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RESEARCH NOTES

A NOTE ON GLOBAL EXISTENCE FOR BOUNDARY VALUE PROBLEMS

CHUAN J. CHYAN and JOHNNY HENDERSON

Department of Algebra, Combinatorics and Analysis
Auburn University
Auburn, Alabama 36849 U.S.A.

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ABSTRACT. Upper and lower solutions are used in establishing global existence results for certain two-point boundary value problems for $y''' = f(x, y, y', y'')$ and $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

KEY WORDS AND PHRASES. Boundary value problem, global existence.
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1. INTRODUCTION.

In this paper, we will be concerned primarily with the global existence of solutions of boundary value problems for the third order ordinary differential equation

$$y''' = f(x, y, y', y''), \tag{1.1}$$

satisfying boundary conditions of the form

$$y(a) = y_1,\ y'(a) = y_2,\ y'(b) = y_3,\ a < b. \tag{1.2}$$

The result we obtain for (1.1), (1.2) is an extension, in some sense, of those for boundary value problems for second order equations which appeared in a recent paper by Umamaheswaram and Suhasini [1]. The results in [1] made use of, or were compared to, results dealing with upper and lower solutions for second order equations obtained by Jackson and Schrader [2], Lees [3], and Schrader [4-6]. In [1, Theorem 1], the following is proved.

THEOREM 1.1. Assume that with respect to the second order equation, $y'' = g(x, y, y')$, the following are satisfied:

(A.1) $g: [\alpha, \beta] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous.

(B.1) Solutions of initial value problems exist on $[\alpha, \beta]$ or become unbounded.

(C.1) There exists a sequence $\{M_j\}$ of real numbers $\to +\infty$, such that $f(x, M_j, 0) \geq 0$, for every $j \geq 1$ and all $\alpha \leq x \leq \beta$.

(D.1) There exists a sequence $\{N_j\}$ of real numbers $\to -\infty$, such that $f(x, N_j, 0) \leq 0$, for every $j \geq 1$ and all $\alpha \leq x \leq \beta$.

Then the boundary value problem

$$y'' = g(x, y, y'),$$
$$y(x_1) = y_1,\ y(x_2) = y_2,$$

where $\alpha \leq x_1 < x_2 \leq \beta$, and $y_1, y_2 \in \mathbf{R}$, has a solution.

In Section 2, we extend Theorem 1.1 to boundary value problems (1.1), (1.2).

For this extension, we generalize (C.1) and (D.1) so that the conditions set forth by Klaasen [7] for (1.1), (1.2) are satisfied for any $y_i \in \mathbf{R}$, $i=1, 2, 3$.

In Section 3, the results we obtained for (1.1), (1.2) are generalized somewhat to boundary value problems for the nth order equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{1.3}$$

satisfying

$$y^{(i-1)}(a) = y_i,\ 1 \le i \le n-1,\ y^{(n-2)}(b) = y_n,\quad a < b. \tag{1.4}$$

We conclude Section 3 with an example.

2. GLOBAL EXISTENCE FOR (1.1), (1.2).

In this section, a theorem is proved concerning the global existence of solutions of (1.1), (1.2). We assume in this section that with respect to (1.1), the following are satisfied.

(A.2) $f(x, u_1, u_2, u_3)$: $[a, b] \times \mathbf{R}^3 \to \mathbf{R}$ is continuous.

(B.2) Solutions of initial value problems for (1.1) extend to $[a, b]$ or become unbounded.

(C.2) There exist sequences $\{L_j\}$ and $\{M_j\}$ of real numbers with both $L_j \to +\infty$ and $M_j \to +\infty$, such that $f(x, M_j x + L_i, M_j, 0) \ge 0$, for all $i, j \ge 1$ and all $a \le x \le b$.

(D.2) There exist sequences $\{K_j\}$ and $\{N_j\}$ of real numbers, with both $K_j \to -\infty$ and $N_j \to -\infty$, such that $f(x, N_j x + K_i, N_j, 0) \le 0$, for all $i, j \ge 1$ and all $a \le x \le b$.

THEOREM 2.1. Assume that (A.2) - (D.2) are satisfied and that $f(x, u_1, u_2, u_3)$ is nonincreasing in u_1 for each fixed x, u_2, u_3. Then the boundary value problem (1.1), (1.2) has a solution for any choice of $y_1, y_2, y_3 \in \mathbf{R}$.

PROOF. Let $y_1, y_2, y_3 \in \mathbf{R}$ be given. By hypotheses (C.2) and (D.2), there exist $I, J \in \mathbf{N}$ such that

$$N_J a + K_I \le y_1 \le M_J a + L_I,$$

and

$$N_J \le \min\{y_2, y_3\} \le \max\{y_2, y_3\} \le M_J.$$

Defining $\gamma(x) \equiv N_J x + K_I$ and $\Psi(x) \equiv M_J x + L_I$, it follows $\gamma^{(i)}(x) \le \Psi^{(i)}(x)$ on on $[a, b]$, for $i = 0, 1$. Furthermore, by (C.2) and (D.2), $\gamma(x)$ and $\Psi(x)$ are lower and upper solutions, respectively, of (1.1) on $[a, b]$.

It follows from results due to Klaasen [7] that there exists a solution $y(x)$ of (1.1), (1.2), for this choice of y_1, y_2, y_3, and furthermore $\gamma(x) \le y(x) \le \Psi(x)$ and $N_J \le y'(x) \le M_J$ on $[a, b]$. The proof is complete.

3. GLOBAL EXISTENCE FOR (1.3), (1.4).

In this section, we will be concerned with the existence of solutions of (1.3), (1.4). For this consideration, results due to Kelley [8] will be used. We assume here that with respect to (1.3), the following are satisfied.

(A.3) $f(x, u_1, u_2, \ldots, u_n)$: $[a, b] \times \mathbf{R}^n \to \mathbf{R}$ is continuous.

(B.3) Solutions of initial value problems for (1.3) extend to $[a, b]$ or become unbounded.

(C.3) There exist sequences $\{M_{1,j}\}, \{M_{2,j}\}, \ldots, \{M_{n-1,j}\}$ of real numbers, such that $M_{k,j} \to +\infty$, $1 \le k \le n-1$, and such that if $p_{j_1 j_2 \ldots j_{n-1}}(x) = \sum_{k=1}^{n-1} M_{k,j_k} x^{k-1}$, then $f(x, p_{j_1 \cdots j_{n-1}}(x), p'_{j_1 \cdots j_{n-1}}(x), \ldots, p^{(n-2)}_{j_1 \cdots j_{n-1}}(x), 0) \geq 0$, for all $j_1, \ldots, j_{n-1} \geq 1$ and all $a \leq x \leq b$.

(D.3) There exist sequences $\{N_{1,j}\}, \{N_{2,j}\}, \ldots, \{N_{n-1,j}\}$ of real numbers, such that $N_{k,j} \to -\infty$, $1 \leq k \leq n-1$, and such that if $q_{j_1 j_2 \cdots j_{n-1}}(x) = \sum_{k=1}^{n-1} N_{k,j_k} x^{k-1}$, then $f(x, q_{j_1 \cdots j_{n-1}}(x), q'_{j_1 \cdots j_{n-1}}(x), \ldots, q^{(n-2)}_{j_1 \cdots j_{n-1}}(x), 0) \leq 0$, for all $j_1, \ldots, j_{n-1} \geq 1$ and all $a \leq x \leq b$.

THEOREM 3.1. Assume in addition to conditions (A.3) - (D.3) that, if $y(x)$ is a solution of (1.3) with maximal interval of existence $I \subseteq [a, b]$ such that $y^{(n-2)}(x)$ is bounded on I, then $y^{(n-1)}(x)$ is bounded on I. Furthermore, assume that for each $1 \leq i \leq n-2$, $f(x, u_1, u_2, \ldots, u_n)$ is nonincreasing in u_i, for each fixed $x, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$. Then the boundary value problem (1.3), (1.4) has a solution for any choice of $y_i \in \mathbb{R}$, $1 \leq i \leq n$.

PROOF. Let $y_i \in \mathbb{R}$, $1 \leq i \leq n$, be given. It follows from (C.3) and (D.3) that there exist $j_1, j_2, \ldots, j_{n-1} \in \mathbb{N}$ such that

$$q^{(i-1)}_{j_1 \cdots j_{n-1}}(a) \leq y_i \leq p^{(i-1)}_{j_1 \cdots j_{n-1}}(a)\ , \quad 1 \leq i \leq n-2,$$

$$q^{(n-2)}_{j_1 \cdots j_{n-1}}(a) = (n-2)!\, N_{n-1,j_{n-1}} \leq \min\{y_{n-1}, y_n\}$$

$$\leq \max\{y_{n-1}, y_n\} \leq (n-2)!\, M_{n-1,j_{n-1}} = p^{(n-2)}_{j_1 \cdots j_{n-1}}(a).$$

Defining $\gamma(x) \equiv q_{j_1 \cdots j_{n-1}}(x)$ and $\Psi(x) \equiv p_{j_1 \cdots j_{n-1}}(x)$, it follows from $\Psi^{(n-2)}(x) - \gamma^{(n-2)}(x) \equiv \Psi^{(n-2)}(a) - \gamma^{(n-2)}(a) \geq 0$, for all $a \leq x \leq b$, and from $\Psi^{(i-1)}(a) \geq \gamma^{(i-1)}(a)$, $1 \leq i \leq n-2$, that $\gamma^{(i-1)}(x) \leq \Psi^{(i-1)}(x)$ on $[a, b]$, for $1 \leq i \leq n-1$. Furthermore, from (C.3) and (D.3), $\gamma(x)$ and $\Psi(x)$ are lower and upper solutions, respectively, of (1.3) on $[a, b]$. It follows from the other hypotheses of the Theorem and from a result due to Kelley [8] that there exists a solution $y(x)$ of (1.3), (1.4), for this choice of $y_i \in \mathbb{R}$, $1 \leq i \leq n$. Moreover, $\gamma^{(i-1)}(x) \leq y^{(i-1)}(x) \leq \Psi^{(i-1)}(x)$ on $[a, b]$, for $1 \leq i \leq n-1$. This completes the proof.

EXAMPLE. Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(u) = \begin{cases} \operatorname{Sin}(\pi u/e^{\pi}) & , u \leq 0, \\ -2u & , 0 \leq u \leq e^{\pi}, \\ -2e^{\pi} + 2e^{\pi} \operatorname{Sin}(\pi u/e^{\pi}), & u \geq e^{\pi}, \end{cases}$$

and let $f(x, u_1, \ldots, u_n)\colon [0, \pi] \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x, u_1, \ldots, u_{n-1}, u_n) = g(u_{n-1}) + 2u_n.$$

The conditions of Theorem 3.1 are satisfied with respect to the differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) = g(y^{(n-2)}) + 2y^{(n-1)}. \tag{3.1}$$

In particular, $\frac{\partial f}{\partial u_i}$, $1 \leq i \leq n$, are piecewise continuous and bounded on $[0, \pi] \times \mathbb{R}^n$, hence initial value problems of (3.1) exist on $[0, \pi]$. Also, the sequences $\{M_{k,j}\} = \{j\}$, for $1 \leq k \leq n-2$, and $\{M_{n-1,j}\} = \{\frac{(1+4j)e^{\pi}}{2(n-2)!}\}$ satisfy condition (C.3), whereas, the sequences $\{N_{k,j}\} = \{-j\}$, for $1 \leq k \leq n-2$, and $\{N_{n-1,j}\} = \{\frac{-je^{\pi}}{(n-2)!}\}$ satisfy condition (D.3). Hence, by Theorem 3.1, boundary value problems for (3.1) satisfying

$$y^{(i-1)}(0) = y_i, \ 1 \leq i \leq n-1, \ y^{(n-2)}(\pi) = y_n$$

are solvable.

In fact,

$$y(x) = \begin{cases} \dfrac{C}{(-4)^k} \left[e^x \sin x + \sum_{i=1}^{k} (-1)^i \left[2^{2i-1}\left(\dfrac{x^{4i-1}}{(4i-1)!} + \dfrac{x^{4i-2}}{(4i-2)!}\right) + 2^{2i-2} \dfrac{x^{4i-3}}{(4i-3)!}\right]\right], & n = 4k+2, \ k = 0, 1, 2, \ldots, \\ \dfrac{C}{2(-4)^k} \left[e^x(\sin x - \cos x) + \sum_{i=1}^{k} (-1)^i \left[2^{2i}\left(\dfrac{x^{4i}}{(4i)!} + \dfrac{x^{4i-1}}{(4i-1)!}\right) + 2^{2i-1} \dfrac{x^{4i-2}}{(4i-2)!}\right] + 1\right], & n = 4k+3, \ k = 0, 1, 2, \ldots, \\ \dfrac{C}{2(-4)^k} \left[-e^x \cos x + \sum_{i=1}^{k} (-1)^i \left[2^{2i}\left(\dfrac{x^{4i+1}}{(4i+1)!} + \dfrac{x^{4i}}{(4i)!}\right) + 2^{2i-1} \dfrac{x^{4i-1}}{(4i-1)!}\right] + (x+1)\right], & n = 4(k+1), \ k = 0, 1, 2, \ldots, \\ \dfrac{C}{(-4)^k} \left[e^x(\sin x + \cos x) + \sum_{i=1}^{k} (-1)^i \left[2^{2i-1}\left(\dfrac{x^{4i-2}}{(4i-2)!} + \dfrac{x^{4i-3}}{(4i-3)!}\right) + 2^{2i-2} \dfrac{x^{4i-4}}{(4i-4)!}\right]\right], & n = 4k+1, \ k = 1, 2, \ldots, \end{cases}$$

where $0 \leq C \leq 1$, are infinitely many solutions of (3.1) satisfying $y^{(i-1)}(0) = y^{(n-2)}(\pi) = 0, \ 1 \leq i \leq n-1$.

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