Internat. J. Math. & Math. Sci.
VOL. 11 NO. 4 (1988) 823-824

A NOTE ON THE SPACES $\mathcal{O}_M$ AND $\mathcal{O}'_M$

CARLOS BOSCH* - JAN KUCERA**

*Instituto de Matemáticas
México, D.F. 04510 MEXICO

**Department of Pure and Applied Mathematics
Washington State University
Pullman, WA 99164-2930 U.S.A.

ABSTRACT. The spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ of multiplication and convolution operators on temperate distributions, together with their strong duals $\mathcal{O}'_M$ and $\mathcal{O}_C$, are Montel and distinguished.

KEY WORDS AND PHRASES. *Temperate distribution, multiplication and convolution operators, inductive and projective limits, Montel space, distinguished space*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. *Primary 46F10 Secondary 46A09*

Let $\mathcal{S}$, resp. $\mathcal{S}'$, be the space of rapidly decreasing functions, resp. temperate distributions, on $\mathbb{R}^n$. Then $\mathcal{O}_M$ is the space of all functions $f \in C^\infty$ for which the map $\varphi \mapsto \varphi D^\alpha f : \mathcal{S} \to \mathcal{S}$ is continuous for each $\alpha \in \mathbb{N}^n$. The topology of $\mathcal{O}_M$ is generated by a family of seminorms $f \mapsto \max\{|\varphi(x)D^\alpha f(x)|;\ x \in \mathbb{R}^n\}, \varphi \in \mathcal{S}, \alpha \in \mathbb{N}^n$. Its strong dual is denoted by $\mathcal{O}'_M$.

For each $q \in \mathbb{N}$ the space

$$L_q = \{f : \mathbb{R}^n \longrightarrow \mathbb{C};\ \|f\|_q^2 = \sum_{|\alpha+\beta|\le q} \int_{\mathbb{R}^n} |x^\alpha D^\beta f(x)|^2 dx < \infty\}$$

is Hilbert. If we denote its dual by L_{-q} we have $\mathcal{S} = \operatorname{proj}\lim_{q\to\infty} L_q$ and $\mathcal{S}' = \operatorname{ind}\lim_{q\to\infty} L_{-q}$.

Put $W(x) = (1 + |x|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^n$. Then for each integer k (positive or negative) the map $T_k : f \to W^k f : \mathcal{S}' \to \mathcal{S}'$ is bijective. We denote by $W^k L_m$, $k, m \in \mathbb{Z}$, the image of L_m under T_k and provide it with the topology which makes $T_k : L_m \to W^k L_m$ a topological isomorphism.

Let $\mathcal{O}_q = \operatorname{ind}\lim_{p\to\infty} W^p L_q$, $q \in \mathbb{N}$, and $\mathcal{O}_{-q}$ be its strong dual. It is proved in [4] that $\mathcal{O}_{-q} = \operatorname{proj}\lim_{p\to\infty} W^{-p} L_{-q}$. Also, $\mathcal{O}_M = \operatorname{proj}\lim_{q\to\infty} \mathcal{O}_q$ and $\mathcal{O}'_M = \operatorname{ind}\lim_{q\to\infty} \mathcal{O}_{-q}$, see [3 & 5].

PROPOSITION 1. The spaces $\mathcal{O}_M$ and $\mathcal{O}'_M$ are Montel.

PROOF. First we prove that $\mathcal{O}'_M$ is Montel. It is ultrabornological, [5; Th. 4] and barreled [1; 3-15, Ex. 9]. Hence it is infrabarreled. Further $\mathcal{O}'_M$ is complete and Schwartz, [5; Ths. 2 & 3] and therefore it is semi-Montel, [1; 3-15, Prop. 4], [6; II, § 4, No. 4, Th. 16]. As infrabarreled semi-Montel space, it is Montel.

$\mathcal{O}_M$ is Montel as a strong dual of the reflexive space $\mathcal{O}'_M$, [5, Th. 1], [1; 3-9, Prop. 9].

PROPOSITION 2. The spaces $\mathcal{O}_M$ and $\mathcal{O}'_M$ are distinguished.

PROOF. Both $\mathcal{O}_M$ and $\mathcal{O}'_M$ are ultrabornological and reflexive, [5; Ths. 1 & 3]. Hence they are strongly ultrabornological and strongly barreled, [1; 3-15, Ex. 9]. By [1; 3-16, Prop. 1], they are distinguished.

Let $\mathcal{O}_c$ be the strong dual of the space $\mathcal{O}'_C$ of convolution operators on $\mathcal{S}'$. Then Fourier transformations $\mathcal{F} : \mathcal{O}_M \to \mathcal{O}'_C$ and $\mathcal{F} : \mathcal{O}'_M \to \mathcal{O}_C$ are topological isomorphisms and we have

COROLLARY. The spaces $\mathcal{O}_C$ and $\mathcal{O}'_C$ are both Montel and distinguished.

REFERENCES

[1] HORVATH, J. Topological Vector Spaces and Distributions, Addison Wesley, Reading 1966.

[2] KUCERA, J. On Multipliers of Temperate Distributions, Czech. Math. J. 21, 96 (1971), 610-618.

[3] KUCERA, J., MCKENNON, K. Certain Topologies on the Space of Temperate Distributions and its Multipliers, Indiana Univ. Math. J. 24(8), (1975), 773-775.

[4] KUCERA, J., MCKENNON, K. The Topology of Certain Spaces of Multipliers of Temperate Distributions, Rocky Mountain J. of Math. 7(2)(1977) 377-383.

[5] KUCERA, J. The Spaces $\mathcal{O}_M$ and $\mathcal{O}_C$ are ultrabornological, a new proof. Internat. J. Math. & Math. Sci. 8(4)(1985), 813-816.

[6] GROTHENDIECK, A., Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Am. Math. Soc. 16,(1955).