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# RESEARCH NOTES

# ON LEGENDRE NUMBERS OF THE SECOND KIND

**PAUL W. HAGGARD**
Department of Mathematics
East Carolina University
Greenville, NC 27858 U.S.A.

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ABSTRACT. The Legendre numbers of the second kind, an infinite set of rational numbers, are defined from the associated Legendre functions. An explicit formula and a partial table for these numbers are given and many elementary properties are presented. A connection is shown between Legendre numbers of the first and second kinds. Extended Legendre numbers of the first and second kind are defined in a natural way and these are expressed in terms of those of the second and first kind, respectively. Two other sets of rational numbers are defined from the associated Legendre functions by taking derivatives and evaluating these at $x = 0$. One of these sets is connected to Legendre numbers of the first find while the other is connected to Legendre numbers of the second kind. Some series are also discussed.

*KEY WORDS AND PHRASES. Associated Legendre functions, extended Legendre numbers of the first and second kinds, gamma function, infinite products, Legendre numbers of the first and second kinds, Legendre's associated differential equation, Legendre's differential equation, spherical functions.*
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## 1. INTRODUCTION.

In [1], the Legendre numbers (of the first kind) were defined from the associated Legendre functions, $P_n^m(x)$. Since there is another type of the associated Legendre functions, denoted by $Q_n^m(x)$, we define the Legendre numbers of the second kind from these functions. These numbers have many properties, as we shall see. Both kinds will be extended in a natural way when the superscript is a negative integer. Furthermore, two other sets of numbers will be defined and connections between all of the sets of numbers will be investigated. It will be shown that the sum of the non-zero entries in the first column of Table 1 exists while the sum of the non-zero entries in each of the other columns does not exist.

## 2. LEGENDRE NUMBERS OF THE SECOND KIND.

For $m$ and $n$ non-negative integers and for $x$ real consider the special case of Legendre's associated differential equation,

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + [n(n + 1) - \frac{m^2}{1 - x^2}]\,y = 0. \tag{2.1}$$

The solutions of (2.1) are spherical functions corresponding to the rotation group of 3-dimensional space. When $m = 0$, (2.1) becomes

$$(1 - x^2)\frac{dy^2}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0, \tag{2.2}$$

and is known as Legendre's differential equation. The fundamental system of solutions of (2.2) is given by two kinds of functions that we denote by $P_n(x)$ and $Q_n(x)$, see [2]. These have many forms, two of which are, see [3],

$$\begin{cases} P_n(x) = \dfrac{1}{n!2^n}\dfrac{d^n}{dx^n}(x^2 - 1)^n \\ Q_n(x) = \dfrac{1}{2}P_n(x)\log\dfrac{1 + x}{1 - x} - \displaystyle\sum_{i=1}^{n}\dfrac{P_{i-1}(x)\,P_{n-i}(x)}{i}, \quad -1 < x < 1. \end{cases} \tag{2.3}$$

For $m \neq 0$, the solutions of (2.1) are denoted by the associated Legendre functions $P_n^m(x)$ and $Q_n^m(x)$ where these can be expressed in the forms

$$\begin{aligned} P_n^m(x) &= (-1)^m(1 - x^2)^{\frac{m}{2}}\frac{d^mP_n(x)}{dx^m} \\ Q_n^m(x) &= (-1)^m(1 - x^2)^{\frac{m}{2}}\frac{d^mQ_n(x)}{dx^m}. \end{aligned} \tag{2.4}$$

In [1], the Legendre numbers ( of the first kind) were defined as the values of $P_n^m(x)$ for $x = 0$. A similar definition is made for Legendre numbers of the second kind.

Definition 1. For m and n non-negative integers, the Legendre numbers of the second kind are

$$Q_n^m = \begin{cases} 0 \text{ for } m,n \text{ of the same parity} \\ Q_n^m(0) \text{ for } m,n \text{ of different parity.} \end{cases}$$

From this definition and (2.4) one can obtain the explicit formula, see [3],

$$Q_n^m = \begin{cases} 0 \quad \text{for } m,n \text{ of the same parity} \\ \dfrac{-2^{m-1}\sqrt{\pi}\,\sin(\frac{m+n}{2}\pi)\,\Gamma(\frac{m+n+1}{2})}{\Gamma(\frac{n-m+2}{2})}, \quad m,n \text{ of different parity} \end{cases} \tag{2.5}$$

for the Legendre numbers of the second kind. Note that the second part of (2.5) could be used to define $Q_n^m$ for $m,n$ of the same parity if $m \leq n$ but not for $m \geq n + 2$. Considering Table 1, it seems reasonable that we define $Q_n^m$ to be $0$ for $m,n$ of the same parity so that if a diagonal or alternate diagonal of the table has a zero on it all entries on it are zero. Recall the gamma function is not defined for zero or negative integers.

By definition $Q_n^m = 0$, an integer, for $m,n$ of the same parity. It is easy to show that $Q_n^m$ is a non-zero integer for $m,n$ of different parity and $n = 0$ and also for $n \geq 1$, $m \geq n - 1$. There are other entries in Table 1 that are integers but we won't classify them here. One could classify those values of $Q_n^m$ that are rational but not integers.

| $n \backslash Q_n^m$ | $Q_n^0=Q_n$ | $Q_n^1$ | $Q_n^2$ | $Q_n^3$ | $Q_n^4$ | $Q_n^5$ | $Q_n^6$ | $Q_n^7$ | $Q_n^8$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | -1 | 0 | -2 | 0 | -24 | 0 | -720 | 0 | |
| 1 | -1 | 0 | 2 | 0 | 8 | 0 | 144 | 0 | 5760 | |
| 2 | 0 | 2 | 0 | -8 | 0 | -48 | 0 | -1152 | 0 | |
| 3 | $\frac{2}{3}$ | 0 | -8 | 0 | 48 | 0 | 384 | 0 | 11520 | |
| 4 | 0 | $-\frac{8}{3}$ | 0 | 48 | 0 | -384 | 0 | -3840 | 0 | $\cdots$ |
| 5 | $-\frac{8}{15}$ | 0 | 16 | 0 | -384 | 0 | 3840 | 0 | 46080 | |
| 6 | 0 | $\frac{16}{5}$ | 0 | -128 | 0 | 3840 | 0 | -46080 | 0 | |
| 7 | $\frac{16}{35}$ | 0 | $-\frac{128}{5}$ | 0 | 1280 | 0 | -46080 | 0 | 645120 | |
| 8 | 0 | $-\frac{128}{35}$ | 0 | 256 | 0 | -15360 | 0 | 645120 | 0 | |
| 9 | $-\frac{128}{315}$ | $\cdots$ | | | | | | | | |
| $\vdots$ | | $\vdots$ | | | | | | | | |

TABLE 1. LEGENDRE NUMBERS OF THE SECOND KIND

## 3. SOME BASIC PROPERTIES.

Many of the following properties of the Legendre numbers of the second kind can be observed from Table 1 and can be easily proved using (2.5).

The non-zero entries in the first row are

$$Q_0^m = -(m-1)!,\ m \geq 1,\ m \text{ odd}. \tag{3.1}$$

The entries on the main diagonal are

$$Q_n^n = 0,\ n \geq 0. \tag{3.2}$$

For $m = n - 1$, we have

$$\begin{cases} Q_1^0 = -1 \\ Q_n^{n-1} = (-1)^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n-2),\ n \geq 2. \end{cases} \tag{3.3}$$

The entries adjacent to the main diagonal and on slant lines through it are equal. Thus,

$$Q_n^{n-1} = Q_{n-1}^n,\ n \geq 1. \tag{3.4}$$

Each entry can be expressed in terms of one in the first column or first row, respectively, as

$$Q_n^m = (-1)^m (n+m-1)(n+m-3)\cdots(n-m+1)Q_{n-m},\ n > m \tag{3.5}$$

$$Q_n^m = (-1)^{m+1}(n+m-1)(n+m-3)\cdots(m-n+1)Q_0^{m-n},\ m > n. \tag{3.6}$$

Entries in the first row are connected to those in the first column by

$$Q_0^m = m(m-2)(m-4)\cdots(-m+2)Q_m^0, \tag{3.7}$$

and, more generally,

$$Q_n^m = (m - n)(m - n - 2)(m - n - 4)\cdots(n - m + 2)Q_m^n, \quad m > n. \tag{3.8}$$

Each entry can be expressed in terms of the one just above and to the left, just above and to the right, second one above, and second one to the left, respectively, as

$$Q_n^m = -(n + m - 1)Q_{n-1}^{m-1}, \quad n \geq 2 \tag{3.9}$$

$$Q_n^m = \frac{Q_{n-1}^{m+1}}{n - m}, \quad m \neq n \tag{3.10}$$

$$Q_n^m = -\frac{n + m - 1}{n - m} Q_{n-2}^m, \quad m \neq n, \ n \geq 2 \tag{3.11}$$

$$Q_n^m = -(n + m - 1)(n - m + 2)Q_n^{m-2}, \quad m \geq 2. \tag{3.12}$$

PROOF OF (3.12). If $n$ and $m$ of the same parity, then $n$ and $m - 2$ are of the same parity. Thus the two sides of (3.12) have value 0 and the result holds. Assume $n$ and $m$ are of different parity and $m \geq 2$. Then $n$ and $m - 2$ are of different parity and by (2.5) and properties of the gamma function, we have,

$$-(n+m-1)(n-m+2)Q_n^{m-2} = -(n+m-1)(n-m+2)\left[\frac{-2^{m-3}\sqrt{\pi}\,\sin(\frac{m+n-2}{2}\pi)\Gamma(\frac{m+n-1}{2})}{\Gamma(\frac{n-m+4}{2})}\right]$$

$$= -(n+m-1)(n-m+2)\left[\frac{-2^{m-1}\sqrt{\pi}\,\sin(\frac{m+n}{2}-1)\pi\ \Gamma(\frac{m+n+1}{2})}{2^2(\frac{m+n-1}{2})(\frac{n-m+2}{2})\Gamma(\frac{n-m+2}{2})}\right], \tag{3.13}$$

which simplifies easily to $Q_n^m$. Thus, (3.12) is proved.

The non-zero entries of Table 1 are given by the explicit formula

$$Q_n^m = \frac{(-1)^{n+1}(m+n-1)!}{(m-n)(m-n+2)\cdots(m+n-2)}, \quad m + n \text{ odd}. \tag{3.14}$$

As a special case of (3.14), the non-zero entries in the first column are

$$Q_n^0 = \frac{(n-1)!}{(-n)(-n+2)\cdots(n-2)}, \quad n \text{ odd}. \tag{3.15}$$

Consider an alternate diagonal of Table 1. Starting with $Q_n^0$ for $n$ odd, the other entries on this alternate diagonal are obtained in order by multiplying by $n, n - 2, n - 4, \cdots, 1, -1, -3, \cdots, -(n - 2)$ in this order. For example, $Q_{n-1}^1 = nQ_n^0$, $Q_{n-2}^2 = (n - 2)Q_{n-1}^1, \cdots, Q_0^n = -(n - 2)Q_1^{n-1}$.

An entry can be expressed in another way, see (3.5), in terms of an entry in the first column as

$$Q_n^m = (n + m)(n + m - 2)\cdots(n - m + 2)Q_{m+n}, \quad m \geq 1. \tag{3.16}$$

An interesting result involving products of numbers symmetric with respect to a given position in Table 1 is

$$Q_{n-i}^n \cdot Q_{n+i}^n = Q_n^{n-i} \cdot Q_n^{n+i}, \quad n - i \geq 0. \tag{3.17}$$

4. A CONNECTION BETWEEN LEGENDRE NUMBERS OF THE FIRST AND SECOND KINDS.

Recall, see [1], that the Legendre numbers of the first kind are given by the explicit formula

$$P_n^m = \begin{cases} 0, & m+n \text{ odd} \\ 0, & m > n \\ \dfrac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n(\frac{n+m}{2})!(\frac{n-m}{2})!}, & m+n \text{ even}, m \le n. \end{cases} \tag{4.1}$$

One can easily verify that

$$Q_n^0 \cdot P_n^1 = -1, \text{ n odd}, n \ge 1. \tag{4.2}$$

We next define the extended Legendre numbers of the first find and obtain a result more general than (4.2). A further extension to $P_{-n}^{-m}$ is possible but will not be presented here.

Definition 2. For $m$ and $n$ non-negative integers with $m \le n$,

$$P_n^{-m} = \frac{(-1)^m\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m.$$

Using (2.5) and Definiton 2, we can prove that for $m + n$ odd,

$$Q_n^m \cdot P_n^{-m+1} = -1, \ m \ge 0, \ n \ge 1, \ m < n. \tag{4.3}$$

PROOF OF (4.3). Now,

$$Q_n^m \cdot P_n^{-m+1} = Q_n^m \cdot P_n^{-(m-1)}$$

$$= \frac{(-2)^{m-1}\sqrt{\pi}\sin(\frac{m+n}{2}\pi)\Gamma(\frac{m+n+1}{2})}{\Gamma(\frac{n-m+2}{2})} \cdot \frac{(-1)^{m-1}(n-m+1)!}{(n+m-1)!} \cdot \frac{(-1)^{\frac{n-m+1}{2}}(n+m-1)!}{2^n(\frac{n+m-1}{2})!(\frac{n-m+1}{2})!}. \tag{4.4}$$

Since $\Gamma(\frac{m+n+1}{2}) = (\frac{n+m-1}{2})!$, we have

$$Q_n^m \cdot P_n^{-m+1} = \frac{-(-1)^{\frac{n+m-1}{2}} 2^{m-1}\sin(\frac{m+n}{2}\pi)(n+m-1)!}{2^n(\frac{n-m+1}{2})! \ \Gamma(\frac{n-m+2}{2})}$$

$$= \frac{-(-1)^{\frac{n+m-1}{2}} 2^{m-1}\sqrt{\pi}\sin(\frac{m+n}{2}\pi)(n-m+1)!}{2^n[(\frac{n-m+1}{2})(\frac{n-m-1}{2})\cdots\frac{2}{2}][(\frac{n-m}{2})(\frac{n-m-2}{2})\cdots\frac{1}{2}]\Gamma(\frac{1}{2})}. \tag{4.5}$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and there are $\frac{n-m+1}{2}$ factors in each bracket, (4.5) reduces to

$$Q_n^m \cdot P_n^{-m+1} = -(-1)^{\frac{n+m-1}{2}}\sin(\frac{n+m}{2}\pi). \tag{4.6}$$

Note that for $n + m = 2k + 1$,

$$\begin{cases} \sin(\frac{n+m}{2}\pi) = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases} \\ (-1)^{\frac{n+m-1}{2}} = (-1)^k = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases} \end{cases} \tag{4.7}$$

Using (4.7) in (4.6), the proof of (4.3) is complete.

A table of the numbers $P_n^{-m}$ would lead to many properties of these numbers. However, we will not examine these numbers further here.

5. NUMBERS RELATED TO LEGENDRE NUMBERS.

The Legendre numbers of the second kind can be extended as were those of the first kind, see Definition 2. The extention to $Q_{-n}^{-m}$ will not be presented here.

Definition 3. For m and n non-negative integers

$$Q_n^{-m} = \begin{cases} 0 & \text{for } m \text{ and } n \text{ of the same parity} \\ 0 & \text{for } n \text{ odd, } m \text{ even} \\ \dfrac{(-1)^m \Gamma(n-m+1)}{\Gamma(n+m+1)} Q_n^m & \text{for } n \text{ even, } m = 1,3,5,\cdots,n-1. \end{cases}$$

These extended numbers can be obtained explicitly by replacing m with -m in (2.5) as

$$Q_n^{-m} = \begin{cases} 0, & m,n \text{ of the same parity} \\ 0, & n \text{ odd, } m \text{ even} \\ \dfrac{-2^{-m-1}\sqrt{\pi}\,\sin\left(\frac{-m+n}{2}\pi\right)\Gamma\left(\frac{-m+n+1}{2}\right)}{\Gamma\left(\frac{n+m+2}{2}\right)}, & n \text{ even, } m = 1,3,5,\cdots,n-1. \end{cases} \tag{5.1}$$

An argument similar to that given for (4.3) would show that (5.1) is equivalent to Definition 3. A result similar to (4.3), with m replaced with -m,

$$Q_n^{-m} \cdot P_n^{m+1} = -1, \; n \text{ even, } m = 1,3,\cdots,n-1 \tag{5.2}$$

is easy to prove by evaluating the left side of (5.2).

Another interesting set of numbers is obtained by taking the first derivatives of the associated Legendre functions and evaluating these at $x = 0$. From [3],

$$\frac{d\,P_n^m(0)}{dx} = \frac{\sqrt{\pi}\,2^{m+1}}{\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{-n-m}{2}\right)}, \; m+n \text{ odd, } n \geq 1, \; m \leq n. \tag{5.3}$$

Now, we define the set of numbers $p_n^m$, for m,n non-negative integers by

$$p_n^m = \begin{cases} 0, & m \geq n \\ 0, & m+n \text{ even} \\ \dfrac{d\,P_n^m(0)}{dx}, & m+n \text{ odd, } n \geq 1, \; m \leq n. \end{cases} \tag{5.4}$$

A connection between these numbers and the Legendre numbers of the first kind is

$$p_n^m = (-1)^n P_n^{m+1}. \tag{5.5}$$

The proof of (5.5) consists of evaluating both sides of the equation.

Proceeding in the same way for the functions $Q_n^m(x)$ with m,n non-negative integers, from [3], with $m + n$ even or $m + n$ odd with $m < n$,

$$\frac{d\,Q_n^m(0)}{dx} = \frac{2^m\sqrt{\pi}\,\cos\left(\frac{n+m}{2}\pi\right)\Gamma\left(\frac{n+m+2}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}. \tag{5.6}$$

For $m$ and $n$ non-negative integers we define the set of numbers

$$q_n^m = \begin{cases} 0, & m + n \text{ odd} \\ \dfrac{d\, Q_n^m(0)}{dx}, & m + n \text{ even.} \end{cases} \tag{5.7}$$

An almost trivial argument shows that these numbers are connected to the Legendre numbers of the second kind by

$$q_n^m = -Q_n^{m+1} \, . \tag{5.8}$$

6. SOME SERIES OF LEGENDRE NUMBERS.

Our aim here is to show that the sum of the non-zero entries in column one of Table 1 exists while those of the other columns do not exist. The work will involve infinite products and the reader is referred to [2] and [6] for short introductions to this topic and to [6] for the following theorem.

THEOREM 1. If each $a_n \geq 0$, then the product $\prod_{n=1}^{\infty}(1 - a_n)$ converges if and only the series $\sum_{n=1}^{\infty} a_n$ converges.

Consider the sum of the non-zero entries in the first column of Table 1. This sum can be expressed as the infinite series

$$\sum_{k=0}^{\infty} Q_{2k+1}. \tag{6.1}$$

From (3.15) with $n = 2k + 1$, a general term of this series can be simplified to

$$a_{2k+1} = \begin{cases} -1, & k = 0 \\ \dfrac{(-1)^{k+1} 2k(2k-2)\cdots 2}{(2k+1)(2k-1)\cdots 3}, & k \geq 1. \end{cases} \tag{6.2}$$

Clearly the series is alternating with decreasing positive terms. Furthermore, the limit of a general term exists and is non-negative since this sequence is a decreasing sequence and is bounded below by 0. If we show that

$$\lim_{k\to\infty} \frac{2k(2k-2)\cdots 2}{(2k+1)(2k-1)\cdots 3} = 0, \tag{6.3}$$

the series converges by the alternating series test. Now, the left side of (6.3) can be expressed in the equivalent forms

$$\prod_{k=1}^{\infty} \frac{2k}{2k+1} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{2k+1}\right). \tag{6.4}$$

Since the series

$$\sum_{k=1}^{\infty} \frac{1}{2k+1} \tag{6.5}$$

clearly diverges, Theorem 1 gives that the infinite products in (6.4) diverge. From this and the remarks after (6.2), this divergence is to zero. Therefore, (6.3) holds and the series (6.1) converges. An argument using asymptotic functions might prove interesting. What the series in (6.1) converges to is an open question.

The product of all the odd positive integers divided by the product of all the even positive integers can be expressed in the forms

$$\prod_{k=1}^{\infty} \frac{2k+1}{2k} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{2k}\right). \tag{6.6}$$

This infinite product can be shown to diverge to $+\infty$.

For $i > 0$, the series involving the non-zero entries in the other columns of Table 1,

$$\sum_{k=0}^{\infty} Q_{2k+1}^{2i}, \quad \sum_{k=0}^{\infty} Q_{2k}^{2i-1} \tag{6.7}$$

diverge because the limit of the nth terms is not zero. This is easily seen by expressing a general term in terms of an entry in the first column using (3.16).

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