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SOME APPLICATIONS OF SCHWARZ LEMMA FOR OPERATORS

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ABSTRACT. A generalized Schwarz lemma and some Harnack type inequalities for operators have been obtained in this paper.

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1. INTRODUCTION.

Let A be a bounded linear operator on a complex Hilbert space H. For a complex valued function f analytic on a domain E of the complex plane containing the spectrum $\sigma(A)$ of A, let f(A) denote the operator on H defined by the Riesz Dunford integral ([2, p.568]).

$$f(A) = \frac{1}{2\pi i} \int_C f(z)\ (zI-A)^{-1} dz,$$

where C is a postively oriented simple closed rectifiable contour containing $\sigma(A)$ in its inside domain Ω and satisfying $C \cup \Omega \subset E$. Fan[3] has obtained Schwarz lemma for f(A) and has given several applications of his results including the Harnack's inequalities for operators in [3,4].

In this paper, we obtain a generalized Schwarz lemma and some further Harnack type inequalities for operators.

2. SOME PRELIMINARY LEMMAS.

We need the following lemmas.

LEMMA 1. Let a,b,c,d be complex numbers such that $ad - bc \neq 0$, $c \neq 0$ and let T be a bounded linear operator on a Hilbert space H such that $-d/c$ is not in $\sigma(T)$. Then

$$\| (aT + bI)\ (cT + dI)^{-1} \| \leq r \tag{2.1}$$

for $0< r<| a| \quad | c|$ if and only if

$$\| T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2 |c|^2} I \| \leq \frac{r|ad - bc|}{|a|^2 - r^2 |c|^2} . \tag{2.2}$$

Equality holds in (2.1) and (2.2) simultaneously.

PROOF. The inequality (2.1) is true if and only if

$$r^2 I - (\bar{c}T^* + \bar{d}I)^{-1} (\bar{a}T^* + \bar{b}I) \ (aT + bI) \ (cT + dI)^{-1} \geq 0$$

or

$$(\bar{c}T^* + \bar{d}I)^{-1} [r^2(\bar{C}T^* + \bar{d}I) \ (cT + dI) - (\bar{a}T^* \ \bar{b}I)(aT + bI)] \ (cT+dI)^{-1} \geq 0$$

The operator inside the square brackets can be written as

$$\frac{r^2|ad-bc|}{|a|^2 - r^2|c|^2} I - \{T^*T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2|c|^2} T^* + \frac{a\bar{b} - r^2 c\bar{d}}{|a|^2 - r^2|c|^2} T + \frac{|\bar{a}b - r^2 \bar{c}d|^2}{(|a|^2 - r^2|c|^2)^2} I \}$$

or

$$\frac{r^2|ad - bc|}{|a|^2 - r^2|c|^2} I - [T^* + \frac{a\bar{b} - r^2 c\bar{d}}{|a|^2 - r^2|c|^2} I] \ [\ T + \frac{\bar{a}b - r^2 \bar{c}d}{|a|^2 - r^2|c|^2} I \ ] \ .$$

This last expression is a positive operator if and only if (2.2) holds. This completes the proof.

LEMMA 2. Let a,b,c,d and T be as in Lemma 1. Then,

$$\|(aT + bI)(cT + dI)^{-1} - \frac{b\bar{d} - r^2 a\bar{c}}{|d|^2 - r^2|c|^2} I\| \leq \frac{r|ad - bc|}{|d|^2 - r^2|c|^2} \tag{2.3}$$

for $0 < r < |d| / |c|$ if and only if $\|T\| \leq r$. Equality holds in (2.3) if and only if $\|T\| = r$.

PROOF. The inequality (2.3) is equivalent to

$$\|\{(aT+bI)(|d|^2 - r^2|c|^2) - (b\bar{d} - r^2 a\bar{c}) \ (cT + dI)\} \ (cT + dI)^{-1}\| \leq r|ad - bc| \ .$$

After simplication the above can be written as

$$\| \ (\bar{d}T + r^2 \bar{c}I) \ (cT + dI)^{-1}\| \leq r \ . \tag{2.4}$$

Now an application of Lemma 1 shows that (2.4) is equivalent to $\| T \| \leq r$.

3. A GENERALIZED SCHWARZ LEMMA.

Let D denote the open unit disc $\{z: | z | < 1\}$ in the complex plane and let H(D) be the class of complex valued functions analytic in D. Further, let $B(D) = \{f \in H(D): |f(z)| < 1, z \in D\}$ and let $B_o(D) = \{f \in B(D): f(0) = 0\}$.

THEOREM 1. Let f be in B(0) and let A be a proper contraction on a Hilbert space H. Then,

$$\frac{\| A \| - |f(0)|}{1-|f(0)|\ \| A\|} \leq \|f(A)\| \leq \frac{\| A \| + |f(0)|}{1+ |f(0)|\ \|A\|} . \tag{3.1}$$

PROOF. Since f is in B(D) and A is a proper contraction, by a result of F and ([3, Theorem 1, p.276]), T = f(A) is also a proper contraction. Now, if we define the complex valued function g by $g(z) = (f(z)- f(0))\ (1-\overline{f(0)}\ f(z))^{-1}$ then g is in $B_o(D)$ and $g(A) = (T-f(0)I)\ (I-\overline{f(0)}\ T)^{-1}$ is also a proper contraction. Further, by the operator version of Schwarz lemma ([3, Corollary 2, p.280]),

$$\| g(A)\| \leq \| A \|. \tag{3.2}$$

If we take a = d = 1, b = f(0), c = $-\overline{f(0)}$ and r = $\|A\|$ in Lemma 1 then (3.2) is equivalent to

$$\| f(A) - \frac{1-\|A\|^2\ f(0)}{1-|f(0)|^2\|A\|^2} I\| \leq \frac{(1-|f(0)|^2)\| A\|}{1 - |f(0)|^2\|A\|^2} .$$

Using triangle inequality we get both the inequalities in (3.1).

CORALLARY 1. Let f in $B_o(D)$ be given by the series $f(z) = bz^n+ ....$, $b\neq 0$, and let A be a proper contraction on a Hilbert space H. Then

$$\| A^n \| \left(\frac{\| A \| - |b|}{1 -|b|\ \|A\|}\right) \leq \| f(A) \| \leq \|A^n\| \left(\frac{\| A\| +|b|}{1+|b|\ \|A \|}\right) . \tag{3.3}$$

PROOF. The function g, defined by $g(z) = (f(z)/z^n)$, $z \neq 0$ and g(0)=b, is in B(D) and $f(A) = A^n g(A)$. Hence the result follows from Theorem 1.

REMARK. The author learned from Professor R. Finn that Theorem 1 follows independently from some results of K. Fan that are now in press.

4. SOME HARNACK TYPE INEQUALITIES.

Let $P(\alpha, \beta)$, $0 \leq \alpha<1$, $0< \beta \leq 1$, denote the subclass of functions p in H(D) satisfying p(0) = 1 and

$$\frac{| p(z) - 1|}{|(2\beta -1)p(z) + (1- 2\alpha\beta) |} < 1, \text{ z in D} .$$

This class of functions have been introduced and studied by Juneja and Mogra [5]. It has been shown in [5] that the n^{th} Taylor coefficient a_n of a function p in $P(\alpha,\beta)$ satisfies the sharp inequality $|a_n|\leq 2\beta\ (1-\alpha)$. Observe that

$P(0,1) =\{p \in H(D): p(0) =1, \text{Re } p(z) > 0, \text{ z in D }\}$,

$P(\alpha,1) =\{p \in H(D): p(0) =1, \text{Re } p(z) > \alpha, \text{ z in D }\}$,

$P(0,\beta) =\{p \in H(D): p(0) =1, |p(z)- \frac{1}{2(1-\beta)}| < \frac{1}{2(1-\beta)} \}$, and

$P(\alpha,\beta)\subset P(0,1)$, for all admissible choices of α and β . We prove the following theorem which extends a distortion theorem by Kapoor and the author ([6, Theorem 1, p.86]).

THEOREM 2. Let p in $P(\alpha,\beta)$ be given by the series $p(z) = 1+2b(1-\alpha)\beta z^a+\ldots$, $0 < |b| \leq 1$, z in D and let A be a proper contraction on a Hilbert space H. Then,

$$\| p(A) \| \leq \frac{1 + \|A\| \, |b| + (1-2\alpha\beta)(\|A\| + |b|)\| A^n \|}{1 + \| A \| |b| + (1-2\beta)(\| A \| + |b|)\| A^n \|}, \tag{4.1}$$

$$\| P(A) \| \geq \frac{1+ \| A \| |b| - (1-2\alpha\beta)(\| A \| + |b|)\| A^n \|}{1+\| A \| |b| -(1-2\beta)(\| A \|+|b|)\|A^n\|}, \tag{4.2}$$

$$\frac{1+\| A\| |b| - (1-2\alpha\beta)(\| A\| + |b|)\| A^n\|}{1+\| A\| |b| -(1-2\beta)(\| A \|+|b|)\|A^n\|} I \leq \mathrm{Re}\, p(A), \tag{4.3}$$

$$\mathrm{Re}\, p(A) \leq \frac{1+\| A\| |b| + (1-2\alpha\beta)(\| A\| +|b|)\| A^n \|}{1+\| A\| |b|+ (1-2\beta)(\| A\| +|b|)\|A^n\|} I , \tag{4.4}$$

$$\pm \mathrm{Im}\, p(A) \leq \frac{2\beta(1-\alpha)(\| A\| +|b|)(1+\| A\| |b|)\| A^n\|}{(1+\| A\| |b|)^2 - (1-2\beta)^2(\|A\| +|b|)\| A^n\|^2} I . \tag{4.5}$$

PROOF. From the definition of $P(\alpha,\beta)$, it follows that there exists a function w in $B_o(D)$ such that

$$p(z) = \{1+ (1-2\alpha\beta) w(z)\} \{1+ (1-2\beta) w(z) \}^{-1}, \text{ z in D}$$

and

$$p(A) = \{1+ (1-2\alpha\beta) T\} \{1+ (1-2\beta) T\}^{-1}, \text{ where } T= w(A).$$

Further, it is observed that $w(z)= bz^n+ \ldots .$, where z in D. Hence by Corollary 1., we can say

$$\| T \| = \| w(A) \| \leq \| A^n \| \left(\frac{\| A\| +| b |}{1+\| A\| |b|} \right) = r . \tag{4.6}$$

Now, choosing $a = 1-2\alpha\beta$, $c= 1-2\beta$, $b= d-1$ in Lemma 2, (4.6) is equivalent to

$$\left\|p(A) - \frac{1-r^2 (1-2\alpha\beta)(1-2\beta)}{1-r^2(1-2\beta)^2}\right\| \leq \frac{2r(1-\alpha)\beta}{1-r^2(1-2\beta)^2} .$$

Hence

$$\frac{1-(1-2\alpha\beta) r}{1-(1-2\beta)r} \leq \| p(A) \| \leq \frac{2+ (1-2\alpha\beta)r}{2+(1-2\beta)r} .$$

Substituting the value of r in the above inequality, we get (4.1) and 4.2). Also,

$$\pm \mathrm{Re}\ [p(A) - \frac{1-r^2(1-2\alpha\beta)\ (1-2\beta\ )}{1-(1-2\ \beta)^2\ \ r^2} I] \leqq \| p(A) - \frac{1-r^2(1-2\ \alpha\beta)(1-2\beta)}{1-r^2(1-2\ \beta)^2} I \|\ I$$

$$\leqq \frac{2r(1-\alpha\ )^{\beta}}{1-r^2(1-2\beta)^2} I \quad .$$

This gives (4.3) and (4.4). Similarly,

$$\pm\ \mathrm{Im}\ p(A) = \pm\ \mathrm{Im}\ [p(A) - \frac{1-r^2\ (1-2\alpha\beta\ )\ (\ 1-2\beta\ )}{1-r^2(1-2\ \beta)^2} I\ ]$$

$$\leqq \| p(A) - \frac{1-r^2(1-2\ \alpha\beta)(1-2\beta\ )}{1-r^2(1-2\beta\ )^2} I \|\ I \ \leqq \ \frac{2\beta\ (1-\alpha)\ r}{1-r^2(1-2\beta)^2} I\ .$$

From this (4.5) follows. This completes the proof.

REMARK. The right hand side of (4.1) and (4.2) are increasing and decreasing function of $|\ b\ |$, respectively. For the case $|\ b| = 1$, $\alpha = 0$ and $\beta = 1$, our Theorem 2 includes some results of Fan ([3, Corollary 3,P281], [4, Proposition 2,P.335]).

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