show that every x-continuous function f: $X \times Y \times Z \to M$ has a "substantial" set C(f) of points of continuity. Some odds and ends concerning a related earlier result shown by the authors are presented. Further, a generalization of S. Kempisty's ideas of generalized continuity on products of finitely many spaces is offered. As a corollary from the above results, a partial answer to M. Talagrand's problem is provided.

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1. x-CONTINUITY.

The notion of symmetric quasi-continuity introduced by S. Kempisty [1] has been generalized in Lee and Piotrowski [2], to x-continuity. In what follows let X, Y, Z and T be spaces. Following Lee and Piotrowski [2] a function f: $X \times Y \times Z \to T$ is *x-continuous* if for every $(p,q,r) \in X \times Y \times Z$, for every neighborhood $U \times V \times W$ of (p,q,r) and for every neighborhood N of f(p,q,r) there exists a neighborhood U' of p with $U' \subset U$ and nonempty open sets V' and W' with $V' \subset V$ and $W' \subset W$ such that for all $(x,y,z) \in U' \times V' \times W'$ it follows that $f(x,y,z) \in N$.

We shall first show that under certain general assumptions concerning the spaces, x-continuous functions have "large" sets of points of joint continuity. In order to do this we first list some necessary definitions.

Let A be an open covering of a space X. Then a subset S of X is said to be *A-small* if S is contained in a member of A. A space X is called *strongly countably complete* if there exists a sequence $\{A_i: i=1,2,...\}$ of open coverings of X such that and sequence $\{F_i\}$ of A_i-small, closed subsets of X for which $F_i \supset F_{i+1}$ has a nonempty intersection.

The class of strongly countably complete spaces include countably compact and complete metric spaces. This fact follows easily from a theorem due to A. Arhandel'skiǐ [3] and Z. Frolík [4] which states that in the class of completely regular spaces, Čech-complete and strongly countably complete spaces coincide (Engelking [5]), see also Frolík [4], where some other properties of these spaces such as their invariance under taking closed, open subspaces or products are discussed.

A space X is called *quasi-regular*, (Oxtoby [6]) if for every nonempty open set u, there is a nonempty open set V such that $clV \subset u$. Obviously, every regular space is quasi-regular.

Let us recall that a function $f\colon X \times Y \to Z$ is said to be *quasi-continuous with respect to x*, (Kempisty [1] p.188,) if for every $(p,q) \in X \times Y$, fore very neighborhood N of $f(p,q)$ and every neighborhood $U \times V$ of (p,q) there exists a neighborhood U' of p with $U' \subset U$ and a nonempty open set $V' \subset V$ such that for all $(x,y) \in U' \times V'$ we have $f(x,y) \in N$. Quasi-continuity with respect to y can be defined similarly.

LEMMA 1. (Lee and Piotrowski [2], Lemma 3 p. 383). Let X, Y, Z and T be spaces and let $F\colon X \times Y \times Z \to T$ be a function. Then f is x-continuous if and only if $g\colon X \times S \to T$ is quasi-continuous with respect to x, where $S = Y \times Z$ and $g(x,(y,z)) = f(x,y,z)$.

THEOREM 2. Let X be a space, Y and Z be spaces such that $Y \times Z$ is quasi-regular, strongly countably complete and let M be metric. If $f\colon X \times Y \times Z \to M$ is x-continuous, then for every $x \in X$, the set C(f) of continuity points of f is dense G_δ subset in $\{x\} \times Y \times Z$.

PROOF. In view of Lemma 1 it is sufficient to prove the following:

CLAIM. Let X be a space, Y be a quasi-regular, strongly countably complete and Z be metric. If $f\colon X \times Y \to Z$ is quasi-continuous with respect to x, then for all $x \in X$ the set of points of joint continuity of f is a dense G_δ subset of $\{x\} \times Y$.

PROOF. First we will prove that the set of points of joint continuity of f is dense in $\{x\} \times Y$. Let $x \in X$, $y \in Y$ and $U \times V$ be any neighborhood U of x, contained in U, and a nonempty open set $V^1 \subset V$ such that for all (x',y') and (x'',y'') in $U^1 \times V^1$, we have $\rho(f(x',y'), f(x'',y'')) < 1$. Without loss of generality we may assume that V^1 is contained in an element A_1 of the covering $\mathcal{A}_1$ of Y. Let W^1 be a nonempty open set such that $cl\, W^1 \subset V^1$. So $cl\, W^1$ is $\mathcal{A}_1$-small. Then $U^1 \times W^1$ is a neighborhood of (x,y_1), where $u_1 \in W^1$, and since f is quasi-continuous with respect to x at (x,y_1), there is a neighborhood U^2 of x, contained in U^1 and a nonempty open set $V^2 \subset W^1$, such that for all (x',y') and (x'',y'') in $U^2 \times V^2$ we have $\rho(f(x',y'), f(x'',y'')) < \frac{1}{2}$. Similarly, we may assume that V^2 is contained in an element A_2 of the covering $\mathcal{A}_2$. Let W^2 be a nonempty open set such that $cl\, W^2 \subset V^2$. We see, that $cl\, W^2$ is $\mathcal{A}_2$-small.

Now, proceeding by induction we get a neighborhood $U^n \times V^n$ of (x,y_n), $y_n \in V^n$, such that for all (x',y') and (x'',y'') in $U^n \times V^n$, we have $\rho(f(x',y'), f(x'',y'')) < \frac{1}{n}$ and that V^n is contained in an element A_n of the covering $\mathcal{A}_n$ of Y. Moreover, there is a nonempty open sets W^n such that $V^{n+1} \subset cl\, W^n \subset V^n$. Thus each $cl\, W^n$ is $\mathcal{A}_n$-small, obviously $cl\, W^n \supset cl\, W^{n+1}$. Since Y is strongly countably complete $\bigcap_{n=1}^{\infty} cl\, W^n \neq \emptyset$. Let

$y^* \in \bigcap_{n=1}^{\infty} \operatorname{cl} W^n$. Then

$$(x,y^*) \in \bigcap_{n=1}^{\infty} (U^n \times \operatorname{cl} W^n) \subset \bigcap_{n=1}^{\infty} (U^n \times V^n) \subset U \times V .$$

Thus $(x,y^*) \in (U \times V) \cap (\{x\} \times Y)$ and (x,y^*) is a point of joint continuity of f. This shows the density of the set of points of joint continuity of f in the set $\{x\} \times Y$.

The proof that this set is G_δ subset of $\{x\} \times Y$ easily follows, when we recall that the function f takes values in the metric space Z. This completes the proof of Claim.

Thus, Theorem 2 is shown.

The forthcoming, Proposition 3 is contained in Lemma 5.1 of [6], since any quasi-regular strongly countably complete space is pseudo-complete; take $B(n)$ = the class of all nonempty open sets that are A_n-small. Then $\{B(n)\}$ is a sequence of (pseudo-) bases that shows X to be pseudo-complete.) We would like to thank the referee who make the above observation.

PROPOSITION 3. (Oxtoby [6], Lemma 5.1) Every quasi-regular strongly countably complete space X is a Baire space.

REMARK 4. Observe that neither base countability nor metrizability assumptions are made on the considered spaces X, Y, Z in Theorem 1 while in Theorem 2 of [2] the same conclusion concerning the set of points of continuity is obtained under an *extra* assumption that X is first countable, Y is Baire, Z is second countable in a neighborhood of any of its points and such that Y Z is Baire.

2. CONDITIONS IMPLYING x-CONTINUITY - COUNTER-EXAMPLES.

Given spaces X and Y; a function $f\colon X \to Y$ is said to be quasi-continuous (Martin [8], compare Kempisty [1]) if for every $x \in X$ and for every neighborhood U of x and for every neighborhood V of f(x) have: $U \cap \operatorname{Int} f\ (V) \neq \emptyset$.

The main result of Lee and Piotrowski [2] is the following:

THEOREM A. (Lee and Piotrowski [2], Theorem 1, p. 383). Let X be first countable, Y be Baire, Z be second countable such that $Y \times Z$ is Baire and let T be regular. If $f\colon X \times Y \times Z \to T$ is:

(1) continuous at $X \times \{y\} \times \{z\}$, $y \in Y$, $z \in Z$, and

(2) quasi-continuous at points of $\{x\} \times Y \times \{z\}$ for all $x \in X$ and $z \in Z$, and

(3) quasi-continuous at points of $\{x\} \times \{y\} \times Z$ for all $x \in X$ and $y \in Y$

then f is x-continuous.

The first natural question which comes up is to check whether the converse of Theorem A is true. Apparently, the following Example 5 settles this question in the negative.

EXAMPLE 5. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = \begin{cases} \sin \dfrac{1}{x^2+y^2+z^2}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, \text{ otherwise} & \end{cases}$$

The function f is x-continuous, however, fixing y = 0 = z we obtain that f(x,0,0) is not continuous.

Now we shall investigate the necessity of the assumptions in Theorem A, in particular:

(*) - *continuity* of f at points of $X \times \{y\} \times \{z\}$

(**) - *quasi-continuity* of f at points of $\{x\} \times Y \times \{z\}$, and

(***) - *quasi-continuity* of f at points of $\{x\} \times \{y\} \times Z$.

In what follows (Examples 6 and 7) such constructions will be provided.

EXAMPLE 6. The assumption (*) is essential. In fact, let us consider a function f: $[-1,1]^3 \to \mathbb{R}^3$ given as follows

$$f(x,y,z) = \begin{cases} (x,y,z+1), & \text{if } (x,y,z) \in [0,1] \times [0,1] \times [0,1] \\ (x,y,z-1), & \text{if } (x,y,z) \in [-1,0] \times [-1,0] \times [-1,0] \\ (x,y,z), & \text{otherwise} \end{cases}$$

A standard verification that f has the required property, (namely f is not x-continuous at (0,0,0)) is left to the reader. Using somewhat more complex, but still elementary techniques we shall show that also (**) (as well as (***)) is essential. In fact, we have

EXAMPLE 7. Consider the function $g:[-1,1]^3 \to \mathbb{R}^3$ given as follows:

$$g(x,y,z) = \begin{cases} (x,y,z+1) \text{ if } (x,y,z) \in [-1,1] \times [-\frac{1}{2},1] \times \\ \qquad \times \{([-\frac{1}{2},\frac{1}{2}) \cap \mathbb{Q}) \cup [\frac{1}{2},1]\} \\ (x,y,z), \text{ otherwise} \end{cases}$$

Again, we leave to the interested reader a standard verification that f is not x-continuous at (0,0,0).

3. ONE-PROMISING HYPOTHESIS.

Observe that the definition of x-continuity at (p,q,r) requires the existence of a "small" neighborhood U' of p and "small" nonempty open sets V' and W' such that q and r "clusters" to V' and W' respectively and such that the set $f(U' \times V' \times W')$ is contained in a "small", previously chosen, open set N. This observation prompts us to label this kind of product almost continuity as *1-3-continuity* - since we require the existence of only <u>one</u> "small" neighborhood U' (around p) of the three neighborhoods U, V, W.

The term "1-3-continuity" has been used already, in a different sense in Breckenridge and Nishiura [9].

So, now let us consider "2-3-continuity".

More precisely, given spaces X, Y, Z and T, we say that f: $X \times Y \times Z \to T$ is *2-3-continuous* or more specifically *xy-continuous*, if for every $(p,q,r) \in X \times Y \times Z$, for every neighborhood $U \times V \times W$ of (p,q,r) and for every neighborhood N of f(p,q,r) there is a neighborhood U' of p, with $U' \subset U$, there is a neighborhood V^1 of q, with $V^1 \subset V$ and a nonempty open set W^1, with $W^1 \subset W$ such that for all $(x,y,z) \in U^1 \times V^1 \times W^1$ we have $f(x,y,z) \in N$.

Now, 3-3-continuity can be defined easily; the set W^1 in definition of 2-3-continuity is assumed to be a neighborhood of r - not just only a nonempty open subset of W.

Clearly, every 3-3-continuous (≡ continuous) function is 2-3-continuous; 2-3-continuous functions are 1-3-continuous and the latter are in turn 0-3-continuous (≡ quasi-continuous).

It now follows from a result of T. Neubrunn [10] that if X, Y, Z are "nice" (e.g. Baire, second countable), T-regular then if f: $X \times Y \times Z \to T$ is separately quasi-continuous then it is (jointly) quasi-continuous.

We can present this fact in the following symbolic equality:

"0 + 0 + 0 = 0",

where the numbers (0 or 1) on the left side of the equality stand for quasi-continuity (0) or continuity (1) of the corresponding sections and the numbers on the right (i = 0, 1, 2 or 3) denote the corresponding i-3-continuity of f as a function of three variables.

Theorem A implies that if X, Y, Z and T are as above and if f: $X \times Y \times Z \to T$ is continuous in x and is quasi-continuous in y and is quasi-continuous in z, then f is 1-3-continuous. Consequently, we get:

"1 + 0 + 0 = 1".

In view of the above considerations it is now natural to state the following:

HYPOTHESIS. Let X, Y and Z be Baire, second countable spaces and let T be regular. If f: $X \times Y \to Z$ T is:

1) continuous in x, and
2) continuous in y, and
3) quasi-continuous in z,

Then f is 2-3-continuous;
In other words:

"1 + 1 + 0 = 2"

We shall resolve this Hypothesis in the *negative* in the forthcoming Example 8.

Now we shall exhibit two examples of i-3-continuous functions which are not (i + 1)-3-continuous, i = 1,2.

EXAMPLE 8. <u>A 1-3-continuous functin which is not 2-3-continuous</u>. Let f: $\mathbb{R}^3 \to \mathbb{R}$ be given by $f(x_1,x_2,x_3) = g(x_1,x_2)$ where g is an arbitrary separately continuous function which is discontinuous at (0,0).

EXAMPLE 9. A 2-3-continuous function which is not 3-3-continuous (≡ continuous). Take f: $\mathbb{R}^3 \to \mathbb{R}$ to be $f(x_1,x_2,x_3) = h(x_3)$, where h is any function which is continuous except for 0.

Using the above pattern the reader will easily construct 0-3-continuous function (≡ quasi-continuous) which is not 1-3-continuous.

Apparently, the above constructions can be illustrated with the following very specific <u>formula-ready</u> example.

EXAMPLE 10. Let f: $\mathbb{R}^3 \to \mathbb{R}$ be a function.

$f(x_1,x_2,x_3) = g_i^3(x_1,\dots,x_i)$, $i = 1,2$ where

$$g_i^3(x_1,\dots,x_i) = \begin{cases} \dfrac{\prod_{j=1}^{i} x_j}{\sum_{j=1}^{i} (x_j)^i}, & \text{if } \sum_{j=1}^{i} (x_j)^i \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then f is i-3-continuous which is not (i + 1)-3-continuous, i = 1,2.

4. FURTHER GENERALIZATION OF i-3-CONTINUITY.

Having defined 1-3 and 2-3-continuity for $f\colon X_1 \times X_2 \times X_3 \to T$, we shall now extend these ideas to a general case.

Namely, let n be an arbitrary natural number. We say that f function $f\colon \prod_{i=1}^{n} X_i \quad T$ is *A-n-continuous* if for every $(p_1, p_2,\dots, p_n) \prod_{i=1}^{n} X_i$ and for every neighborhood $U_1 \times U_2 \times \dots \times U_n$ of $(p_1, p_2,\dots, p_n)$ and for every neighborhood N of $f(p_1, p_2,\dots, p_n)$ there are neighborhoods $U'_{i,s}$ $(1 \leq s \leq k)$ of the first k out of n points $p_1, p_2,\dots, p_n$ with $U'_{i,s} \subset U_i$ and there are (n-k) nonempty open sets $V'_{i,m}$ with $V'_{i,m} \subset U_i$ $1 \leq m \leq n-k$ such that for all $(x_1, x_2,\dots, x_n) \in \prod_{s=1}^{k} U'_{i,s} \times \prod_{m=1}^{n-k} V'_{i,m}$ we have $f(x_1, x_2, \dots, x_n) \in N$.

An interested reader will easily observe that the formula

$$g_k^n(x_1, \dots, x_k) = \begin{cases} \dfrac{\prod_{i=1}^{k} x_i}{\sum_{i=1}^{k} (x_i)^k}, & \text{if } \sum_{i=1}^{k} (x_i)^k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ describes a k-n-continuous function f given by $f(x_1, \dots, x_n) = g_k^n(x_1, \dots, x_k)$, $k = 1, 2, 3, \dots n-1$.

One can also give analogues of Example 8 and 9 for k-n-continuity.

Studies of C(f) in hyperspaces for separately continuous functions and related ones were done also in Bögel [11] and Hahn [12].

5. A PARTIAL SOLUTION TO A PROBLEM OF M. TALAGRAND.

M. Talagrand ([13] Problem 3 p. 160) asked whether if X is Baire, Y is compact and $f\colon X \times Y \to \mathbb{R}$ is any separately continuous function, is there the set C(f) of points of continuity of f nonempty.

We shall answer this question in the positive if a compact space Y is additionally *first countable*.

In fact, we have shown the following result:

LEMMA 11. (Lee and Piotrowski [2], Lemma 2 p. 381). Let X be Baire, Y be first countable and Z be regular. If $f\colon X \times Y \to Z$ is a function such that all its x-sections f_X are continuous with the exception of a first category set, and all its y-sections f_Y are quasi-continuous, then f is quasi-continuous with respect to y.

It follows from the definition that

REMARK 12. Every quasi-continuous function with respect to y is quasi-continuous.

LEMMA 13. (Marcus [14]). Let X be a Baire, M be metric. If $f\colon X \to M$ is quasi-continuous, then C(f), the set of point of continuity of f is dense G_δ subset of X.

PROPOSITION 14. Let X be Baire, Y be compact first countable and let $f: X \times Y \rightarrow \mathbb{R}$ be any separately continuous function. Then $C(f) \neq 0$.

PROOF. By Lemma 11 and Remark 12 such f is quasi-continuous. Now, since the Cartesian product of a compact space and a Baire space is Baire, we are done by Lemma 13.

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