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SMOOTH STRUCTURES ON SPHERE BUNDLES OVER SPHERES

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ABSTRACT. In [1] R. De Sapio gave a classification of smooth structures of a p-sphere bundle over a q-sphere with one cross-section and $p < q$. In [2] J. Munkres also gave a classification up to concordance of differential structures in the case where the bundle has at least two cross-sections. In [3] R. Schultz gave a classification in the case $p \geq q$. Here we will give a classification of the p-sphere bundle over a q-sphere without any cross-section and $p < q$.

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1. INTRODUCTION

Let E represent p-sphere bundle over a q-sphere with $\beta \in \Pi_{q-1}SO(p+1)$ the characteristic class of the corresponding p+1-disc bundle over the q-sphere. In [4] R. De Sapio gave a complete classification of the special case where $\beta = 0$. In [5] and [6] Kawakubo and Schultz respectively also gave a classification of E for this special case. This author in [7] gave a generalization of this special case to product of three ordinary spheres. In [1] a classification of E was given for $p < q - 1$ and where E has a cross-section and $\beta \neq 0$. In [3] Schultz gave a classification of E for $p \geq q$ and E is without cross-section. We shall here remove the fact that E has a cross-section so that not every element of $\pi_{q-1}SO(p+1)$ can be pulled back to the element $\pi_{q-1}SO(p)$ in the homomorphism $S_* : \pi_{q-1}SO(p) \to \pi_{q-1}SO(p+1)$ induced by the inclusion $s : SO(p) \to SO(p+1)$. S^n denotes the unit n-sphere with the usual differential structure in the Euclidean (n+1)-space R^{n+1} . Σ^n denotes an homotopy n-sphere and θ^n denotes the group of homotopy n-spheres. H(p,k) denotes the subset of θ^p which consists of those homotopy p-sphere Σ^p such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [4, Lemma 4], H(p,k) is a subgroup of θ^p and it is not always zero and in fact in [7] we showed that if $k \geq p-3$, $H(p,k) = \theta^p$. We shall adopt the notation $E(\Sigma^q)$ to represent the total space of a p-sphere bundle over a homotopy q-sphere Σ^q. We will then prove the following:

THEOREM. If M is a smooth, n-manifold homeomorphic to a p-sphere bundle over a q-sphere with total space E where $n = p+q \geq 6$ and $p < q$ then there exists homotopy spheres Σ^q and Σ^n such that M is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$. We shall define a pairing

$$G : \pi_p SO(q) \times \pi_{q-1} SO(p+1) \to \theta^{p+q}$$

and show that if $\beta \in \pi_{q-1} SO(p+1)$ is the characteristic class of a p-sphere bundle over an homotopy q-sphere Σ^q, then $G(\pi_p SO(q), \beta)$ equals the inertial group of $E(\Sigma^q)$. The above theorem together with the latter will give us the following.

THEOREM. Let E be the total space of a p-sphere bundle over a q-sphere then the diffeomorphism classes of (p+q)-manifolds that are homeomorphic to E are in one-to-one correspondence with the group

$$\frac{\theta^q}{H(p,q)} \times \frac{\theta^n}{\text{Image } G_\beta} \quad \text{where} \quad n = p+q \geq 6 \quad \text{and} \quad p < q.$$

2. CLASSIFICATION THEOREM

In this section, we will prove the classification theorem for any manifold M^n homeomorphic to E. We will apply the obstruction theory to smoothing of manifolds developed by Munkres in [8]. Since $p+q \geq 6$ and $2 \leq p < q$ then E is simply-connected and the homology of E has no 2-torsion, hence the "Hauptvermutung" of D. Sullivan [9] applies and this means that piecewise linear homeomorphism can be replaced by homeomorphism, we shall not distinguish the two.

DEFINITION. Let M and N be smooth closed n-manifolds and L a closed subset of M of dimension less than n. Let $f : M \to N$ be a homeomorphism such that for each simplex γ of L, $\bar{\gamma}$ and $f(\bar{\gamma})$ are contained in coordinate systems under which they are flat. f is said to be a diffeomorphism modulo L if $f|(M-L)$ is a diffeomorphism and each simplex γ of L has a neighborhood V such that f is smooth on V-L near γ. By [8, Theorem 2.8], if M and N are homeomorphic then there is a diffeomorphism modulo (n-1)-skeleton of M. If $f : M \to N$ is a diffeomorphism modulo m-skeleton $m < n$ then the obstruction to deforming f to a diffeomorphism modulo (m-1)-skeleton $g : M \to N$ is an element $\lambda(f) \in H_m(M,\Gamma^{n-m})$ where Γ^{n-m} is a group of diffeomorphism of S^{n-m-1} modulo those that extend to diffeomorphisms of D^{n-m}. g is called the smoothing of f. If $\lambda(f) = 0$ then by [8, §4] smoothing g exist.

THEOREM 2.1. If M is a smooth n-manifold homeomorphic to E where E denotes the total space of a p-sphere bundle over a q-sphere, $2 \le p < q$ and $n = p + q$ then there exist homotopy spheres Σ^q and Σ^n such that M is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$ where $E(\Sigma^q)$ denotes the total space of a p-sphere bundle over the homotopy q-sphere Σ^q.

PROOF. E is the total space of a p-sphere bundle over a q-sphere with characteristic class $[b] \in \pi_{q-1}SO(p+1)$ then $E = D^q \times S^p \bigcup_{f_b} D^q \times S^p$ where $f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ is a diffeomorphism defined by $f_b(x,y) = (x,b(x)\cdot y)$, $(x,y) \in S^{q-1} \times S^p$

$$H_i(E) = \begin{cases} Z & \text{for } i = 0,p,q,p+q \\ 0 & \text{elsewhere} \end{cases}$$

Since M^n is homeomorphic to E where $n = p+q \ge 6$ $2 \le p < q$, then M^n is simply connected and since $H_3(M,Z)$ has no 2-torsion, then "Hauptvermutung" of D. Sullivan [9] implies that there is a piecewise linear homeomorphism $h : M^n \to E$ which by [8, §5] is a diffeomorphism modulo (n-1)-skeleton. Since $H_i(M,Z) = 0$ for $n-p+1 \le i \le n-1$ then we can assume that h is a diffeomorphism modulo $n-p = q$ skeleton. The obstruction to a diffeomorphism modulo q-1 skeleton is $\lambda(h) \in H_q(M,\Gamma^p) = \Gamma^p$. If $[\phi] = \lambda(h) \in \Gamma^p$ where $\phi : S^{p-1} \to S^{p-1}$ is a diffeomorphism that represents $\lambda(h)$ and let Σ^p denote the homotopy p-sphere where $\Sigma^p = D_1^p \bigcup_{\phi} D_2^p$. We define a map

$$j : S^p \to \Sigma^p \quad \text{where} \quad S^p = D_1^p \bigcup_{\text{id.}} D_2^p$$

such that

$$j(x) = \begin{cases} x & \text{if} \quad x \in D_1^p \\ x\, \phi^{-1}(\frac{x}{|x|}) & \text{if} \quad x \in D_2^p. \end{cases}$$

So j is an homeomorphism which is identity on D_1^p and the radial extension of ϕ^{-1} on D_2^p and so the first obstruction $\lambda(j)$ to deforming j to a diffeomorphism is $[\phi^{-1}] = -\lambda(h)$. We then define $\text{id} \times j : D^q \times S^p \to D^q \times \Sigma^p$ where id is the identity, then $\text{id} \times j$ is a homeomorphism and it follows from [8, Def. 3.4] that the first obstruction $\lambda(\text{id}\, x j)$ to deforming id xj to a diffeomorphism is also $-\lambda(h)$. We can form a manifold E' by identifying two copies of $D^q \times \Sigma^p$ along their common boundaries $S^{q-1} \times \Sigma^p$ by the diffeomorphism $f_b : S^{q-1} \times \Sigma^p \to S^{q-1} \times \Sigma^p$ where $f_b(x,y) = (x,b(x)\ y)$ and $[b] \in \pi_{q-1}SO(p+1)$. So $E' = D^q \times \Sigma^p \underset{f_b}{\cup} D^q \times \Sigma^p$. We define a map $g : E = (D^q \times S^p)_1 \underset{f_b}{\cup} (D^q \times S^p)_2 \to (D^q \times \Sigma^p)_1 \underset{f_b}{\cup} (D^q \times \Sigma^p)_2 = E'$ by $g(x,y) = id \times j(x,y)$ on both $(D^q \times \Sigma^p)_1$, and $(D^q \times S^p)_2$, the map looks like

$$\begin{array}{ccccccc}
E = (D^q \times S^p)_1 \underset{f_b}{\cup} (D^q \times S^p)_2 & = & (D^q \times S^p)_1 & \underset{f_b}{\cup} & S^{q-1} \times S^p & \underset{id}{\cup} & (D^q \times S^p)_2 \\
\downarrow g & = & \downarrow id \times j & & \downarrow id x j & & \downarrow id \times j \\
E' = (D^q \times \Sigma^p)_1 \underset{f_b}{\cup} (D^q \times \Sigma^p)_2 & = & (D^q \times \Sigma^p)_1 & \underset{f_b}{\cup} & S^{q-1} \times S^p & \underset{id}{\cup} & (D^q \times \Sigma^p)_2
\end{array}$$

g is an homeomorphism and the first obstruction to a diffeomorphism is $\lambda(id\ xj) = -\lambda(h)$. It follows that the obstructions to smoothing the composition $g \cdot h : M \to E'$ is $\lambda(g \cdot h) = \lambda(g) + \lambda(h) = -\lambda(h) + \lambda(h) = 0$. It follows that g.h. : $M \to E'$ is a diffeomorphism modulo (q-1)-skeleton. However in [7, Remark 1] we showed that $D^q \times \Sigma^p$ is diffeomorphic to $D^q \times S^p$ if $p \le q + 2$ and so by our hypothesis $p < q$ then it follows that $D^q \times \Sigma^p$ is diffeomorphic to $D^q \times S^p$. This implies that E and E' are diffeomorphic hence $g' : M \to E$ is a diffeomorphism modulo (q-1)-skeleton. Since $H_i(M,Z) = 0$ for $p + 1 < i < q-1$, there is no more obstruction to deforming g' to a diffeomorphism until we get to (p-1) skeleton. We can then assume that g' is a diffeomorphism modulo p-skeleton. The first obstruction to deforming g' to a diffeomorphism modulo (p-1)-skeleton is $\lambda(g';) \in H_p(M,\Gamma^q) = \Gamma^p$. Let $[\phi] = \lambda(g') \in \Gamma^q$ where $\phi : S^{q-1} \to S^{q-1}$ is a diffeomorphism which represents $\lambda(g') \in \Gamma^q$. We define $(\phi \times id) : S^{q-1} \times S^p \to S^{q-1} \times S^p$ where $(\phi \times id)(x,y) = (\phi(x),y)$ and if $\beta = [b] \in \pi_{q-1}SO(p+1)$ we also define $f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ where $f_b(x,y) = (x,b(x) \cdot y)$. We then have two orientation preserving diffeomorphisms of $S^{q-1} \times S^p$ unto itself which we can compose to get $(\phi \times id) \cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ where $(\phi \times id) \cdot f_b(x,y) = (\phi(x),b(x) \cdot y)$. We then construct a manifold by attaching two copies of $D^q \times S^p$ along their common boundary $S^{q-1} \times S^p$ using the diffeomorphism $(\phi \times id) \cdot f_b$ to have $D_1^q \times S^p \underset{(\phi \times id) \cdot f_b}{\cup} D_2^q \times S^p$. Notice that this manifold is a p-sphere bundle over a homotopy q-sphere $\Sigma^q = D_1^q \underset{\phi}{\cup} D_2^q$ whose characteristic map is

$\beta = [b] \in \pi_{q-1}SO(p+1)$. We define a map

$$h : D^q \times S^p \underset{f_b}{\cup} D_2^q \times S^p \to D_1^q \times S^p \underset{(\phi \times id)\cdot f_b}{\cup} D_2^q \times S^p$$

by

$$h(x,y) = \begin{cases} (x,y) & \text{if} \quad (x,y) \in D_1^q \times S^p \\ (x\cdot\phi^{-1}(\frac{x}{|x|}),y) & \text{if} \quad (x,y) \in D_2^q \times S^p \end{cases}$$

Hence h is identity on $D_1^q \times S^p$ and a radial extension of ϕ^{-1} on D_2^q. It then follows that h is an homeomorphism with the first obstruction to a diffeomorphism being $[\phi^{-1}] = -\lambda(g')$. Then by [8, 3.8] the first obstruction to deforming the composition $g' \circ h = g : M \to D_1^q \times S^p \underset{(\phi \times id)\cdot f_b}{\cup} D_2^q \times S^p$ into a diffeomorphism is $\lambda(g) = \lambda(g'\circ h) = \lambda(g') + \lambda(h) = -\lambda(h) + \lambda(h) = 0$ and hence g is a diffeomorphism modulo (p-1)-skeleton. Since $H_i(M,Z) = 0$ for $0 < i < p$ then we can assume that g is a diffeomorphism modulo one point. Since $D_1^q \times S^p \underset{(\phi \times id)\cdot f_b}{\cup} D_2^q \times S^p$ is a p-sphere bundle over a homotopy q-sphere $\Sigma_1^q = D_2^q \underset{\phi}{\cup} D^q$ with characteristic map $[b] \in \pi_{q-1}SO(p+1)$, we shall denote it by $E(\Sigma^q)$. Since g is a diffeomorphism modulo one point then it is known that there is an homotopy n-sphere Σ^n such that M is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$. Hence the proof.

3. INERTIAL GROUPS

Since by Theorem 2.1, every manifold homeomorphic to E is diffeomorphic to $E(\Sigma^q) \# \Sigma^n$ for some homotopy spheres Σ^q, Σ^n, classification of such manifolds reduces to classification of manifolds of the form $E(\Sigma^q) \# \Sigma^n$. To complete this classification, we then need to investigate what happens when we vary the homotopy spheres and in particular we need to investigate the Inertial group of $E(\Sigma^q)$. We will investigate these in this section.

LEMMA 3.1. Let Σ_1^q and Σ_2^q be homotopy q-spheres such that $\Sigma_i^q = D_1^q \underset{\phi_i}{\cup} D_2^q$ $i = 1, 2$ then $E(\Sigma_1^q)$ is diffeomorphic to $E(\Sigma_2^q)$ if and only if $\Sigma_1^q \pm \Sigma_2^q \in H(q,p)$.

PROOF. Suppose $E(\Sigma_1^q)$ is diffeomorphic to $E(\Sigma_2^q)$. This means that $D_1^q \times S^p \underset{(\phi_1 \times id)\cdot f_b}{\cup} D_2 \times S^p$ is diffeomorphic to $D_1^q \times S^p \underset{(\phi_2 \times id)\cdot f_b}{\cup} D_2^q \times S^p$ where $\phi_i \times id : S^{q-1} \times S^p \to S^{q-1} \times S^p$ is the diffeomorphism defined by $\phi_i(x,y) = (\phi_i(x),y)$ and $f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ is defined by $f_b(x,y) = (x,b(x)\cdot y)$ where $[b] = \beta \in \pi_{q-1}SO(p+1)$ is the characteristic map of the bundle. The manifold $E(\Sigma_2^q)$ can be regarded as the boundary of the (p+1)-disc bundle over Σ_2 which is denoted by

$D_1^q \times D^{p+1} \bigcup_{(\phi_2\times id)\cdot f_b} D_2^q \times D^{p+1} = D(\Sigma_2^q)$. So if $E(\Sigma_1^q)$ is diffeomorphic to $E(\Sigma_2)$ then since Σ_1^q can be embedded in $E(\Sigma_1^q)$ it follows that Σ_1^q embedds in $E(\Sigma_2^q)$. But Σ_2^q naturally embedds in $E(\Sigma_2^q)$ and so we have Σ_1^q and Σ_2^q sitting in $E(\Sigma_2^q)$, if we translate Σ_1^q away from Σ_2^q we can run a tube between them to obtain an embedding $\Sigma_1^q \# (-\Sigma_2^q) \to E(\Sigma_2^q)$ so that the embedding is homotopically trivial and so by the engulfing result of [10, chapter 7] it means that $\Sigma_1^q \# (-\Sigma_2^q)$ can be embedded in the interior of a (p+q+1)-disc in $E(\Sigma_2^q)$ and by [11, 3.5] the embedding is isotopic to a nuclear embedding into the interior of $S^q \times D^{p+1}$. However the embedding $\Sigma_1^q \# (-\Sigma_2^q) \to S^q \times D^{p+1}$ is an homotopy equivalence, it then follows by Smale's theorem [12, Theorem 4.1] that $\Sigma_1^q \# (-\Sigma_2^q) \times D^{p+1}$ is diffeomorphic to $S^q \times D^{p+1}$ and so it follows that $\Sigma_1^q \# (-\Sigma_2^q) \times S^p$ is diffeomorphic to $S^q \times S^p$ hence $\Sigma_1^q \# (-\Sigma_2^q) \varepsilon H(q,p)$. Conversely suppose $\Sigma_1^q \# (-\Sigma_2^q) \varepsilon H(q,p)$ then this implies $(\Sigma_1^q \# (-\Sigma_2^q)) \times S^p$ is diffeomorphic to $S^q \times S^p$. Since $S^q \times S^p$ embedds in R^{p+q+1} with trivial normal bundle then it follows that $\Sigma_1^q \# (-\Sigma_2^q)$ embedds in R^{p+q+1} with trivial normal bundle. This shows that each Σ_i^q for i = 1, 2 embedds in R^{p+q+1} with trivial normal bundle and by [11, §3.5] the embedding is isotopic to an embedding of Σ_i^q into the interior of $S^q \times D^{p+1}$. However for i = 1, 2 the embedding $\Sigma_i^q \to S^q \times D^{p+1}$ is an homotopy equivalence hence it follows from [12, Theorem 4.1] that $\Sigma_i^q \times D^{p+1}$ is diffeomorphic to $S^q \times D^{p+1}$ which implies that $\Sigma_1^q \times D^{p+1}$ is diffeomorphic to $\Sigma_2^q \times D^{p+1}$. Now since $\Sigma_i^q = D_1^q \bigcup D_2^q$ where $\phi_i : S^{q-1} \to S^{q-1}$ represents $\Sigma_i^q \varepsilon \Gamma^q$ i = 1, 2, then we can write $\Sigma_i^q \times D^{p+1} \stackrel{\phi_i}{=} D_1^q \times D^{p+1} \bigcup_{\phi_i\times id} D_1^q \times D^{p+1}$ where we identify two copies of $D^q \times D^{p+1}$ along $S^{q-1} \times D^{p+1}$ by the diffeomorphism $\phi_i \times id : S^{q-1} \times D^{p+1} \to S^{1-1} \times D^{p+1}$ defined by $(\phi_i \times id)(x,y) = (\phi_i(x),y)$ where $(x,y) \varepsilon S^{q-1} \times D^{p+1}$. So $\Sigma_1^q \times D^{p+1}$ is diffeomorphic to $\Sigma_2^q \times D^{p+1}$ implies $D_1^q \times D^{p+1} \bigcup_{\phi_1\times id} D_2^q \times D^{p+1}$ is diffeomorphic to $D_1^q \times D^{p+1} \bigcup_{\phi_2\times id} D_2^q \times D^{p+1}$.

Now consider the manifold $D(S^q) = D_+^q \times D^{p+1} \bigcup_{f_b} D_-^q \times D^{p+1}$ which is a (p+1)-disc bundle over a q-sphere with characteristic map $[b] \varepsilon \pi_{q-1}SO(p+1)$. We then form the quotient space

$$D(S^q) \bigcup \Sigma^q{}_1 \times D^{p+1} = (D_+^q \times D^{p+1} \bigcup_{f_b} D_-^q \times D^{p+1}) \bigcup (D_1^q \times D^{p+1} \bigcup_{\phi_1\times id} D_2^q \times D^{p+1})$$

by identifying $D_-^q \times D^{p+1} \subset D(S^q)$ and $D_1^q \times D^{p+1} \subset \Sigma_1^q \times D^{p+1}$ by the relation $(x,y) = (x,y)(x \varepsilon D_-^q = D_1^q,\ y \varepsilon D^{p+1})$. The manifold $D(S^q) \cup \Sigma_2^q \times D^{p+1}$ is similarly constructed. Since $\Sigma_1^q \times D^{p+1}$ is diffeomorphic to $\Sigma_2^q \times D^{p+1}$. Let $d : \Sigma_1^q \times D^{p+1} \to \Sigma_2^q \times D^{p+1}$ be the diffeomorphism and since any diffeomorphism fixes a disc, we can assume that d is identity on the disc $D^{p+q+1} = D_1^q \times D^{p+1}$, then we can define a diffeomorphism.

$$g : D(S^q) \cup \Sigma_1^q \times D^{p+1} \to D(S^q) \cup \Sigma_2^q \times D^{p+1}$$

where

$$g(x) = \begin{cases} d(x) & \text{for} \quad x \in \Sigma_1^q \times D^{p+1} \\ x & \text{for} \quad x \in D(S^q). \end{cases}$$

This means that $g = d$ on $\Sigma_1^q \times D^{p+1}$ and identity on $D(S^q)$. g is well defined because d is identity on the disc connecting $D(S^q)$ and $\Sigma_1^q \times D^{p+1}$ and g is a diffeomorphism. The manifold $D(S^q) \cup \Sigma_i^q \times D^{p+1}$ can be clearly seen as follows. Let $(\phi_i \times id) \cdot f_b : S^{q-1} \times D^{p+1} \to S^{q-1} \times D^{p+1}$ be the diffeomorphism defined by $((\phi_i \times id) \cdot f_b)(x,y) = (\phi_i(x), b(x) \cdot y)$, $(x,y) \in S^{q-1} \times D^{p+1}$ then attaching two manifolds $D_+^q \times D^{p+1}$ and $D_-^q \times D^{p+1}$ by the diffeomorphism $(\phi_i\ id) \cdot f_b$ we have $D_+^q \times D^{p+1} \underset{(\phi_i \times id) \cdot f_b}{\cup} D_-^q \times D^{p+1}$ we get a (p+1)-disc bundle over the homotopy q-sphere $\Sigma^q{}_i = D_i^q \underset{\phi_i}{\cup} D_2^q$ $i = 1, 2$. However, from the way $D(S^q) \cup \Sigma_i^q \times D^{p+1}$ is constructed it is easily seen that $D(S^q) \cup \Sigma_i^q \times D^{p+1} = D_+^q \times D^{p+1} \underset{(\phi_i \times id) \cdot f_b}{\cup} D_-^q \times D^{p+1} = D(\Sigma_i^q)$ hence g is the diffeomorphism of $D(\Sigma_1^q)$ onto $D(\Sigma_2^q)$ then it follows that $\partial(D(\Sigma_1^q)) = E(\Sigma_1^q)$ is diffeomorphic to $\partial D(\Sigma_2^q) = E(\Sigma_2^2)$.

Hence the theorem is proved.

REMARK 1. This theorem implies that $E(\Sigma_1^q)$ is diffeomorphic to $E(\Sigma_2^q)$ if and only if Σ_1^q and Σ_2^q are equivalent in the quotient group $\theta^q/_{H(q,p)}$.

To complete this classification, we need to determine the inertial group of $E(\Sigma^q)$. The inertial group $\iota(M)$ of an oriented closed smooth n-dimensional manifold M is defined to be the subgroup of θ^n consisting of those homotopy n-spheres Σ^n such that $M \# \Sigma^n$ diffeomorphic to M.

Let E_β represent the total space of a p-sphere bundle over a real q-sphere with characteristic class $\beta \in \pi_{q-1}SO(p+1)$. In [13] we defined a map $G_\beta : \pi_p SO(q) \to \theta^{p+n}$ and showed that the image of this map equals the inertial group of E_β where $p < q$ and E_β has no cross-section. We shall similarly define a map $G_{\phi \cdot \beta} : \pi_p SO(q) \to \theta^{p+q}$ and show that the image of this map equals the inertial group of $E(\Sigma^q)$ where $E(\Sigma^q)$ is the total space of p-sphere bundle over a homotopy sphere $\Sigma^q = D_1^q \underset{\phi}{\cup} D_2^q$. Let $\alpha \in \pi_p SO(q)$ we define $G_{\phi \cdot \beta}(\alpha) = S^{q-1} \times D^{p+1} \underset{f_{a^{-1}}(\phi \times id) \cdot f_b}{\cup} D^q \times S^p$ where $[a] = \alpha$ and $[b] = \beta \in \pi_{q-1}SO(p+1)$ and

$f_{a-1}(\phi\times id)\cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ is a diffeomorphism defined by $f_{a-1}(\phi\times id)\cdot f_b(x,y) = (a^{-1}(b(x)\cdot y) \cdot (x), b(x)\cdot y)$. One can easily show that $G_{\phi\cdot\beta}$ is well-defined and that its image is an homotopy (p+q)-sphere as similarly shown in [13].

LEMMA 3.2. Let $E(\Sigma^q)$ denote the total space of a p-sphere bundle over an homotopy q-sphere $\Sigma^q = D_1^q = D_1^q \bigcup_\phi D_2^q$ with characteristic class $\beta \in \pi_{q-1}SO(p+1)$ then $G_{\phi\cdot\beta}\pi_p(SO(q)) = I(E(\Sigma^q))$.

PROOF. If $\Sigma^{p+q} \in I(E(\Sigma^q))$ then this means there is a diffeomorphism $d : E(\Sigma^q) \# \Sigma^{p+q} \to E(\Sigma^q)$, that is,

$$d : (D_1^q\times S^p \bigcup_{(\phi\times id)\cdot f_b} D_2^q\times S^p) \# \Sigma^{p+q} \to D_1^q \times S^p \bigcup_{(\phi\times id)\cdot f_b} D_2 \times S^p$$

since $p < q$ then $\pi_p(E(\Sigma^q))$ is infinitely cyclic and $d(o\times S^q)$ represents a generator and so is homotopic to the inclusion $0 \times S^p \to E(\Sigma^q)$. By Haefliger's theorem [14], $d|0 \times S^p$ and the inclusion $0 \times S^p \to E(\Sigma^q)$ are isotopic and by isotopy extension theorem and tubular neighborhood theorem, d is isotopic to a map which we shall again denote by d such that $d|D^q \times S^p = D^q \times S^p$ where $d(x,y) = (a(y)\cdot x,y)$ for $[a] \in \pi_pSO(q)$ and $(x,y) \in D^q \times S^p$. We now remove $D^q \times S^p$ from $E(\Sigma^q) \# \Sigma^{p+q} = (D^q\times S^p \bigcup_{(\phi\times id)\cdot f_b} D^q\times S^p) \# \Sigma^{p+q}$ by surgery away from the connected sum and replace it with $S^{q-1} \times D^{p+1}$. After this operation on the summand $E(\Sigma^q)$ of the connected sum, we have the manifold $S^{q-1} \times D^{p+1} \bigcup_{(\phi\times id)\cdot f_b} D^q \times S^p$. Since the diffeomorphism $(\phi\times id)\cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ extend to the diffeomorphism of $S^{q-1} \times D^{p+1}$ onto itself then $S^{q-1} \times D^{p+1} \bigcup_{(\phi\times id)\cdot f_b} D^q \times S^p$ is diffeomorphic to $S^{q-1} \times D^{p+1} \bigcup_{id} D^q \times S^p$, the diffeomorphism g is defined thus

$$\begin{array}{ccc} S^{q-1} \times D^{p+1} & \bigcup_{id} & D^q \times S^p \\ {\scriptstyle (\phi\times id)\cdot f_b}\downarrow & & \downarrow \\ S^{q-1} \times D^{p+1} & \bigcup_{(\phi\times id)\cdot f_b} & D^q \times S^p \end{array}$$

where

$$g(x,y) = \begin{cases} (x,y) & \text{if} \quad (x,y) \in D^q \times S^p \\ ((\phi\times id)\cdot f_b)(x,y) & \text{if} \quad (x,y) \in S^{q-1} \times D^{p+1}. \end{cases}$$

However, by [7, Lemma 2.1.2], $S^{q-1} \times D^{p+1} \bigcup_{id} D^q \times S^p$ is diffeomorphic to the standard (p+q)-sphere S^{p+q}, hence after this surgery $E(\Sigma^q)$ is reduced to S^{p+q} and so $E(\Sigma^q) \# \Sigma^{p+q}$ is reduced to $S^{p+q} \neq \Sigma^{p+q} = \Sigma^{p+q} = \Sigma^{p+q}$.

We perform the corresponding modification (under d) on $E(\Sigma^q)$ to remove the p-sphere $0 \times S^p$ with product structure $d(D_1^q \times S^p)$ in $E(\Sigma^q)$. From this modification we obtain a manifold $S^{q-1} \times D^{p+1} \cup_{\psi} D^q \times S^p$ where $\psi = (d^{-1}|S^{q-1} \times S^p) \cdot (\phi \times id) \cdot f_b$ and this is diffeomorphic to Σ^{p+q} because of the way we performed the surgery using d. However, this manifold $S^{q-1} \times D^{p+1} \cup_{\psi} D^q \times S^p = G_{\phi \cdot \beta}(\alpha)$ by the definition of $G_{\phi \cdot \beta}$, thus there exists an element $\alpha \in \pi_p SO(q)$ (namely) $d|(D_1^q \times S^p)$ which gives $\alpha \in \pi_p SO(q)$ such that $\Sigma^{p+q} = G_{\phi \cdot \beta}(\alpha)$ and so $\Sigma^{p+q} \in G_{\phi \cdot \beta}(\pi_p SO(q))$, hence $I(E(\Sigma^q)) \subset G_{\phi \cdot \beta}(\pi_p SO(q))$. Conversely suppose $\Sigma^{p+q} \in G_{\phi \cdot \beta}(\pi_p SO(q))$ then for some $\alpha \in \pi_p SO(q)$, $\Sigma^{p+q} = S^{q-1} \times D^{p+1} \cup_{f_{a^{-1}} \cdot (\phi \times id) \cdot f_b} D^q \times S^p$ where ϕ is a diffeomorphism of S^{q-1} onto itself representing $\Sigma^q = D_1^q \cup_{\phi} D_2^q$ and $f_{a^{-1}}$ and f_b are as defined earlier. Notice that $G_{\phi \cdot \beta}(\alpha)$ is thus the obstruction to the construction of a diffeomorphism $S^{p+q} \to \Sigma^{p+q}$. To construct a diffeomorphism from $S^{p+q} \to \Sigma^{p+q}$, we map $S^{q-1} \times D^{p+1} \subset S^{p+q}$ to itself using $(\phi \times id) \cdot f_b$ to have

$$\begin{array}{c} S^{p+q} = S^{q-1} \times D^{p+1} \cup_{id} D^q \times S^p \\ \downarrow (\phi \times id) \cdot f_b \\ \Sigma^{p+q} = S^{q-1} \times D^{p+1} \cup_{f_{a^{-1}} \cdot (\phi \times id) \cdot f_b} D^q \times S^p \end{array}$$

and try to extend it to $D^q \times S^p$. On the boundary $S^{q-1} \times S^p$ of $D^q \times S^p$, the map is $f_{b^{-1}} \cdot (\phi^{-1} \times id) \cdot f_a \cdot (\phi \times id) \cdot f_b$. So this means that $\Sigma^{p+q} = G_{\phi \cdot \beta}(\alpha)$ is the obstruction to extending the diffeomorphism $f_{b^{-1}} \cdot (\phi^{-1} \times id) \cdot f_a \cdot (\phi \times id) \cdot f_b : S^{q-1} \times S^p \to S^{q-1} \times S^p$ to a diffeomorphism of $D^q \times S^p$ onto itself. We can then define a map $E(\Sigma^p) \to E(\Sigma^q)$ using the diffeomorphism $f_a : D_1^q \times S^p \to D_1^q \times S^p$ where $f_a(x,y) = (a(y) \cdot x, y)$ $(x,y) \in D_1^q \times S^p$ we then have

$$\begin{array}{c} E(\Sigma^q) = D_1^q \times S^p \cup_{(\phi \times id) \cdot f_b} D_2^q \times S^p \\ \downarrow f_a \\ E(\Sigma^q) = D_1^q \times S^p \cup_{(\phi \times id) \cdot f_b} D_2^q \times S^p \end{array}$$

On the boundary $S^{q-1} \times S^p$ of $D_1^q \times S^p$, this map is $f_{b^{-1}} \cdot (\phi^{-1} \times id) \cdot f_a \cdot (\phi \times id) \cdot f_b$ and the obstruction to extending this to a diffeomorphism of $E(\Sigma^q)$ onto itself is the obstruction to extending the map $f_{b^{-1}}\cdot(\phi^{-1}\times id)\cdot f_a\cdot(\phi\times id)\cdot f_b$ to the diffeomorphism of $D^q_2 \times S^p$ onto itself which is Σ^{p+q}. It then follows that $E(\Sigma^q) \to E(\Sigma^q) \# \Sigma^{p+q}$ is a diffeomorphism and so $\Sigma^{p+q} \in I(E(\Sigma^q))$ hence

$$E(E(\Sigma^q)) = G_{\phi\cdot\beta}\pi_p(SO(q)$$

REMARK 2. We note that if p = 2, 4, 5, 6 (mod 8) and p < q-1 then $\Pi_p SO(q) = 0$ and so the image of G is trivial and hence in this particular case, the inertial group of $E(\Sigma^q)$ is trivial and this coincides with the result of [4, Proposition 1].

REMARK 3. By [15], intertial group I(M) of a smooth manifold M is a diffeotopy invariant of M. So if $2p \geq q+1$ then we can deduce that the inertial group $I(E(\Sigma^q))$ of a p-sphere bundle over an homotopy q-sphere Σ^q is equal to the inertial group $I(E_\beta)$ of a p-sphere bundle over the standard q-sphere, where $\beta \in \Pi_{q-1}SO(p+1)$ classifies the associated disc bundle. Let $D(\Sigma^q)$ be the associated (p+1)-disc bundle over the homotopy q-sphere where $E(\Sigma^q)$ is the boundary of $D(\Sigma^q)$. Σ^q has the homotopy type of $D(\Sigma^q)$ and Σ^q has the homotopy type of S^q, it follows that S^q has the homotopy type of $D(\Sigma^q)$. Since $2p \geq q+1$ then it follows that $2(p+q+1) \geq 3q + 3$ and since $p + q > 5$ and $p \geq 3$ then $D(\Sigma^q)$ and $E(\Sigma^q)$ are simply connected and from [12: Theorem 4.4], it follows that $D(\Sigma^q)$ is diffeomorphic to a (p+1)-disc bundle $D(S^q)$ over the q-sphere S^q hence the boundary $\partial D(\Sigma^q) = E(\Sigma^q)$ of $D(\Sigma^q)$ is diffeomorphic to the boundary $\partial D(S^q) = E_\beta$ of $D(S^q)$. It then follows by [15] that $I(E(\Sigma^q)) = I(E_\beta)$. This means that the inertial group of S_β in [13] coincides with Lemma 3.2.

Combination of Lemmas 3.1 and 3.2 give the following.

THEOREM 3.3. Let E be the total space of a p-sphere bundle over a q-sphere with characteristic map $\beta \in \Pi_{q-1}SO(p+1)$ then the diffeomorphism classes of p+q-manifolds that are homeomorphic to E are in one-to-one correspondence with the group

$$\frac{\theta^q}{H(q,p)} \times \frac{\theta^n}{\text{Image } G_\beta}$$

where $p+q = n \geq 6$ and $p < q$.

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