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SUFFICIENT CONDITIONS FOR TWO-DIMENSIONAL POINT DISSIPATIVE NON-LINEAR SYSTEMS

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ABSTRACT. A two-dimensional autonomous system

$$\dot{x} = AX + (x^T B^1 x,\ x^T B^2 x)^T$$

of differential equations with quadratic non-linearity is point dissipative, if there exists a positive number γ such that the symmetric matrices B^1 and B^2 are of the form

$$B^1 = \begin{pmatrix} 0 & b^1_{12} \\ b^1_{12} & b^1_{22} \end{pmatrix}, \qquad B^2 = -\gamma \begin{pmatrix} 2b^1_{12} & \frac{1}{2}b^1_{22} \\ \frac{1}{2}b^1_{22} & 0 \end{pmatrix}$$

and $b^T \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} Ab < 0$, where $b^T = \left(b^1_{22}, -2b^1_{12}\right)$.

KEY WORDS AND PHRASES. Point dissipative, quadratic form, quadratic non-linearity, positive semi-orbit, limit set, positive definite, negative definite, autonomous system, symmetric matrices, Level set, critical points.

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I. INTRODUCTION.

Consider the following two-dimensional autonomous dynamical system

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b^1_{11}x_1^2 + 2b^1_{12}x_1x_2 + b^1_{22}x_2^2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b^2_{11}x_1^2 + 2b^2_{12}x_1x_2 + b^2_{22}x_2^2 \end{aligned} \tag{1.1}$$

with quadratic non-linearity, where at least one of the symmetric matrices

$$B^1 = \begin{pmatrix} b^1_{11} & b^1_{12} \\ b^1_{12} & b^1_{22} \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} b^2_{11} & b^2_{12} \\ b^2_{12} & b^2_{22} \end{pmatrix}$$

is non-zero. We are interested in deriving sufficient conditions on the matrices

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, B^1 and B^2, so that the system (1.1) is point dissipative. That is,

there exists a bounded set G such that the orbit of each solution of (1.1) eventually enters the set G and remains there.

II. THEOREM 1.

The system (1.1) is point dissipative if the following conditions are satisfied:

There exists a number $\gamma > 0$ such that (i) the matrices B^1 and B^2 are of the form

$$B^1 = \begin{pmatrix} 0 & b^1_{12} \\ b^1_{12} & b^1_{22} \end{pmatrix}, \quad B^2 = -\gamma \begin{pmatrix} 2b^1_{12} & \frac{1}{2}b^1_{22} \\ \frac{1}{2}b^1_{22} & 0 \end{pmatrix}$$

and (ii) $b^T \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} Ab < 0$, where the vector b is given by $b^T = \left(b^1_{22}, -2b^1_{12}\right)$.

In order to prove the above theorem, we need the following lemma:

LEMMA. If the matrices A, B^1, and B^2 satisfy the conditions (i) and (ii) in Theorem 1, then it is possible to construct a function (Lyapunov) of the form

$$V = \tfrac{1}{2}\rho(x_1-\alpha_1)^2 + \tfrac{1}{2}(x_2-\alpha_2)^2 - \tfrac{1}{2}\rho\alpha_1^2 - \tfrac{1}{2}\alpha_2^2$$

(i.e. to choose the real numbers $\rho > 0$, α_1, α_2) so that the set $S = \{x \mid \dot{V}(x) \geq 0\}$, where $\dot{V}$ is the derivative of V with respect to the system (1.1), is bounded.

PROOF OF THE LEMMA. First, we choose $\rho = \gamma$, where γ is the positive number given in conditions (i) and (ii) of Theorem 1. $\dot{V}$, for yet unspecified α_1 and α_2, is given by

$$\begin{aligned} \dot{V} &= \operatorname{grad} V \cdot (\dot{x}_1, \dot{x}_2) \\ &= -\alpha^T \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} Ax + x^T \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} A - \gamma\alpha_1 B^1 - \alpha_2 B^2 \right\} x, \end{aligned}$$

where $\alpha^T = (\alpha_1, \alpha_2)$. The cubic terms in $\dot{V}$ cancelled out because of condition (i). [Note that without the vanishing of the cubic terms there is no possiblity that the set S can be bounded.] Let

$$C = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} A - \gamma\alpha_1 B^1 - \alpha_2 B^2.$$

We would like to show that C is negative definite. This we will accomplish by showing that -C is positive definite. Again $-C = ((P_{ij}))$ is positive definite if and only if the symmetric matrix

$\hat{C} = \left(\left(\dfrac{P_{ij} + P_{ji}}{2}\right)\right)$ is positive definite. Now

$$\hat{C} = \begin{pmatrix} -2\gamma\alpha_2 b^1_{12} - \gamma a_{11} & \frac{1}{2}\left(2\gamma\alpha_1 b^1_{12} - \gamma\alpha_2 b^1_{22} - \gamma a_{12} - a_{21}\right) \\ \frac{1}{2}\left(2\gamma\alpha_1 b^1_{12} - \gamma\alpha_2 b^1_{22} - \gamma a_{12} - a_{21}\right) & \gamma\alpha_1 b^1_{22} - a_{22} \end{pmatrix}.$$

Necessary and sufficient conditions for $\hat{C}$ to be positive definite are

$$-2\gamma\alpha_2 b^1_{12} - \gamma a_{11} > 0 \ , \ \gamma\alpha_1 b^1_{22} - a_{22} > 0 \tag{2.1}$$

and det $(\hat{C}) > 0$. That is

$$\left(-2\gamma\alpha_2 b^1_{12} - \gamma a_{11}\right)\left(\gamma\alpha_1 b^1_{22} - a_{22}\right) > \frac{1}{4}\left(2\gamma\alpha_1 b^1_{12} - \gamma\alpha_2 b^1_{22} - \gamma a_{12} - a_{21}\right)^2 \tag{2.2}$$

We need to show that α_1 and α_2 can be chosen so that both the inequalities (2.1) and (2.2) are satisfied. Setting $-2\gamma\alpha_2 b^1_{12} - \gamma a_{11} = \varepsilon_2$, $\gamma\alpha_1 b^1_{22} - a_{22} = \varepsilon_1$, where ε_1 and ε_2 are two positive numbers, the inequality (2.2) becomes

$$\varepsilon_1\varepsilon_2 > \frac{1}{16(b^1_{12})^2\,(b^1_{22})^2}\left\{b^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ab + 4(b^1_{12})^2\,\varepsilon_1 + (b^1_{22})^2\,\varepsilon_2\right\}^2 \tag{2.3}$$

for the case $b^1_{12} \neq 0$, $b^1_{22} \neq 0$.

[Note that the inequality (2.3) cannot be satisfied if $b^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ab \geq 0$, for

$\dfrac{1}{16(b^1_{12})^2(b^1_{22})^2}\left\{4(b^1_{12})^2\,\varepsilon_1 + (b^1_{22})^2\,\varepsilon_2\right\}^2 \geq \varepsilon_1\varepsilon_2$, using the standard inequality $a^2 + b^2 \geq 2\,|a|\,|b|$].

Since by condition (ii) $b^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ab < 0$, letting

$$\varepsilon_1 = \frac{-b^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ab}{8(b^1_{12})^2}\ ,\quad \varepsilon_2 = \frac{-b^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ab}{2(b^1_{22})^2} \tag{2.4}$$

the inequality (2.3) becomes $\varepsilon_1\varepsilon_2 > 0$. Hence both the inequalities (2.1) and (2.2) are satisfied for these choices of ε_1 and ε_2. Again this implies that inequalities (2.1) and (2.2) are satisfied for

$$\alpha_1 = \frac{a_{22} + \varepsilon_1}{\gamma b^1_{22}}\ ,\qquad \alpha_2 = -\ \frac{\gamma a_{11} + \varepsilon_2}{2\gamma b^1_{12}}\ ,$$

where ε_1 and ε_2 are given by (2.4). Other choices of α_1 and α_2 are certainly possible. Thus C is negative definite for the above choices of α_1 and α_2.

The case where only one of b^1_{12} or b^1_{22} is zero can be disposed of similarly. Note that both b^1_{12} and b^1_{22} cannot be zero, for in that case both the matrices B^1 and B^2 become zero matrices contradicting our assumption. Now to see that the set S is bounded we come back to $\dot{V} = -\alpha^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ax + x^TCx$. Since the quadratic form x^TCx is negative definite and $-\alpha^T\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}Ax$ is linear, there exists $R_0 > 0$ such that $\dot{V} < 0$ for all x with $\|x\| > R_0$. Hence the set $S = \{x \mid \dot{V}(x) \geq 0\}$ must lie inside the circle $S(0,R_0)$ and therefore bounded. Note that the set S contains all the critical points of the system (1.1).

PROOF OF THEOREM 1. To show that the system (1.1) is point dissipative under conditions (i) and (ii), we need to exhibit a bounded set G so that the positive semi-orbit of each solution of (1.1) eventually enters the set G and remains there. Using the lemma we first construct the function

$$V = \frac{1}{2}\rho(x_1-\alpha_1)^2 + \frac{1}{2}(x_2-\alpha_2)^2 - \frac{1}{2}\rho\alpha_1^2 - \frac{1}{2}\alpha_2^2$$

so that the set $S = \{x \mid \dot{V}(x) \geq 0\}$ is bounded. We can choose $r_0 > 0$, sufficiently large, so that the level set (ellipse) $V = r_0$ contains in its interior the compact set S. We choose the interior of $V = r_0$ as our bounded set G. Let P_0 be a point outside of G and $\phi(t,P_0)$ be the solution of (1.1) with $\phi(0,P_0) = P_0$. Let $V = r_1$ be the level set of V passing through P_0. Clearly $r_1 > r_0$. Let H be the ring-shaped closed region formed by the two concentric ellipses $V = r_0$ and $V = r_1$. Since S lies inside the ellipse $V = r_0$, $\dot{V} < 0$ on H. Therefore $V(\phi(t,P_0))$ is a decreasing function of t on H. Hence the positive semi-orbit C^+ of $\phi(t,P_0)$ must enter the ellipse $V = r_1$ and cannot go outside of $V = r_1$ at any time $t > 0$. Suppose that C^+ cannot enter the region G. Then C^+ must remain in H for all time $t \geq 0$. We need a contradiction resulting from this hypothesis. C^+ must have limit points in H. Let $L(C^+)$ be the set of all limit points of C^+. $L(C^+) \subset H$. We would like to show that V is constant on $L(C^+)$. Let P_1 and P_2 be any two points in $L(C^+)$, then there exists sequences $\{t_n\}$ and $\{s_n\}$ such that

$$\lim_{n\to\infty} \phi(t_n,P_0) = P_1 \quad , \quad \lim_{n\to\infty} \phi(s_n,P_0) = P_2.$$

Since $V(\phi(t,P_0))$ is decreasing in H and by continuity V has a lower bound in H, $\lim_{t\to\infty} V(\phi(t,P_0))$ must exist. Let this limit be q. Then

$$q = \lim_{n\to\infty} V(\phi(t_n,P_0)) = \lim_{n\to\infty} V(\phi(s_n,P_0))$$

and so by the continuity of V, $V(P_1) = V(P_2) = q$. That is $V(P) = q$ on $L(C^+)$. Let $P \varepsilon L(C^+)$ and $\psi(t,P)$ be the solution of (1.1) with $\psi(0,P) = P$. Then $\psi(t,P) \subset L(C^+)$. But $\dot{V}(P) = \dot{V}(\psi(0,P) = \frac{d}{dt}(V(\psi(t,P)))\,|_{t=0} = \frac{dq}{dt} = 0$ which implies a contradiction of $\dot{V} < 0$ on H. Hence C^+ must enter G eventually and cannot go out of G by the decreasing property of $V(\phi(t,P_0))$ and therefore remains in G. This completes the proof.

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