Internat. J. Math. & Math. Sci.
VOL. 12 NO. 3 (1989) 493-502

A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

M.K. AOUF

Faculty of Science
Department of Mathematics
University of Mansoura
Mansoura, Egypt

(Received January 6, 1987)

ABSTRACT. Let T_p be the class of analytic and p-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}, \ |z| < 1.$$

The subclasses $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ of T_p have been considered. Sharp results concerning coefficient estimates, distortion and covering theorems are obtained. The radius of convexity for the class $T_p^*(A,B,\alpha)$ is determined. It is further proved that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under arithmetic mean and convex linear combinations.

KEY WORDS AND PHRASES. P-valent, Analytic, Radius of Convexity.
1980 AMS SUBJECT CLASSIFICATION CODES. 30A32, 30A36.

1. INTRODUCTION.

Let $S_p (p \geq 1)$ denote the class of functions of the form

$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ which are analytic and p-valent in the unit disc $U = \{z: |z| < 1\}$. A function f is said to be subordinate to a function $F(f \prec F)$ if there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq |z|$, $z \in U$, such that $f = f.\phi$.

For A,B fixed, $-1 \leq A < B \leq 1$, and $0 \leq \alpha < p$, we say that $f \in S_p^*(A,B,\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{p + [pB+(A-B)(p-\alpha)]z}{1 + Bz}, \ z \in U,$$

or equivalently $f \in S_p^*(A,B,\alpha)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{B\frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]} \right| < 1, \ z \in U.$$

Further f is said to belong to the class $K_p(A,B,\alpha)$ if and only if $\frac{zf'(z)}{p} \in S_p^*(A,B,\alpha)$.

Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| \, z^{p+n}, \text{ and we set}$$

$$T_p^*(A,B,\alpha) = S_p^*(A,B,\alpha) \quad T_p \text{ and } C_p(A,B,\alpha) = Kp(A,B,\alpha) \quad T_p.$$

Silverman [3], Gupta and Jain [2] and Silverman and Silvia [4,5] have studied Certain subclasses of univalent functions with negative coefficients. Also Goel and Sohi [1] have studied certain subclasses of multivalent functions with negative coefficients. In this paper we obtain coefficient estimates, distortion and covering theorems for the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$. We also determine the radius of convexity for the class $T_p^*(A,B,\alpha)$. It is further shown that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under arithmetic mean and convex linear combinations. By taking $\alpha=0$, we get results due to Goel and Sohi [1] and by assigning specific values to A and B and taking $p=1$, we get results due to Silverman [3] and Gupta and Jain [2].

2. COEFFICIENT INEQUALITIES.

THEOREM 1. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}| z^{p+n}$ is in $T_p^*(A,B,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)] |a_{p+n}| \leq (B-A)(p-\alpha). \tag{2.1}$$

The result is sharp.

PROOF. Let $|z| = 1$, then

$$|zf'(z) - pf(z)| - |B\, zf'(z) - [pB+(A-B)(p-\alpha)]\, f(z)|$$

$$= \left| \sum_{n=1}^{\infty} -n\, |a_{p+n}| \, z^{p+n} \right| - \Big| (B-A)(p-\alpha)z^p$$

$$- \sum_{n=1}^{\infty} [nB + (B-A)(p-\alpha)] \, |a_{p+n}| \, z^{p+n} \Big|$$

$$\leq \sum_{n=1}^{\infty} [(1+B)n + (B-A)(p-\alpha)] |a_{p+n}| - (B-A)(p-\alpha) \leq 0.$$

Hence by the principle of maximum modulus $f(z) \in T_p^*(A,B,\alpha)$.

Conversely, suppose that

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{B.\frac{zf'(z)}{f(z)} - [pB + (A-B)(p-\alpha)]}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty} n|a_{p+n}| z^{p+n}}{(B-A)(p-\alpha)z^p - \sum_{n=1}^{\infty} [nB+(B-A)(p-\alpha)]|a_{p+n}|z^{p+n}}\right| < 1, \qquad z \in U.$$

Since $|\text{Re } z| \leq |z|$ for all z, we have

$$\text{Re}\left\{\frac{\sum_{n=1}^{\infty} n |a_{p+n}| z^{p+n}}{(B-A)(p-\alpha)z^p - \sum_{n=1}^{\infty} [nB+(B-A)(p-\alpha)]|a_{p+n}| z^{p+n}}\right\} < 1. \tag{2.2}$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=1}^{\infty} n|a_{p+n}| \leq \{(B-A)(p-\alpha) - \sum_{n=1}^{\infty} [nB + (B-A)(p-\alpha)] |a_{p+n}|\}$$

which implies that

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)(p-\alpha)] |a_{p+n}| \leq (B-A)(p-\alpha).$$

The function

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)n + (B-A)(p-\alpha)} z^{p+n}$$

is an extremal function.

COROLLARY 1. If $f \in T_p^*(A,B,\alpha)$ then $|a_{p+n}| \leq \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)}$, with

equality only for functions of the form $f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)n + (B-A)(p-\alpha)} z^{p+n}$.

COROLLARY 2. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{p+n}|z^{p+n}$ is in $C_p(A,B,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} (\frac{n+p}{p}) [(1+B)n+(B-A)(p-\alpha)]|a_{p+n}| \leq (B-A)(p-\alpha).$$

PROOF. It is well known that $f \in C_p(A,B,\alpha)$ if and only if $\frac{zf'(z)}{p} \in T_p^*(A,B,\alpha)$.

Since $$\frac{zf'(z)}{p} = z^p - \sum_{n=1}^{\infty} (\frac{n+p}{p}) |a_{p+n}|z^{p+n}$$

we may replace $|a_{p+n}|$ with $(\frac{n+p}{p}) |a_{p+n}|$ in Theorem 1.

3. REPRESENTATION FORMULA.

THEOREM 2. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ is in $T_p^*(A,B,\alpha)$ if and only if

$$f(z) = z^p \exp \{(B-A)(p-\alpha) \int_0^z \frac{\phi(t)}{1-B\, t\phi(t)} dt\}, \tag{3.1}$$

where $\phi(z)$ is analytic in U and satisfies $|\phi(z)| < 1$, $z \in U$.

PROOF. Let $f(z) \in T_p^*(A,B,\alpha)$, then

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{B \frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]} \right| < 1, \; z \in U.$$

Since the absolute value vanishes for $z = o$, we have

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{B. \frac{zf'(z)}{f(z)} - [pB+(A-B)(p-\alpha)]} \right| = h(z) \tag{3.2}$$

where $h(z)$ is analytic in U and $|h(z)| < 1$ for $z \in U$. Integrating (3.2) with $h(z) = z\phi(z)$ we find that

$$f(z) = z^p. \exp \{(B-A)(p-\alpha) \int_o^z \frac{\phi(t)}{1-Bt\phi(t)} dt\} .$$

The converse is obtained by differentiating (3.1).

4. DISTORTION AND COVERING THEOREMS FOR $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$.

THEOREM 3. If $f(z) \in T_p^*(A,B,\alpha)$, then

$$r^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} \leq |f(z)| \leq$$

$$r^p + \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} \quad (|z| = r), \tag{4.1}$$

with equality for $f(z) = z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} (z=\pm r)$.

PROOF. From Theorem 1, we have

$$[1+B+(B-A)(p-\alpha)] \sum_{n=1}^{\infty} |a_{p+n}| \leq \sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)] |a_{p+n}| \leq (B-A)(p-\alpha).$$

This implies that

$$\sum_{n=1}^{\infty} |a_{p+n}| \leq \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} . \tag{4.2}$$

Thus

$$|f(z)| \leq |z|^p + \sum_{n=1}^{\infty} |a_{p+n}| \, |z|^{p+n}$$

$$\leq r^p(1 + r \sum_{n=1}^{\infty} |a_{p+n}|)$$

$$\leq r^p + \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} .$$

Similarly,

$$|f(z)| \geq |z|^p - \sum_{n=1}^{\infty} |a_{p+n}| \cdot |z|^{p+n}$$

$$\geq r^p(1-r \sum_{n=1}^{\infty} |a_{p+n}|)$$

$$\geq r^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^{p+1} .$$

COROLLARY 3. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in C_p(A,B,\alpha)$, then

$$r^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} r^{p+1} \leq |f(z)| \leq$$

$$r^p + \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} r^{p+1} \quad (|z| = r),$$

with equality for

$$f(z) = z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} (z = \pm r).$$

THEOREM 4. The disc $|z| < 1$ is mapped onto a domain that contains the disc $|w| < \frac{1 + B}{1+B+(B-A)(p-\alpha)}$ by any $f \in T_p^*(A,B,\alpha)$, and onto a domain that contains the disc $|w| < \frac{p+[pB+(A-B)(p-\alpha)]}{p.[1+B+(B-A)(p-\alpha)]}$ by any $f \in C_p(A,B,\alpha)$.

The theorem is sharp, with extremal functions

$$z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} \in T_p^*(A,B,\alpha) \text{ and}$$

$$z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} \in C_p(A,B,\alpha).$$

PROOF. The results follow upon letting $r \to 1$ in Theorem 3 and Corollary 3.

THEOREM 5. If $f \in T_p^*(A,B,\alpha)$, then

$$pr^{p-1} - \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p \leq |f'(z)| \leq$$

$$pr^{p-1} + \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p \quad (|z| = r).$$

Equality holds for

$$f(z) = z^p - \frac{(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} z^{p+1} \quad (z = \pm r).$$

PROOF. We have

$$|f'(z) \le pr^{p-1} + \sum_{n=1}^{\infty} (p+n) \ |a_{p+n}| \ r^{p+n-1}$$

$$\le pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n) \ |a_{p+n}|$$

$$= r^{p-1} \ [p+r \sum_{n=1}^{\infty} (p+n) \ |a_{p+n}|]. \tag{4.3}$$

In view of Theorem 1,

$$\sum_{n=1}^{\infty} (1+B) \ [n+p- \frac{p(1+B)+(A-B)(p-\alpha)}{1+B}] \, |a_{n+p}|$$

$$\le (B-A)(p-\alpha)$$

or

$$\sum_{n=1}^{\infty} (1+B)(n+p) \ |a_{n+p}| \le (B-A)(p-\alpha) +$$

$$[p(1+B)+(A-B)(p-\alpha)] \sum_{n=1}^{\infty} |a_{n+p}| \tag{4.4}$$

(4.4) with the help of (4.2) implies that

$$\sum_{n=1}^{\infty} (n+p) \ |a_{n+p}| \le \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} . \tag{4.5}$$

A substitution of (4.5) into (4.3) yields the right-hand inequality.

On the other-hand

$$|f'(z)| \ge r^{p-1}[p-r \sum_{n=1}^{\infty} (p+n) \ |a_{p+n}|]$$

$$\ge pr^{p-1} - \frac{(p+1)(B-A)(p-\alpha)}{1+B+(B-A)(p-\alpha)} r^p.$$

This completes the proof.

COROLLARY 4. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \in C_p(A,B,\alpha)$, then

$$pr^{p-1} - \frac{(B-A)(p-\alpha)(p+1)^2}{p[1+B+(B-A)(p-\alpha)]} r^p \leq |f'(z)| \leq$$

$$pr^{p-1} + \frac{(B-A)(p-\alpha)(p+1)^2}{p[1+B+(B-A)(p-\alpha)]} r^p \quad (|z| = r).$$

Equality holds for $f(z)=z^p - \frac{(B-A)(p-\alpha)(p+1)}{p[1+B+(B-A)(p-\alpha)]} z^{p+1} (z= \pm r)$.

5. RADIUS OF CONVEXITY FOR THE CLASS $T_p^*(A,B,\alpha)$.

THEOREM 6. If $f(z) \in T_p^*(A,B,\alpha)$, then f(z) is p-valently convex in the disc

$$|z| < R_p = \inf_n \left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}} \quad (n=1,2,\ldots). \tag{5.1}$$

The result is sharp, with the extremal function

$$f(z) = z^p - \frac{(B-A)\ (P-\alpha)}{(1+B)\ n + (B-A)(p-\alpha)} z^{p+n} .$$

PROOF. It is sufficient to show that $\left|(1+ \frac{zf''(z)}{f'(z)}) - p\right| \leq p$ for $|z| < R_p$.

We have

$$\left|(1+ \frac{zf''(z)}{f'(z)}) -p\right| = \left| \frac{-\sum_{n=1}^{\infty} n(n+p)\ |a_{n+p}| z^n}{p - \sum_{n=1}^{\infty} (n+p)\ |a_{n+p}| z|^n} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(n+p)\ |a_{n+p}|\ |z|^n}{p- \sum_{n=1}^{\infty} (n+p)\ |a_{n+p}||z|^n} .$$

Thus

$$\left|(1 + \frac{zf''(z)}{f'(z)}) - p\right| \leq p \text{ if}$$

$$\sum_{n=1}^{\infty} (n+p)^2 |\ a_{n+p}|\ |z|^n \leq p^2$$

or

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^2 |a_{n+p}|\ |z|^n \leq 1. \tag{5.2}$$

According to Theorem 1, $\sum_{n=1}^{\infty} \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} |a_{n+p}| \leq 1$.

Hence (5.2) will be true if

$$\left(\frac{n+p}{p}\right)^2 |z|^n \leq \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)}$$

or if

$$|z| \leq \left[\frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \cdot \left(\frac{p}{n+p}\right)^2\right]^{\frac{1}{n}} \quad (n=1,2,\ldots). \tag{5.3}$$

The theorem follows easily from (5.3).

6. CLOSURE THEOREMS.

In this section we shall prove that the classes $T_p^*(A,B,\alpha)$ and $C_p(A,B,\alpha)$ are closed under convex linear combinations.

THEOREM 7. If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ and $g(z) = z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}$ are in $T_p^*(A,B,\alpha)$, then $h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} |a_{n+p} + b_{n+p}| z^{n+p}$ is also in $T_p^*(A,B,\alpha)$.

PROOF. Since f(z) and g(z) are in $T_p^*(A,B,\alpha)$, we have

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|a_{n+p}| \leq (B-A)(p-\alpha) \tag{6.1}$$

and

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)]|b_{n+p}| \leq (B-A)(p-\alpha). \tag{6.2}$$

From (6.1) and (6.2) we get

$$\frac{1}{2} \sum_{n=1}^{\infty} [(1+B)n+(B-A)(p-\alpha)] |a_{n+p}+b_{n+p}| \leq (B-A)(p-\alpha)$$

which implies that $h(z) \in T_p^*(A,B,\alpha)$.

The following theorem can be proven similarly.

THEOREM 8. $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ and $g(z)=z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}$ are in $C_p(A,B,\alpha)$, then $h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} |a_{n+p}+b_{n+p}| z^{n+p}$ is also in $C_p(A.B.\alpha)$.

THEOREM 9. Let $f_p(z) = z^p$, $f_{n+p}(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} z^{n+p}$ $(n=1,2,3,\ldots)$.

Then $f \in T_p^*(A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} f_{n+p}(z) \text{ where } \lambda_{n+p} \geq o \text{ and } \sum_{n=o}^{\infty} \lambda_{n+p}=1.$$

PROOF. Suppose $f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} f_{n+p}(z) = z^p \sum_{n=o}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} \lambda_{n+p} z^{n+p}$.

Then

$$\sum_{n=1}^{\infty} [\lambda_{n+p} \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} \cdot (\frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)})]$$

$$= \sum_{n=1}^{\infty} \lambda_{n+p} \leq 1 - \lambda_p \leq 1.$$

So by Theorem 1, $f(z) \in T_p^*(A,B,\alpha)$.

Conversely suppose $f(z) \in T_p^*(A,B,\alpha)$. Then

$$|a_{n+p}| \leq \frac{(B-A)(p-\alpha)}{(1+B)n+(B-A)(p-\alpha)} .$$

Setting $\lambda_{n+p} = \frac{(1+B)n+(B-A)(p-\alpha)}{(B-A)(p-\alpha)} |a_{n+p}|$ $(n=1,2,\ldots)$,

and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{n+p},$$

we have

$$f(z) = \sum_{n=o}^{\infty} \lambda_{n+p} \quad f_{n+p}(z).$$

This completes the proof of the theorem.

REMARKS. (1) Putting $\alpha=o$ in the above theorems we get the results obtained by R.M. Goel and N.S. Sohi [1].

(2) Putting $p=1$ and taking $A= -\beta$, $B=\beta$, where $o < \beta \leq 1$, in the above theorems we get the results obtained by Gupta and Jain [2].

(3) Putting $p=1$ and taking $A= -1$, $B=1$ in the above theorems we get the results obtained by Silverman [3].

ACKNOWLEDGMENT. In conclusion, I would like to thank Professor Dr. H. Silverman for reading the manuscript and for helpful suggestions.

REFERENCES

1. GOEL, R.M. and SOHI, N.S. Multivalent functions with negative coefficients, Indian J. Pure appl. Math., 12(7) (1981), 844-853.

2. GUPTA, V.P. and JAIN, P.K. Certain classes of univalent functions with negative coefficients , Bull. Austr. Math. Soc., 14 (1976), 409-416.

3. SILVERMAN, H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975), 109-116.

4. SILVERMAN, H. and SILVIA, E.M. Convex families of starlike functions, Houston J. Math., 4, (1978), 263-268.

5. SILVERMAN, H. and SILVIA, E.M. Pre-Starlike functions with negative coefficients Int. J. Math. Sci., 2 (1979), 427-439.