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HOW MANY NUMBERS SATISFY THE 3X + 1 CONJECTURE?

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ABSTRACT. Let $\theta(x)$ be the number of numbers not exceeding x satisfy the 3X + 1 conjecture. We obtain a system of difference inequalities on functions closely related to θ. Solving this system in the simplest case, we establish $\theta(x) > cx^{\frac{3}{7}}$. This improves a result of Crandall [1].

KEY WORDS AND PHRASES. 3X + 1 conjecture, residue class, difference inequality.
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1. INTRODUCTION.

The famous conjecture of Collatz-Kakutani, also known as the Syracuse or the "3X + 1" problem, claims that the sequence

$$\alpha_{n+1} = T(\alpha_n) = \begin{cases} \frac{3\alpha_n + 1}{2}, & \alpha_n \equiv 1 \pmod 2 \\ \frac{\alpha_n}{2}, & \alpha_n \equiv 0 \pmod 2 \end{cases} \tag{1.1}$$

converges to the cycle (1,2) for any $\alpha_0 \in Z^+$.

The following well-known heuristic argument serves as an evidence for its validity. Consider T as though it were a random walk. It is natural to suppose that odd and even numbers appear independently, with probability 1/2 at each jump.

Then $T^{(n)}(\alpha_0)$ should converge since the mathematical expectation

of $\frac{T(\alpha)}{\alpha}$ is about $(\frac{3}{2} \cdot \frac{1}{2})^{1/2} < 1$.

Although this conjecture seems to be intractable at present, some supporting results have been obtained. An interesting review on this problem can be found in [2]. In particular, Crandall [1] proved that the conjecture is true for many values of α_0. Namely, set $\theta(x) = |\{u : T^{(k)}(u) = 1 \text{ for some } k \geq 0 \text{ and } u \leq x\}|$. Thus, $\theta(x)$ is just the number of numbers not exceeding x satisfing the conjecture. Then Crandall's result is $\theta(x) > cx^r$, for appropriate constants c, r > 0. However, his proof gives a very poor value for r, about 0.05.

Here we derive a system of difference inequalities on functions closely related to θ (Lemma 4). Solving this system in the simplest case, we establish $\theta(x) > cx^{\frac{3}{7}}$. Actually our proof gives a little more, namely:

given any $v \equiv 1$ or 2 (mod 3) that is not in a cycle, for all $x \geq 1$

$$|\{n \leq vx: T^{(k)}(n) = v \text{ for some } k \geq 1\}| \geq c_0 x^{\frac{3}{7}},$$

where c_0 is a positive constant independent of v.

In some sense the proof may be regarded as an attempt to formalize the above mentioned heuristic argument.

2. RESULTS.

Consider the infinite directed graph G on the vertex set $V = Z^+$ and the edge set $E = \{(T(v), v)\}$, whose edges are oriented from T(v) to v. Denote by G(v,x) an induced subgraph of G whose vertex set consists of all integers n such that some $T^k(n) = v$ and $T^i(n) \leq x$ for $0 \leq i \leq k$. That is, it consists of all integers n whose trajectory hits v and remains below x the entire time. In particular G(v,x) is the empty set if x < v. We also put $G(v) = G(v,\infty)$. Observe that G(v) has at most one cycle since the in degree of each vertex, but may be v, is one. Moreover, if v does not lie in a cycle of G then G(v) is a tree.

Here we prefer to deal with U, the mapping inverse to T, namely:

$$U(\alpha) = \begin{cases} 2\alpha, & \alpha \equiv 0,1 \pmod 3 \\ 2\alpha \cup \frac{2\alpha - 1}{3}, & \alpha \equiv 2 \pmod 3 \end{cases} \tag{2.1}$$

Since only numbers $\alpha \equiv 2$ (mod 3) have two inverses under T, we wish to analyze iterates under $U = T^{-1}$ restricted to integers $\equiv 2$ (mod 3). To do this we must consider values of α (mod 9).

Let S_n be a complete system of residue classes modulo 3^n. We split S_n as follows:

$$S_n = \bigcup_{i=0}^{2} R_n^i, \text{ where } \alpha \in R_n^i \iff \alpha \equiv i \pmod 3).$$

Furthermore, put

$$R_n^2 = Q_n^2 \cup Q_n^5 \cup Q_n^8, \text{ where } \alpha \in Q_n^i \iff \alpha \equiv i \pmod 9).$$

Obviously, $U: R_n^o \to R_n^o$ and $U: R_n^1 \to R_n^2$. The action of U on R_n^2 can be split into the four following operators:

$$U_1: R_n^2 \to R_n^2,\ U_1(\alpha) = 4\alpha$$

$$U_2: Q_n^5 \to R_{n-1}^o,\ U_2(\alpha) = \frac{2\alpha - 1}{3}$$

$$U_3: Q_n^2 \to R_{n-1}^2,\ U_3(\alpha) = \frac{4\alpha - 2}{3}$$

$$U_4: Q_n^8 \to R_{n-1}^2,\ U_4(\alpha) = \frac{2\alpha - 1}{3}$$

The following lemma is an easy exercise in elementary number theory:

LEMMA 1.

(i) U_1 ia a bijection $R_n^2 \leftrightarrow R_n^2$. Moreover, if $\alpha \in R_n^2$ then $\ell = 3^{n-1}$ is the smallest positive integer such that $U_1^{(\ell)}(\alpha) = \alpha$.

(ii) U_3 is a bijection $Q_n^2 \leftrightarrow R_{n-1}^2$.

(iii) U_4 is a bijection $Q_n^8 \leftrightarrow R_{n-1}^2$.

The action of U on R_n^o and R_n^1 is much simpler. Namely, $U: R_n^o \to R_n^o$ and $U: R_n^1 \to R_n^2$ are bijections. Moreover, since $\alpha \in R_n^o$ implies $U(\alpha) = 2\alpha \in R_n^o$ we get

LEMMA 2. If $v \in R_n^o$ then G(v) is a chain.

Now we define the functions we deal with in this paper.
Let $v \equiv m \pmod{3^n}$. We set $f(v,x) = f_n^m(v,x) = |G(v,x)|$. (The reason for using the redundant notation $f_n^m(v,x)$ instead of $f(v,x)$ is to simplify the statement of the difference inequalities that follow.)

Observe that for $v \leq x$

$$f_n^m(v,x) = 1 + [\log_2 \frac{x}{v}],\ m \in R_n^o, \tag{2.2}$$

$$f_n^m(v,x) = 1 + f_n^{2m}(2v,x),\ m \in R_n^1. \tag{2.3}$$

Furthermore, let $W = \{w\}$ be the set of those vertices of G which do not belong to a cycle. For instance, $U^k(4) \in W$ for all $K > 0$. Then $G(w)$ is a tree and we set

$$\phi_n^m(y) = \inf_{v \in W} f_n^m(v, 2^y v) = \inf\{f(v, 2^y v) : v \in W \text{ and } v \equiv m \pmod{3^n}\}.$$

Note that for any $m \equiv 2 \pmod 3$ and n, the set $\{v: v \; G(u), v \equiv m \pmod{3^n}\} \neq \emptyset$ because $2^k v$ is in this set and 2 is a primitive root (mod 3^n) for all n.

LEMMA 3. $\phi_n^m(y)$ is nondecreasing function of y.

PROOF. Obviously, $f_n^m(v,x)$ is a nondecreasing function of x.

Hence, $\phi_n^m(y) = \inf f_n^m(v, 2^y v)$ is nondecreasing function of y.

The following lemma gives important recurrent inequalities on $\phi_n^m(y)$.

LEMMA 4. For $y > 0$,

$$\begin{cases} \phi_n^m(y) \geq \phi_n^{4m}(y-2) + \phi_{n-1}^{\frac{4m-2}{3}}(y+\alpha-2), & m \in Q_n^2 \\ \phi_n^m(y) \geq \phi_n^{4m}(y-2) + \phi_{n-1}^{\frac{2v-1}{3}}(y+\alpha-1), & m \in Q_n^8 \\ \phi_n^m(y) \geq \phi_n^{4m}(y-2) + [y+\alpha], & m \in Q_n^5 \end{cases} \tag{2.4}$$

where $\alpha = \log_2 3 \simeq 1.585$ and

$$\phi_{n-1}^m(y) = \min\left(\phi_n^m(y), \phi_n^{m+3^{n-1}}(y), \phi_n^{m+2\cdot 3^{n-1}}(y)\right). \tag{2.5}$$

PROOF. (2.5) follows immediately from the definition of $\phi_n^m(y)$. Let us demonstrate (2.4). If $v = m \pmod{3^n}$, $m \in Q_n^5$ then, by (2.1), if $v < x$,

$$|G(v,x)| \geq |G(4v,x)| + \left| G\left(\frac{2v-1}{3}, x\right)\right|.$$

If $\frac{2v-1}{3} \equiv 0 \pmod 3$ then $G\left(\frac{2v-1}{3}, x\right)$ is a chain by lemma 2. Thus, by (2.2), if $v < x$,

$$f_n^m(v,x) = f_n^{4m}(4v,x) + 1 + \left[\log_2 \frac{3x}{2v-1}\right].$$

Hence, $\phi_n^m(y) \geq \phi_n^{4m}(y-2) + [y+\alpha]$.

If $m \in Q_n^8$ then $G(v,x)$ is a forest. Hence,

$$|G(v,x)| = |G(4v,x)| + \left|G\left(\frac{2v-1}{3}, x\right)\right|$$

By $\frac{2v-1}{3} < \frac{2v}{3}$ and by lemma 3 we get, if $y \geq 0$ and $x = 2^y v$, then

$$\phi_n^m(y) = \inf f_n^m(v,x) = \inf \left(f_n^{4m}(4v,x) + f_{n-1}^{\frac{2v-1}{3}}\left(\frac{2v-1}{3},x\right)\right) \geq$$

$$\geq \inf f_n^{4m}(4v, x) + \inf f_{n-1}^{\frac{2v-1}{3}}\left(\frac{2v-1}{3},x\right) \geq \phi_n^{4m}(y-2) + \phi_{n-1}^{\frac{2v-1}{3}}(y+\alpha-1).$$

The case $m \in Q_n^2$ may be considered similarly to the case $m \in Q_n^8$. We omit the details

THEOREM 1. $\theta(x) > c_2 x^{\frac{3}{7}}$.

PROOF. For $n = 2$ the system (2.4) becomes for $y \geq 0$,

$$\phi_2^2(y) \geq \phi_2^8(y-2) + \phi_1^2(y+\alpha-2),$$

$$\phi_2^8(y) \geq \phi_2^5(y-2) + \phi_1^2(y+\alpha-1),$$

$$\phi_2^5(y) > \phi_2^2(y-2),$$

where $\phi_1^2(y) = \min(\phi_2^2(y), \phi_2^8(y), \phi_2^5(y))$. Observe that $\phi_2^8(y) > \phi_1^2(y)$ for $y \geq 2$ by

$$\phi_2^8(y) \geq \phi_2^5(y-2) + \phi_1^2(y+\alpha-1) > \phi_1^2(y),$$

since $\phi_1^2(y+\alpha-1) \geq \phi_1^2(y)$ and $\phi_2^5(y-2) > 0$ if $y \geq 2$. Hence,

$$\phi_1^2(y) = \min(\phi_2^2(y), \phi_2^5(y)) \geq \min(\phi_2^2(y), \phi_2^2(y-2)) = \phi_2^2(y-2).$$

This yields if $y \geq 6$,

$$\phi_2^2(y) \geq \phi_2^5(y-4) + \phi_1^2(y+\alpha-1) + \phi_1^2(y+\alpha-2)$$

$$\geq \phi_2^2(y-6) + \phi_1^2(y+\alpha-1) + \phi_1^2(y+\alpha-2)$$

$$\geq \phi_2^2(y-6) + \phi_2^2(y+\alpha-5) + \phi_2^2(y+\alpha-4).$$

The initial conditions $\phi_2^2(0) = 1$ imply $\phi_2^2(y) \geq 1$ for $y \geq 6$, whence one proves by induction on n, that for $n \leq y \leq n+1$, one has $\phi_2^2(y) \geq c_1\lambda^y$, where $\lambda \approx 1.3534$ is the largest root of $1 = \lambda^{-6} + \lambda^{\alpha-5} + \lambda^{\alpha-4}$.

Finally, we obtain $\theta(x) \geq c_2 x^{\log_2 \lambda} > c_2 x^{\frac{3}{7}}$, where $\log_2\lambda \approx 0.436$.

REMARK. Although system (2.4) seems to be very complicated and we were unable to solve it for $n \geq 3$, averaging it over all residue classes modulo 3^{n-1} looks much more attractive. Namely, define

$$F_n(y) = 3^{-n+1} \sum_{m \in R_n^2} \phi_n^m(y).$$

Using lemmas 1 and 4 we get

$$3^{n-1}F_n(y) = \sum_{m \in R_n^2} \phi_n^m(y) \geq \sum_{m \in R_n^2} \phi_n^m(y-2) + \sum_{m \in R_{n-1}^2} \phi_{n-1}^m(y + \alpha - 2) + \sum_{m \in R_{n-1}^2} \phi_{n-1}^m(y + \alpha - 1) =$$

$$= 3^{n-1}F_n(y - 2) + 3^{n-2}F_{n-1}(y + \alpha - 2) + 3^{n-2}F_{n-1}(y + \alpha - 1).$$

Thus,

$$F_n(y) \geq F_n(y - 2) + \frac{1}{3} F_{n-1}(y + \alpha - 2) + \frac{1}{3} F_{n-1}(y + \alpha - 1).$$

Observe that the associated limit equation $1 = \lambda^{-2} + \frac{1}{3}(\lambda^{\alpha-2} + \lambda^{\alpha-1})$ has $\lambda = 2$ as the smallest positive root. Therefore, one might expect that the solution of the difference ineqalities gives $\theta(x) > c_n x^{r_n}$, where $r_n \to 1$ when n tends to infinity.

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