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ON SOME FIXED POINT THEOREMS

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ABSTRACT. In this paper we prove a fixed point theorem for inward mappings using a well-known result of Ky Fan type in Hilbert space setting.

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The following well known theorem of Ky Fan has been of great importance in nonlinear analysis, minimax theory and approximation theory [1].

Let C be a nonempty compact, convex subset of a normed linear space X and let $f:C \to X$ be a continuous mapping. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C) ,$$

where $d(a, B) = \inf\{\|a - b\|/b \in B\}$.
If $fy \in C$, then f has a fixed point.

There have appeared several extensions of Ky Fan theorem. Lin [2] proved an interesting result for densifying mappings. Reich [3] relaxed compactness and proved the result for approximately compact, convex sets. Other results are due to Sehgal [4], Sehgal and Singh [5], Kapoor [6] and Singh and Watson [7].

In the present paper we prove a fixed point theorem for inward mappings using a result of Ky Fan type theorem for Hilbert space.

For definitions and notations we refer to Browder [8]. We will use his results for our theorem.

Let C be a closed, bounded, convex subset of H, a Hilbert space. A function $f: C \to H$ is called semicontractive if there exists a mapping T of $H \times H \to C$ such that

i) $f(x) = T(x, x)$ for $x \in C$, while

ii) for fixed $x \in H$, $T(\cdot, x)$ is nonexpansive,

iii) for fixed $x \in H$, $T(x, \cdot)$ is compact.

Recall that $f: H \to H$ is nonexpansive if $||fx - fy|| \leq ||x - y||$ for all $x, y \in H$.

The following is a special case of a well-known theorem of Browder [8]. (We state it in Hilbert space).

Let C be a closed, bounded, convex subset of a Hilbert space H and let $f: C \to C$ be a semicontractive mapping. Then f has a fixed point.

The following more general result holds.

THEOREM 1. Let C be a nonempty, closed, convex subset of a Hilbert space H and let $f:C \to H$ be semicontractive mapping such that $f(C)$ is bounded. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C) .$$

PROOF: Let $P: H \to C$ be the proximity map. Then P is a nonexpansive map, i.e.

$$\|Px - Py\| \leq \|x - y\| \text{ for all } x, y \in H . \quad \text{(see [9])} .$$

Also,

$$Pof: C \to C .$$

Let $B = \bar{C}_0\ (Pf(C))$, convex closure of $(Pf(C))$.

Then $Pf: B \to B$ is a semicontractive mapping and has a fixed point say $Pfy = y$.

Therefore
$$\|y - fy\| = \|Pfy - fy\|$$
$$= d(fy, C) .$$

COROLLARY 1.

Let C be a closed, bounded and convex subset of H and let $f:C \to H$ be a semicontractive. Then there exists a $y \in C$ such that

$$\|y - fy\| = d(fy, C) .$$

Let us now recall the "inwardness condition". Let K be a closed subset of a Banach space X. We say that $f: K \to X$ is an inward mapping if for every $x \in K$

$$f\,x \in I_k\,(x) = \{z:\ z = x + \alpha(y - x) \in K,\ \alpha \geq 0\}$$

This condition introduced by Halpern [10] and [11] is weaker than $x \in \delta K \Rightarrow f(x) \in K$ and is widely used in order to obtain fixed point results for mappings $f: K \to X$. See e.g. Assad and Kirk [12], Caristi [13], Caristi and Kirk [14], Downing and Kirk [15], S. Reich [3], Downing and Ray [16] and S. Massa [17], [18]. (δK stands for boundary of K).

S. Massa [18] pointed out that if K is a convex set C $(K=C)$ then the inwardness condition is equivalent to

$$x \in C \Rightarrow (x,\ fx] \cap C \neq \phi$$

where $(x,\ y] = [(1 - \alpha)\,x + \alpha y\ ,\ 0 < \alpha \leq 1]$.

THEOREM 2. Let C be a closed, convex subset of a Hilbert space H and $f: C \to H$ be a semicontractive inward mapping with bounded range. Then f has a fixed point.

PROOF. Let $y \in C$ be such that

$$\|y - fy\| = d(fy,\ C)\ . \qquad \text{(By Theorem 1)}.$$

Suppose $y \neq fy$. Then $fy \notin C$ and there exists a $z \in (y,\ fy) \cap C$ We have

$$\|y - fy\| = \|y - z\| + \|z - fy\|\ .$$

Then $d(fy,\ C) \geq \|y - z\| + d(fy,\ C)$ absurd, because $y \neq z$.

COROLLARY 2.

Let C be a closed, convex subset of H and let $f: C \to H$ be semicontractive with bounded range. If $f(\delta C) \subseteq C$, then f has a fixed point.

COROLLARY 3.

Let B_r be a closed ball of radius r and center 0 in a Hilbert space H. Let $f: B_r \to H$ be a semicontractve mapping satisfying the condition: if $fx = \alpha x$ for $x \in \delta\ B_r$ then $\alpha \leq 1$. Then f has a fixed point.

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References

1. FAN, Ky. Extensions of two fixed point theorems of F. E. Browder, Math. Z. 112(1969), 234-240.

2. LIN, T. C. A note on a theorem of Ky Fan, Canadian Math. Bulletin 22(1979), 415-515.

3. REICH, S. Approximate Selections, Best Approximations, Fixed Points and Invariant sets, J. Math. Anal. & Appl. 62(1978), 104-113.

4. SEHGAL, V. M. A Simple Proof of a Theorem of Ky Fan, Proc. Amer. Math. Soc. 63(1977), 368-369.

5. SEHGAL, V. M. and SINGH, S. P. A Variant of a Fixed Point Theorem of Ky Fan, Indian J. of Math. 25(1983), 171-174.

6. KAPOOR, O. P. Two applications of an Intersection Lemma, J. Math. Anal. & Appl. 41(1973), 226-233.

7. SINGH, S. P. and WATSON, B. Proximity maps and Fixed Points, J. Approx. Theory, 39(1983), 72-76.

8. BROWDER, F. E. Fixed Point Theorems for nonlinear semicontractive mappings in Banach spaces, Arch. Rat. Mech. & Anal. 21 (1966), 259-269.

9. CHENEY, E. W. and GOLDSTEIN, A. A. Proximity maps for convex sets, Proc. Amer. Math. Soc. 10, (1959), 448-450.

10. HALPERN, B. Fixed Point Theorems for outward maps, Doctoral thesis, U.C.L.A., (1965).

11. HALPERN, B. and BERGMAN, G. A Fixed point theorem for inward and outward maps. Trans. Amer. Math. Soc. 130(1968), 353-358.

12. ASSAD, N. A. and KIRK, W. A. Fixed Point Theorems for set-valued mappings of contractive type, Pacif. J. Math. 43(1972), 553-562.

13. CARISTI, J. Fixed point theorems for mappings satisfying inward conditions, Trans. Amer. Math. Soc. 215(1976), 241-251.

14. CARISTI, J. and KIRK, W. A. Geometric Fixed Point Theory and Inwardness conditions, The Geometry of Metric & Linear spaces, Proc., Ed. L. M. Kelly, Lecture Notes in Math. Springer-Verlag.

15. DOWNING, D. J. and KIRK, W. A. Fixed Pont Theorems for set-valued mappings in metric and Banach spaces, Math. Japonicae 22(1977), 99-112.

16. DOWNING, D. J. and RAY, W. O. Some Results in set-valued mappings, Nonlinear Analysis, 5(1981), 1367-1377.

17. MASSA, S. Nonexpansive mappings with noncompact values, Atti. Accad. Sci. Torino, Sci. Fis. Mat. Natur, 117(1983), 35-41.

18. MASSA, S. Some Remarks on Opial Spaces, Boll. Un. Mat. Ital., (6), 2-A(1983), 65-70.