Internat. J. Math. & Math. Sci.
VOL. 12 NO. 2 (1989) 247-256

RESEARCH PAPERS

COINCIDENCE AND FIXED POINTS OF NONLINEAR HYBRID CONTRACTIONS

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(Received June 8, 1987 and in revised form November 2, 1987)

ABSTRACT. In this paper, we show the existence of solutions of functional equations $fx \in Sx \cap Tx$ and $x = fx \in Sx \cap Tx$ under certain contraction and asymptotic regularity conditions, where f, S and T are single-valued and multi-valued mappings on a metric space, respectively. We also observe that Mukherjee's fixed point theorem for a single-valued mapping commuting with a multi-valued mapping admits of a counter-example and suggest some modifications. While doing so, we also answer an open question raised in [1] and [2]. Moreover, our results extend and unify a multitude of fixed point theorems for multi-valued mappings.

KEY WORDS AND PHRASES. Asymptotic regularity, coincidence and common fixed points, commuting mappings, functional equations, Hausdorff metric, orbital completeness, contraction, hybrid contraction.
1980 AMS SUBJECT CLASSIFICATION CODE. 54H25.

1. INTRODUCTION.

The study of fixed points of multi-valued mappings using the Hausdorff metric was initiated by Markin [3] and Nadler [4]. Subsequently, a number of generalizations of the multi-valued contraction principle (which states that a multi-valued contraction mapping on a complete metric space having values in the set of all closed and bounded subsets of the metric space possesses a fixed point, [4]) were obtained, among others, by Ciric [5], Khan [6], Kubiak [7], Reich [8], Smithson [9] and Wegrzyk [10]. However, hybrid contractions, viz., contractive conditions involving multi-valued and single-

single-valued mappings have recently been studied by Mukherjee [11], Naimpally et al. [12], Rhoades et al. [1] and Singh et al [2]. In this paper, we consider a very general type of condition involving two multi-valued mappings and a single-valued mapping and establish coincidence and fixed point theorems (cf. Theorems 2.1-2.3) which improve, extend and unify some coincidence theorems and a multitude of known fixed point theorems. At the end, we have compared a few contractive conditions.

Let, (X,d) be a metric space. We shall use the following notation and definitions:

$$CL(X) = \{A: A \text{ is a nonempty closed subset of } X\},$$

$$CB(X) = \{A: A \text{ is a nonempty closed and bounded subset of } X\}$$

and

$$C(X) = \{A: A \text{ is a nonempty compact subset of } X\}.$$

For $A, B \in CL(X)$ and $\varepsilon > 0$

$$N(\varepsilon, A) = \{x \in X: d(x, \alpha) < \varepsilon \text{ for some } \alpha \in A\},$$

$$E_{A,B} = \{\varepsilon > 0: A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}$$

and

$$H(A,B) = \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \Phi, \\ +\infty, & \text{if } E_{A,B} = \Phi. \end{cases}$$

H is called the generalized Hausdorff distance function for $CL(X)$ induced by d, and H defined on $CB(X)$ is said to be the Hausdorff metric induced by d. $D(x,A)$ will denote the ordinary distance between $x \in X$ and a nonempty subset A of X. Let f be a single-valued mapping from X to itself and S, T multi-valued mappings from X to the nonempty subsets of X.

DEFINITION 1.1. If, for $x_o \in X$, there exists a sequence $\{x_n\}$ in X such that $fx_n \in Sx_{n-1}$ if n is odd and $fx_n \in Tx_{n-1}$ if n is even, then $0_f(x_o) = \{fx_n: n=1,2,\ldots\}$ is said to be the orbit for $(S,T;f)$ at x_o. Further, $0_f(x_o)$ is called a regular orbit for $(S,T;f)$ if

$$d(fx_n, fx_{n+1}) \leq \begin{cases} H(Sx_{n-1}, Tx_n), & \text{if } n \text{ is odd}, \\ H(Tx_{n-1}, Sx_n), & \text{if } n \text{ is even}. \end{cases}$$

DEFINITION 1.2. If, for $x_o \in X$, there exists a sequence $\{x_n\}$ in X such that every Cauchy sequence of the form $0_f(x_o)$ converges in X, then X is called $(S,T;f)$-orbitally complete with respect to x_o or simply $(S,T;f,x_o)$-orbitally complete.

If f is the identity mapping on X, then $0_f(x_o)$ is denoted by $0(x_o)$ and $(S,T;f,x_o)$-orbital completeness by $(S,T;x_o)$-orbital completeness.

DEFINITION 1.3. A pair (S,T) is said to be asymptotically regular at $x_o \in X$ if for any sequence $\{x_n\}$ in X and each sequence $\{y_n\}$ in X such that $y_n \in Sx_{n-1} \cup Tx_{n-1}$, $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

We remark the Definitions 1.1 - 1.3 with S = T reduce respectively to Definitions 4, 6 and 7 of Rhoades et al. [1]. A definition of a regular orbit for a multi-valued mapping seems to appear first in [9]. We further remark that orbital completeness need not imply the completeness of the space. Evidently every complete space is orbitally complete.

DEFINITION 1.4. f and S are said to commute at a point $x \in X$ if $fSx \subseteq Sfx$ (f and S are said to commute on X [13] if $fSx \subseteq Sfx$ for every point $x \in X$).

In [14], Sessa introduced the concept of weak commutativity for single-valued mappings on a metric space. Now, we extend this concept to the setting of a single-valued mapping and a multi-valued mapping on a metric space as follows:

DEFINITION 1.5. f and S are said to be commute weakly at $z \in X$ if $H(fSz, Sfz) \leq D(fz, Sz)$. f and S are said to commute weakly on X if they commute weakly at every point in X.

Note that commutativity implies weak commutativity, but the converse need not be true even in the case of single-valued mappings as shown in [14].

EXAMPLE 1.6. Let X = {1, 2, 3, 4}. Define a metric d on X and mappings f, S as follows:

$$d(1,2) = d(3,4) = 2, \; d(1,3) = d(2,4) = 1,$$
$$d(1,4) = d(2,3) = 3/2;$$
$$S1 = S3 = \{4\}, \; S2 = S4 = \{3\};$$
$$f1 = f2 = f3 = 2, \; f4 = 1, \text{ respectively.}$$

We have $Sf1 = \{3\}$ and $fS1 = \{1\}$ and so f and S do not commute at x = 1. But f and S commute weakly at x = 1 since $H(Sf1, fS1) = D(f1, S1) = 1$.

Let F be the family of mappings ϕ from the set R^+ of nonnegative real numbers to itself such that each ϕ is upper-semicontinuous and nondecreasing.

The following theorem appears in [11]:

THEOREM 1.7. Let (X,d) be a complete metric space, f a continuous mapping from X into itself and T a multi-valued mapping from X into CL(X) such that f and T commute. Also suppose, given $x_o \in X$, there is a point $x_1 \in X$ such that $fx_1 \in Tx_o$. Then, if for all $x, y \in X$ and for some $\alpha \in (0,1)$,

$$H(Tx, Ty) \leq \alpha d(fx, fy), \tag{1.1}$$

there is a point $z \in X$ such that $z = fz \in Tz$; that is, z is a common fixed point of f and T. The following example shows that this theorem is false.

EXAMPLE 1.8. [12]. Let $X = [0,\infty)$, $Tx = [1+x,\infty)$ and $fx = 2x$ for $x \in X$. Clearly (1.1) and the other hypotheses hold with $\alpha \in [1/2,1)$. However, Theorem 1.7 is true in as much as f and T have a coincidence point; that is, $fz \in Tz$ for some $z \in X$. Note that $T:X \to CB(X)$ satisfying (1.1) with f = an identity mapping on X is a multi-valued contraction. Theorem 1.7 is true when f is the identity mapping on X.

The following thereom is an interesting result for the existence of coincidence points of hybrid contractions, that is, contractive conditions involving single-valued and multi-valued mappings.

THEOREM 1.9. [1]. Let t be a multi-valued mapping from a metric space X into $CL(X)$. If there exists a mapping f from X into itself such that $T(X) \subseteq f(X)$, for each $x, y \in X$ and $\phi \in F$,

$$H(Tx,Ty) \leq \phi\,(\max(D(fx,Tx),D(fy,Ty),D(fx,Ty),D(fy,Tx),d(fx,fy))), \tag{1.2}$$

$$\phi(t) < qt \text{ for each } t > 0 \text{ and for some } 0 < q < 1, \tag{1.3}$$

there exists a point $x_o \in X$ such that T is asymptotically regular at x_o and $f(X)$ is $(T;f,x_o)$-orbitally complete, (1.4)

then f and T have a coincidence point.

If f is not the identity mapping, then commuting mappings f and T satisfying the hypotheses of Theorems 1.7 and 1.9 need not have a common fixed point. The following question is raised in [1] and [2]: What additional conditions will guarantee the existence of a common fixed point for f and T?

We remark that (1.1) implies (1.2) and Theorem 1.9 gives a solution of the coincidence point equation $fx \in Tx$ for $x \in X$.

In this paper, we investigate different sets of conditions under which the fixed point equation $x = fx \in Sx \in Tx$ for $x \in X$ possesses a solution.

2. THE MAIN THEOREMS.

Now, we are ready to give our main theorems:

THEOREM 2.1. Let S and T be multi-valued mappings from a metric space X into $CL(X)$. If there exists a mapping f from X into itself such that $S(X) \cup T(X) \subseteq f(X)$, for each $x, y \in X$ and $\phi \in F$,

$$H(Sx,Ty) \leq \phi\,(\max(D(fx,Sx),D(fy,Ty),D(fx,Ty),D(fy,Sx),d(fx,fy))), \tag{2.1}$$

$$\phi(t) \leq qt \text{ for each } t > 0 \text{ and for some fixed } q \in (0,1), \tag{2.2}$$

there exists a point $x_o \in X$ such that the pair (S,T) is asymptotically regular at x_o, (2.3)

and $f(X)$ is $(S,T;f,x_o)$-orbitally complete, (2.4)

then f,S and T have a coincidence point. Further, if z is a coincidence point of f,S,T and fz is a fixed point of f, then (a) fz is also a fixed point of S (resp. T) provided f commutes weakly with S (resp. T) at z, and (b) fz is a common fixed point of S and T provided f commutes weakly with each of S and T at z.

PROOF. Let x_o be a point in X satisfying (2.3). Since $S(X) \subseteq f(X)$, we can find a point $x_1 \in X$ such that $fx_1 \in Sx_o$. Noting that $T(X)$ is also a subset of $f(X)$, we, for a suitable point $x_2 \in X$, can choose a point $fx_2 \in Tx_1$ such that

$$d(fx_1, fx_2) \leq q^{-1/2} H(Sx_o, Tx_1).$$

We remark that such a choice is possible by the definition of H since $q^{-1/2} > 1$. In general, we can choose a sequence $\{x_n\}$ in X such that $fx_{2n+1} \in Sx_{2n}$, $fx_{2n+2} \in Tx_{2n+1}$, $fx_{2n+3} \in Sx_{2n+2}$ and

$$d(fx_{2n+1}, fx_{2n+2}) \leq q^{-1/2} H(Sx_{2n}, Tx_{2n+1}),$$

$$d(fx_{2n+2}, fx_{2n+3}) \leq q^{-1/2} H(Tx_{2n+1}, Sx_{2n+2}).$$

By (2.3), $\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0$

Now, we assert that $\{fx_n\}$ is a Cauchy sequence in f(X). Suppose not. Then one of the subsequences $\{fx_{2n}\}$ or $\{fx_{2n-1}\}$ is not a Cauchy sequence. Without loss of generality, we may assume that $\{fx_{2n}\}$ is not a Cauchy sequence. Then there exists a positive number ε such that, for each positive integer 2k, there exist integers 2n(k) and 2m(k) such that

$$2k \leq 2n(k) < 2m(k), \tag{2.5}$$

$$d(fx_{2n(k)}, fx_{2m(k)}) \geq \varepsilon. \tag{2.6}$$

Let $d_{i,j} = d(fx_i, fx_j)$ and $d_i = d(fx_i, fx_{i+1})$. Then for each integer 2k,

$$\varepsilon \leq d_{2n(k),2m(k)} \leq d_{2n(k),2m(k)-2} + d_{2m(k)-2} + d_{2m(k)-1}. \tag{2.7}$$

For each integer 2k, let 2m(k) denote the smallest integer satisfying (2.5) and (2.6). So $d_{2n(k),2m(k)-2} < \varepsilon$, and from (2.7),

$$\lim_{k \to \infty} d_{2n(k),2m(k)} = \varepsilon. \tag{2.8}$$

Using the triangle inequality,

$$\left| d_{2n(k),2m(k)-1} - d_{2n(k),2m(k)} \right| \leq d_{2m(k)-1}$$

and

$$\left| d_{2n(k)+1,2m(k)-1} - d_{2n(k),2m(k)} \right| \leq d_{2n(k)} + d_{2m(k)-1}.$$

These relations, in view of (2.3) and (2.8), yield

$$\lim_{k\to\infty} d_{2n(k),2m(k)-1} = \lim_{k\to\infty} d_{2n(k)+1,2m(k)-1} = \varepsilon.$$

So by (2.1),

$$\begin{aligned} d_{2n(k),2m(k)} &\le d_{2n(k)} + d_{2n(k)+1,2m(k)} \\ &\le d_{2n}(k) + q^{-1/2} H(Sx_{2n(k)},Tx_{2m(k)-1}) \\ &\le d_{2n(k)} + q^{-1/2} \phi(\max(D(fx_{2n(k)},Sx_{2n(k)}), \\ &\quad D(fx_{2m(k)-1},Tx_{2m(k)-1}),D(fx_{2n(k)},Tx_{2m(k)-1}), \\ &\quad D(fx_{2m(k)-1},Sx_{2n(k)}), d_{2n(k),2m(k)-1})) \\ &\le d_{2n(k)} + q^{-1/2} \phi(\max(d_{2n(k)},d_{2m(k)-1},d_{2n(k),2m(k)}, \\ &\quad d_{2m(k)-1,2n(k)+1},d_{2n(k),2m(k)-1})). \end{aligned}$$

Using the upper-semicontinuity of ϕ and letting $k \to \infty$, this yields

$$\varepsilon \le q^{-1/2}\phi(\varepsilon) \le q^{-1/2} q\varepsilon < \varepsilon,$$

since $\varepsilon > 0$ and $q^{-1/2} < 1$. This contradicts the choice of ε, and so the subsequence $\{fx_{2n}\}$ is a Cauchy sequence. Consequently, $\{fx_n\}$ is a Cauchy sequence and, by (2.4), $\{fx_n\}$ has a limit in $f(X)$. Call it u. Hence there is at least one point z in X such that $u = fz$. By (2.1),

$$\begin{aligned} D(fz,Sz) &\le d(fz,fx_{2n+2}) + D(fx_{2n+2},Sz) \\ &\le d(fz,Fx_{2n+2}) + H(Sz,Tx_{2n+1}) \\ &\le d(fz,fx_{2n+2}) + \phi(\max(D(fz,Sz),D(fx_{2n+1},Tx_{2n+1}), \\ &\quad D(fz,Tx_{2n+1}),D(fx_{2n+1},Sz),d(fz,fx_{2n+1}))) \\ &\le d(fz,fx_{2n+2}) + \phi(\max(D(fz,Sz),d(fx_{2n+1},fx_{2n+2}), \\ &\quad d(fz,fx_{2n+2}),d(fx_{2n+1},fz) + D(fz,Sz), \\ &\quad d(fz,fx_{2n+1}))). \end{aligned}$$

Letting $n \to \infty$, this inequality yields

$$D(fz,Sz) \leq \phi(\max(D(fz,Sz),0,0,D(fz,Sz),0)).$$

If $fz \notin Sz$, then $D(fz,Sz) > 0$ and the above inequality implies

$$D(fz,Sz) \leq \phi(D(fz,Sz)) < D(fz,Sz),$$

which is a contradiction. Hence $fz \in Sz$, since Sz is a closed subset of X. Similarly, $fz \in Tz$. Thus z is a coincidence point of f, S and T. If we assume that $u = fz$ is a fixed point of f, then $u = fu = ffz \in fSz$. If f and S commute weakly at $z \in X$, then $fSz = Sfz$ since $fz \in Sz$. Therefore, we have $u \in Su$. Similarly, if f commutes weakly with T at z, then $u \in Tu$. This completes the proof.

Since (1.3) implies (2.2), Theorem 2.1 with $S = T$ improves slightly Theorem 1.9. Replacing the condition $S(X) \cup T(X) \subseteq f(X)$ of Theorem 2.1 by the orbital regularity, clearly we have the following:

THEOREM 2.2. Let S and T be multi-valued mappings from a metric space X into $CL(X)$. If there exists a mapping f and X into itself such that (2.1) and

$\phi(t) < t$ for each $t > 0$ and some $\phi \in F$, (2.9)

for a point $x_o \in X$, there exists a sequence $\{x_n\}$ in X such that the orbit $O_f(x_o)$ is regular, the pair (S,T) is asymptotically regular at x_o and $f(x)$ is $(S,T;f,x_o)$-orbitally complete, (2.4)

then f, S and T have a coincidence point. Further, if the limit of $O_f(x_o)$ is a fixed point of f, then the conclusions (a) and (b) in Theorem 2.1 are also true.

We remark that Theorem 2.2 with $S = T$ is Theorem 2 in [1]. It is well-known that if P is a multi-valued mapping from X into $C(X)$, then for every $y_1,y_2 \in X$ and $z_1 \in Py_1$, there exists a point $z_2 \in Py_2$ such that

$$d(z_1,z_2) \leq H(Py_1,Py_2).$$

This suggests that if S and T are multi-valued mappings from X into $C(X)$, then the orbital regularity condition in Theorem 2.2 can be dropped. Indeed, we have the following:

THEOREM 2.3. Let S and T be multi-valued mappings from a metric space X into $C(X)$. If there exists a mapping f from X into itself such that $S(X) \cup T(X) \subseteq f(X)$, (2.1), (2.9), (2.3) and (2.4), then f, S and T have a coincidence point. Further, if the limit of $O_f(x_o)$ is a fixed point of f, then the conclusions (a) and (b) in Theorem 2.1 are also true.

If, in (2.1), each of the terms $D(fx,Ty)$ and $D(fy,Sx)$ is replaced by $\frac{1}{2}(D(fx,Ty) + D(fy,Sx))$, then the condition of asymptotic regularity of the pair (S,T) can be dropped from Theorems 2.1-2.3. We emphasize that, without the assumption "fz is a fixed point of f" in Theorems 2.1-2.3, f, S and T need not have a common fixed point, even if the mappings are continuous, commuting on X and have fixed points. We are indebted to R.E. Smithson for the following example, which the first author received in a personal communication, though in a different context.

EXAMPLE 2.4. Let $X = [0,1]$ and $Sx = Tx = \{0,1\}$, $fx = 1 - x$ for all $x \in X$. Since $S(x) = \{0,1\} \subseteq f(X) = X$, $H(Sx,Sy) = 0$ for all $x, y \in X$, $f(Sx) = \{0,1\} = S(fx)$ and $f0 = 1 \in S1$, $f1 = 0 \in S0$, all the hypotheses of Theorems 2.1-2.3 are satisfied except that none of the coincidence values, viz., $f0$ or $f1$, is a fixed point of f. Evidently, f and S are continuous, and the only fixed point of f is $1/2$ which is not a fixed point of S.

In Theorem 2.1 taking f the identity mapping on X and defining $\phi(t) = qt$, $0 < q < 1$, we have the following:

COROLLARY 2.5. Let S and T be multi-valued mappings from a metric space X into $CL(X)$. If there exists a number $q \in (0,1)$ such that, for each $x,y \in X$,

$$H(Sx,Ty) \leq q \max(d(x,y),D(x,Sx),D(y,Ty),D(x,Ty),D(y,Sx)), \tag{2.10}$$

there exists a point $x_0 \in X$ such that the pair (S,T) is asymptotically regular at x_0, and (2.11)

X is $(S,T;x_0)$-orbitally complete, (2.12)

then S and T have a common fixed point.

Now, consider the following conditions:

$$H(Sx,Ty) \leq q \max(d(x,y),D(x,Sx),D(y,Ty),D(y,Sx),\tfrac{1}{2}D(x,Ty))$$
$$= q\, A(x,y), \text{ say, and} \tag{2.13}$$

$$H(Sx,Ty) \leq q \max(d(x,y),D(x,Sx),D(y,Ty),\tfrac{1}{2}(D(y,Sx) + D(x,Ty))). \tag{2.14}$$

Note that (2.14) implies (2.10), and (2.13) also implies (2.10). However, in Corollary 2.5, if we replace (2.10) by (2.14), then (2.11) is not needed. In fact, we have the following:

COROLLARY 2.6. Let S and T be multi-valued mappings from a metric space X into $CL(X)$. If there exists a number $q \in (0,1)$ such that for each $x, y \in X$, (2.14) and there exists a point $x_0 \in X$ such that (2.12), then S and T have a common fixed point.

Corollary 2.6 includes a multitude of fixed point theorems for multi-valued mappings such as Nadler's multi-valued contraction principle [4], Reich's fixed point Theorem [8], Ciric's "generalized multi-valued contraction" Theorem 2 [5] and an important result of Kubiak [7, Corollary 1.2]. The following example shows that corollary 2.6, if (2.14) is replaced by (2.13), will be false in general without some additional condition such as (2.11) even if the space X is complete.

EXAMPLE 2.7. Let $X = \{1,2,3,4\}$ and d be the metric on X given in Example 1.6. Define mappings S, T as follows: $S1 = S3 = \{4\}$, $S2 = S4 = \{3\}$; $T1 = T4 = \{2\}$, $T2 = T3 = \{1\}$, respectively. Note that $S(X) = \{3,4\}$, $T(X) = \{1,2\}$, and $H(Sx,Ty) = d(Sx,Ty) \leq 3/2$. Then, since $A(x,y) = 2$, $H(Sx,Ty) \leq q\, A(x,y)$, $q \in [3/4, 1]$, and the condition (2.13) is satisfied but S and T have no coincidence even.

We remark that the conditions (2.13) and (2.14) are independent. Indeed, Kubiak [7] rightly shows in his Example 2 (wherein $d(2,3) = 5/4$ is misprinted as $d(2,3) = 4/5$) that (2.14) need not imply (2.13), but wrongly remarks [7, Remark 3] that (2.13) implies (2.14), for if (2.13) implies (2.14) then mappings of Example 2.7 will satisfy (2.14) and Corollary 2.6 will guarantee a common fixed point of S and T which however will contradict the conclusion of Example 2.7. Moreover, the following example shows that the condition (2.13) need not imply (2.14).

EXAMPLE 2.8. Let $X = \{a,b,c\}$. Define a metric d on X and mappings S, T as follows: $d(b,c) = 2$, $d(a,c) = 3$, $d(a,b) = 4$, $Sa = Sb = Sc = \{a\}$ and $Ta = Tc = \{a\}$, $Tb = \{c\}$. So $H(Sx,Ty) < q\, A(x,Y)$, $q \in [3/4, 1]$, i.e., (2.13) is satisfied but (2.14) is satisfied only for $q > 1$.

The following is the conclusion of the above comparisons.

THEOREM 2.9. (i) (2.13) implies (2.10), but not conversely; (ii) (2.14) implies (2.10), but not conversely; (iii) (2.13) and (2.14) are independent.

ACKNOWLEDGMENT. We thank Professor Sheila Oates Williams and the referee for suggesting improvements in the original manuscript. This research of the second and third authors was supported by the Basic Science Research Institute Program, Ministry of Education, 1987.

REFERENCES

1. RHOADES, B.E., SINGH, S.L. and KULSHRESTHA, Chitra, Coincidence theorems for some multi-valued mappings, Internat. J. Math. Math. Sci. 7 (1984), 429-434.
2. SINGH, S.L. and KULSHRESTHA, Chitra, Coincidence theorems in metric spaces Indian J . Phy. Natur. Sci. 2B (1982), 19-22.
3. MARKIN, J., A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc.74 (1968), 639-640.
4. NADLER, S.B., Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-486.
5. CIRIC, Ljubomir B., Fixed points for generalized multi-valued contractions, Mat. Vesnik. 9(24) (1972), 265-272.
6. KHAN, M.S., Common fixed point theorems for multi-valued mappings, Pacific J. Math. 95 (1981), 337-347.
7. KUBIAK, T., Fixed point theorem for contractive type multi-valued mappings, Math. Japon. 30 (1985), 89-101.
8. REICH, S., Kannan's fixed point theorem, Bull. Un. Mat. Ital. 4 (1971), 1-11.
9. SMITHSON, R.E., Fixed points for contractive multifunctions, Proc. Amer. Math. Soc. 27 (1971), 192-194.
10. WEGRZYK, R., Fixed Point Theorems For Multi-valued Functions and Their Applications To Functional Equations, Dissert. Math., Rozprawy Math., 1982.

11. MUKHERJEE, R.N., On fixed points of single-valued and set-valued mappings, J. Indian Acad. Math. 4 (1982), 101-103.

12. NAIMPALLY, S.A., SINGH, S.L. and WHITFIELD, J.H.M. Coincidence theorems for hybrid contractions, Math. Nachr. 127 (1986), 177-180.

13. ITOH, S. and TAKAHASHI, W. Single-valued mappings, multi-valued mappings and fixed point theorems, J. Math. Anal. Appl. 59 (1977), 514-521.

14. SESSA, S. On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. 32 (46) (1982), 149-153.