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ON THE MINIMAL SPACE OF SURJECTIVITY QUESTION FOR LINEAR TRANSFORMATIONS ON VECTOR SPACES WITH APPLICATIONS TO SURJECTIVITY OF DIFFERENTIAL OPERATORS ON LOCALLY CONVEX SPACES

D.K. COHOON

Mathematics Department 038-16
Temple University
Philadelphia, Pennsylvania 19122

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ABSTRACT. We use transfinite induction to show that if L is an epimorphism of a vector space V and maps a vector subspace W of V into a proper subspace of itself, then there is a smallest subspace E of V containing W such that L(E) = E (or a minimal space of surjectivity or solvability) and we give examples where there are infinitely many distinct minimal spaces of solvability. We produce an example showing that if L_1 and L_2 are two epimorphisms of a vector space V which are endomorphisms of a proper subspace W of V such that $L_1(W) \cap L_2(W)$ is a proper subspace of W, then there may not exist a smallest subspace E of V containing W such that $L_1(E) = E = L_2(E)$. While no nonconstant linear partial differential operator maps the field of meromorphic functions onto itself, we construct a locally convex topological vector space of formal power series containing the meromorphic functions such that every linear partial differential operator with constant coefficients maps this space linearly and continuously onto itself. Furthermore, we show that algebraically there is for every linear partial differential operator P(D) with constant coefficients a smallest extension E of the meromorphic functions in n complex variables, where $D = (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$, with the property that for every f in E, there is a u in E such that P(D)u=f.

KEY WORDS AND PHRASES. Axiom of choice, category theory, epimorphisms of vector spaces, extensions of vector space endomorphisms, linear transformations, partial differential operators with constant coefficients, transpose of a linear partial differential operator, meromorphic functions, formal power series, global solvability of partial differential equations.
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1. INTRODUCTION.

A straightforward argument using transfinite induction and the axiom of choice will show that if L is a linear transformation of the vector space V over the field F onto itself, and W is a subspace of V such that L(W) is a proper subset of W, then there is a subspace E of V containing W such that L(E) = E and such that if F is any proper subspace of E containing W, then L(F) $\neq$ F. That is to say there is a smallest extension E of W in which the inhomogeneous equation

$$Lu = f \tag{1.1}$$

has a solution u in E for every f in E.

In section two of this paper we show that there is a vector space V, a subspace W of V, a pair of linear transformations L_1 and L_2 of V onto itself such that $L_1(W) \cap L_2(W)$ is a proper subspace of W and such that if E is any vector space containing W and contained in V such that $L_1(E) = L_2(E) = E$, then there is also a proper subspace F of E containing W such that $L_1(F) = L_2(F) = F$. Thus the minimal space of surjectivity question for families of mappings is not solvable.

If C is a category whose objects are sets, possibly equipped with some structure, and whose morphisms are mappings between the sets, which preserve the structure, then we can define the minimal space of surjectivity question as follows. Let C be such a category. Let V be an object in C and let W be a subobject of V, a subset of V which has the structure (if any) induced by that of V. Let L be a mapping that is an epimorphism of V in the sense that (e.g. Northcott [1], chapter III) V is the unique object in the category of vector spaces and linear transformations such that $I_V L$ and LI_V are defined where I_V is the identity mapping of V and L(V) = V. Further assume that if U is any subobject of V, then the restriction of L to U defines a morphism of the category whose range can be any subobject of V containing U. Then we say E is a solution of the minimal space of surjectivity problem defined by the triple (V,W,L) satisfying the preceding conditions if L(E) = E, L(W) is a proper subspace of W, and if F is any subspace of E containing W, then L(F) $\neq$ F. If there is a triple for which there is no solution to the minimal space of surjectivity problem we say that for the category the MSS question has a negative answer. If there is a triple (V,W,L) satisfying the above conditions for which the MSS problem does not have a unique answer, we say that there is nonuniqueness for the MSS question for the category.

In section 3 of this paper we show that the MSS question has a positive answer for the category of vector spaces and linear transformations, but in section 4 we show that we have nonuniqueness in this category.

In section 5 of this paper we show, for every nonzero linear partial differential operator with constant coefficients, the existence of a smallest extension of the meromorphic functions on which the operator is an epimorphism. We do this by exhibiting a locally convex topolgoical vector space containing the meromorphic functions on which every linear partial differential operator is an epimorphism and applying the results of the previous sections.

2. NONEXISTENCE OF A MSS FOR SOME FAMILIES OF EPIMORPHISMS.

It can be shown that if V is a vector space, W is a subspace of V, and L is an epimorphism of V such that L(W) is a proper subset of W, then there is a subspace E of V containing W which is minimal with respect to surjectivity in the sense that L(E) = E, but if U is any proper subspace of E containing W, then L(U) does not contain U. It seems natural to ask the same question for families of linear transformations.

MSS Question for Families of Mappings. Let V be a vector space. Let W be a subspace of V. Let $\mathcal{F}$ be a family of epimorphisms of V such that $L(W) \subset W$ for all L in $\mathcal{F}$ and $L(W) \neq W$ for at least one L in $\mathcal{F}$. Does there exist a subspace E of V containing W such that L(E) = E for every L in $\mathcal{F}$ having the additional property that if U is any subspace of E containing W, then $L(U) \neq U$ for some L in $\mathcal{F}$ if $U \neq E$.

The following theorem shows that the MSS question for families of linear transformations fails in general to have a positive answer even if $\mathcal{F}$ contains only two mappings.

Theorem 2.1. There exists a vector space V, two epimorphisms L_1 and L_2 of V, a subspace W of V such that $L_k(W) \subset W$ for k = 1, 2, such that $L_1(W) \cap L_2(W)$ is a proper subset of W, and having the additional property that if E is any subspace of V containing W such that $L_k(E) = E$ for k = 1, 2 then there is a proper subspace U of E containing W such that $L_k(U) = U$ for k = 1, 2.

Proof of Theorem 2.1. Let N denote the set of nonnegative integers. Let Q denote the set of all nonzero integer powers of the prime q. Let F denote an arbitrary field. For convenience we introduce the following.

Definition 2.1. If S is a set without a topology and $\Psi: S \to F$ is a mapping from the set S into a field F, then the support of Ψ is defined by the rule,

$$\operatorname{supp}(\Psi) = \{j \in S : \Psi(j) \neq 0\} .$$

We let V_0 denote the vector space of mappings from {0} into the field F. For every positive integer k we let V_k denote the vector space of mappings from Q into F whose support is a finite subset of Q. Let $P : V_k \to V_k$ denote the projector onto the space of functions whose supports are subsets of

$$N_q = \{q^{-n} : n = 1,2,3, \ldots\} . \tag{2.1}$$

Let $T : P(V_k) \to P(V_k)$ be a linear transformation defined by the rule

$$T(P\Psi_k)(q^{-2n}) = P\Psi_k(q^{-n}) \tag{2.2}$$

and

$$T(P\Psi_k)(q^{-(2n-1)}) = 0 \tag{2.3}$$

for all positive integers n and all functions Ψ_k in V_k.

Lemma 2.1. *If* Ψ *is a function in* $P(V_k)$ *whose support is precisely* $\{q^{-n_1}, q^{-n_2}, \ldots, q^{-n_r}\}$, *then the support of* $T\Psi$ *is exactly* $\{q^{-2n_1}, q^{-2n_2}, \ldots, q^{-2n_r}\}$.

Proof of Lemma 2.1. Suppose q^{-2n} were in the support of $T\Psi$. Then by definition n must be one of $\{n_1, n_2, \ldots, n_r\}$.

As a corollary of Lemma 2.1 we observe that Ker(T) is trivial.

Let $\tilde{P} : V_k \to V_k$ denote the projector defined by the rule

$$\tilde{P}\Psi(q^{2n}) = \Psi(q^{2n}) \tag{2.4}$$

for all positive integers n, and

$$\tilde{P}\Psi(q^{m}) = 0 \tag{2.5}$$

if m is an integer that is not equal to 2n for some positive integer n.

Let $B : V_1 \to V_0$ be defined by the rule,

$$B\Psi = \Phi , \tag{2.6}$$

where

$$\Phi(0) = \sum_{k=0}^{\infty} \Psi(q^{2n+1}) \quad . \tag{2.7}$$

Define a mapping,

$$S:V_k \to (I-P)V_k , \tag{2.8}$$

by the rule,

$$\begin{aligned} S\Psi(q^{2n}) &= \Psi(q^{n}) , \\ S\Psi(q^{2n-1}) &= \Psi(q^{-n}) , \end{aligned} \tag{2.9}$$

and

$$S\Psi(q^{-n}) = 0 \tag{2.10}$$

for all positive integers n.

Let V be the vector space over F defined to be the set of all

$$(\Psi_0, \Psi_1, \ldots, \Psi_k, \ldots)$$

where Ψ_k is a member of V_k for all nonnegative integers k and Ψ_k is identically zero for all but a finite number of nonnegative integers k, and let W denote the space of

$$w = (\Psi_0, \Phi_1, \ldots, 0, \ldots) \tag{2.11}$$

where Ψ_0 is a member of V_0 and Φ_1 is a member of $\bar{P}(V_1) \cap \mathrm{Ker}(B)$ where

$$\bar{P} = I - P - \tilde{P} . \tag{2.12}$$

Define a mapping $L_1 : V \to V$ by the rule,

$$L_1(\Psi_0,\Psi_1,\ldots,\Psi_n,\ldots) = (B(\bar{P}\Psi_1),TP\Psi_1 + \tilde{P}\Psi_1 + \Psi_2,\ldots,TP\Psi_n + \tilde{P}\Psi_n + \Psi_{n+1},\ldots) \tag{2.13}$$

Define a mapping $L_2:V \to V$ by the rule

$$L_2(\Psi_0,\Psi_1,\Psi_2,\ldots,\Psi_n,\ldots) = (\Psi_0,S\Psi_1 + \Psi_2 ,\ldots,S\Psi_n + \Psi_{n+1},\ldots) \tag{2.14}$$

Lemma 2.2. $L_k:V \to V$ is an epimorphism for $k = 1,2$, and $L_2:V \to V$ is an isomorphism.

Proof. That each L_k is an epimorphism is obvious since V_0 is one dimensional and Ψ_{n+1} covers the part of the nth coordinate space not covered by $TP\Psi_n + \tilde{P}\Psi_n$ or $S\Psi_n$, respectively. Also if each $S\Psi_n + \Psi_{n+1}$ is identically zero this implies that each Ψ_n is identically zero, if $\Psi_{n+1} = 0$, since $S:V_n \to (I-P)V_n$ is an isomorphism.

Now let us construct a space of surjectivity, E, for the operators L_1 and L_2 which contains W. Let $(\Psi_0,0,0,\ldots)$ denote a nonzero member of W. Then there exists a vector

$$(\tilde{\Psi}_0,\Psi_1,\ldots,\Psi_n,0,0,\ldots) = \tilde{v} \text{ in } V$$

such that

$$L_1\tilde{v} = (\Psi_0,0,\ldots,0,\ldots) .$$

Since $(-\tilde{\Psi}_0,0,\ldots,0,\ldots)$ is in W and E is a vector space we conclude that

$$v = (0,\Psi_1,\Psi_2,\ldots,\Psi_n,0,0,\ldots) , \tag{2.15}$$

it in E and $L_1v = L_1\tilde{v}$. Now suppose that $n > 1$. Then the nth entry of $L_1(0,\Psi_1,\ldots,\Psi_n,0,0,\ldots)$ is $TP\Psi_n + \tilde{P}\Psi_n$. Since $T:PV_n \to PV_n$ is a monomorphism by Lemma 2.1 we conclude that $P\Psi_n$ and $\tilde{P}\Psi_n$ are identically zero. Now we know that $P\Psi_{n-1} + \tilde{P}\Psi_{n-1} + \Psi_n$ is identically zero. Thus, the fact that $(P+\tilde{P})(P\Psi_{n-1}+\tilde{P}\Psi_n+\Psi_n)$ is identically zero implies that $P\Psi_{n-1} + \tilde{P}\Psi_{n-1}$ is identically zero. Hence, Ψ_n is identically zero. Since n represents an arbitrary integer larger than 2, we conclude that the vector v defined in (2.15) is of the form

$$v = (0,\Psi_1,0,0,\ldots)$$

where $P\Psi_1$ and $\tilde{P}\Psi_1$ are identically zero. We conclude that

$$\mathrm{supp}(\Psi_1) = \{q^{2n_k-1} : k = 1,2,\ldots,r\} \tag{2.16}$$

where $0 < n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots < n_r$. Presumably Ψ_0 is not identically zero and, consequently, $\Psi_1 \notin \mathrm{Ker}(B)$. Let $\mathrm{supp}(\Psi_1)$ denote the support of Ψ_1. Then we observe that $L_2 v \epsilon E$ and that there is some v_1 in E such that $L_2 v_1 = v = (0,\Psi_1,0,\ldots)$. We observe that L_2 is a one-to-one mapping. Hence $L_2^{-1}(\{v\})$ contains only a single element. Observe that if the support of Ψ_1 is defined by (2.16), then

$$L_2^{-1}(v) = (0,\Phi_1,0,0,\ldots,0,\ldots) \tag{2.17}$$

and

$$\text{supp}(\Phi_1) = \{q^{-n_k} : k = 1,2,\ldots,r\} . \tag{2.18}$$

Now we observe that since $P\Phi_1 = \Phi_1$, that

$$L_1^n v_1 = (0,T^n\Phi_1,0,\ldots,0,\ldots) . \tag{2.19}$$

Notice that the support of $T^n\Phi_1$ is given by

$$\text{supp}(T^n\Phi_1) = \{q^{-2^n n_1},q^{-2^n n_2},\ldots,q^{-2^n n_r}\} . \tag{2.20}$$

Now let us look at

$$L_2(0,T^n\Phi_1,0,\ldots,0,\ldots) = (0,ST^n\Phi_1,0,\ldots,0,\ldots) . \tag{2.21}$$

Observe that the support of $ST^n\Phi_1$ is given by

$$\text{supp}(ST^n\Phi_1) = \{q^{22^n n_1-1},\ldots,q^{22^n n_r-1}\} . \tag{2.22}$$

Observe that

$$B(ST^n\Phi_1) = \Psi_0 .$$

Thus, we can say that the vector space generated by the elements we know to be in E has the property that

$$W \subset L_1(E) \cap L_2(E) . \tag{2.23}$$

Indeed

$$L_1(0,ST^n\Phi_1,0,\ldots,0,\ldots) = (\Psi_0,0,0,\ldots,0,\ldots) . \tag{2.24}$$

Now the support of $S^mST^n\Phi_1$ is given by

$$\text{supp}(S^mST^n\Phi_1) = \{q^{2^m(2\cdot 2^n n_1-1)},\ldots,q^{2^m(2\cdot 2^n n_r-1)}\}$$

Now there must be a vector w in E of the form

$$w = (\phi_0,\phi_1,\phi_2,\ldots,\phi_n,\ldots)$$

such that

$$L_1(w) = (0,\Psi_1,0,\ldots,0,\ldots) .$$

Since W is a vector space and $(-\phi_0,0,0,\ldots,0,\ldots)$ is in W we may assume that w is actually of the form $(0,\phi_1,\phi_2,\ldots,\phi_n,\ldots)$, where $\phi_1 \in \mathrm{Ker}(B)$. But $\phi_1 \in \mathrm{Ker}(B)$ and $(P+\tilde{P})\phi_1 = 0$ implies that

$$(0,-\phi_1,0,\ldots,0,\ldots) \in W .$$

Thus, we may assume that

$$w = (0,0,\phi_2,\ldots,\phi_n,\ldots) .$$

We can show that if $n > 2$, that ϕ_n is identically zero. Thus, we may assume that

$$w = (0,0,\phi_2,0,\ldots,0,\ldots) .$$

Since $(P+\tilde{P})\phi_2 = 0$ we deduce that $\phi_2 = \Psi_1$. Hence, we deduce that

$$\{(0,\Psi_1,0,\ldots,0,\ldots), \quad (0,0,\Psi_1,0,\ldots,0,\ldots),\ldots\}$$

are all contained in E. Thus, the set of functions in E whose coordinate functions are in $\bar{P}(\mathrm{Ker}(B))$ or else are modulo a function in $\bar{P}(\mathrm{Ker}(B))$ a function with support sufficiently far out forms a proper subspace of E which is mapped onto itself by L_1 and L_2.

3. THE MINIMAL SPACE OF SURJECTIVITY PROBLEM FOR VECTOR SPACES AND LINEAR TRANSFORMATIONS

Let L be a linear transformation of a vector space V onto itself, and let W be a subspace of V such that L(W) is a proper subspace of W. The solvability of the MSS problem for the category of vector spaces and linear transformations is expressed in the following theorem.

Theorem 3.1. *Let* L, V, *and* W *be as defined in the introduction to this section. Then there is a subspace* E *of* V *containing* W *such that* L(E) = E *and such that if* E^1 *is a proper subspace of* E *containing* W, *then* $L(E^1) \neq E^1$.

Proof of Theorem 3.1. Let W_1 be a subspace of W such that $W = L(W) \oplus W_1$. Let $\mathcal{B}(W_1)$ be a bases for W_1. Let $\mathcal{E}_1(W)$ be a set consisting of precisely one member from each of the sets in the family

$$\{L^{-1}(w) : w \in \mathcal{B}(W_1)\} \tag{3.1}$$

The axiom of choice tells us that this set exists. We need the following result.

Lemma 3.1. Let $L : V \to V$ be a linear transformation of V onto itself. Let $\mathcal{T}$ be a linearly independent subset of V. If $\mathcal{S}$ is a set consisting of precisely one member from each of the sets in the family

$$\{L^{-1}(t) : t \in \mathcal{T}\} \tag{3.2}$$

then $\mathcal{S}$ is a linearly independent set.

Proof of Lemma 3.1. Let $v_1, \ldots, v_m$ be an arbitrary finite subset of $\mathcal{S}$. Let $c_1, \ldots, c_m$ be scalars. Then $c_1v_1 + \ldots + c_mv_m = 0$ implies $c_1L(v_1) + \ldots + c_mL(v_m) = 0$. But $\{L(v_1),\ldots,L(v_m)\}$ is an m-element subset of T and is, therefore, linearly independent. Hence, $c_1 = c_2 = \ldots = c_m = 0$.

Lemma 3.2. Let L, $\mathcal{T}$, and $\mathcal{S}$ be as defined in Lemma 3.1. If $[\mathcal{T}]$ and $[\mathcal{S}]$ denote the vector spaces generated by $\mathcal{T}$ and $\mathcal{S}$, respectively, and if $\mathcal{B}([\mathcal{S}])$ is a basis for $[\mathcal{S}]$, then $L(\mathcal{B}[\mathcal{S}]$ is a bases for $[\mathcal{T}]$.

Proof of Lemma 3.2. Let $\{u_1,\ldots,u_m\}$ be an m-element subset of $\mathcal{B}([\mathcal{S}])$. Write

$$u_j = \sum_{k=1}^{p} a_{(k,j)}v_k \quad (j = 1,\ldots,m) \tag{3.3}$$

where $\{v_1,v_2,\ldots,v_p\}$ is a p-element subset of $\mathcal{S}$ and the matrix

$$A = \begin{pmatrix} a_{(1,1)} & \cdots & a_{(1,m)} \\ \vdots & & \vdots \\ a_{(p,1)} & \cdots & a_{(p,m)} \end{pmatrix} \tag{3.4}$$

is a one-to-one linear transformation from m-dimensional space to p-dimensional space. Then L is a linear implies

$$L(u_j) = \sum_{k=1}^{p} a_{(k,j)}L(v_k) \ (j = 1,\ldots,m) \ . \tag{3.5}$$

Now suppose that we had

$$\sum_{j=1}^{m} c_jL(u_j) = 0 \tag{3.6}$$

Then interchanging summation signs we deduce from combining 3.6 and 3.5 that

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{m} a_{(k,j)}c_j\right) L(v_k) = 0 \tag{3.7}$$

But Lemma 3.1 implies that

$$\sum_{j=1}^{m} a_{(k,j)}c_j = 0 \tag{3.8}$$

for $k = 1,2,\ldots,p$. But the fact that the matrix A defined by 3.4 is one-to-one implies that

$$c_1 = c_2 = \dots = c_m = 0 . \tag{3.9}$$

Hence, the fact that (3.6) implies (3.9) for all m-element subsets $\{u_1,\dots,u_m\}$ of $\mathcal{B}([S])$ tells us that $L(\mathcal{B}[S])$ is a linearly independent set. To see that $L(\mathcal{B}[S])$ generates $[T]$ we let $L(v)$ denote an arbitrary element of T, where $v \in S$. Then there exist $u_1, \dots, u_m$ in $\mathcal{B}[S]$ such that

$$c_1u_1 + \dots + c_mu_m = v \tag{3.10}$$

By linearity of L and (3.10) we see that $L(v)$ is a linear combination of a finite number of elements in $L(\mathcal{B}[S])$.

Let $U_1 = W$ and let E_1 be the vector space generated by $\mathcal{E}(U_1)$ and U_1, where $\mathcal{E}(U_1)$ is an image of the choice function on the family of sets $\{L^{-1}(t):t \in \mathcal{B}(W_1)\}$. Then we may define

$$E_1 = U_1 \oplus [\mathcal{E}(U_1)]$$

where $[S]$ denotes the vector space generated by S for all subsets S of the vector space V.

<u>Lemma 3.3</u>. <u>Suppose</u> L <u>is a linear transformation of</u> V <u>into itself and</u> U_1 <u>is a subspace of</u> V <u>such that</u> $L(U_1)$ <u>is a proper subspace of</u> U_1. <u>Suppose</u> $\mathcal{E}(U_1)$ <u>is the set obtained by taking one member from each set in the family</u>

$$\{L^{-1}(t) : t \in B(W_1)\} \tag{3.11}$$

<u>where</u> $U_1 = L(U_1) \oplus W_1$. Define

$$E_1 = U_1 \oplus [\mathcal{E}(U_1)] . \tag{3.12}$$

<u>Then</u> E^1 <u>is a proper subspace of</u> E_1 <u>containing</u> U_1 <u>implies that</u> $L(E^1)$ <u>does not contain</u> U_1.

<u>Proof of Lemma 3.3</u>. Let π_1 be a projection of E_1 onto $[\mathcal{E}(U_1)]$. Then $\pi_1(E^1)$ must be a proper subspace of $[\mathcal{E}(U_1)]$ since the definition of direct sum and the fact that E^1 is a proper subspace of E_1 containing U_1 implies

$$E^1 = U_1 \oplus \pi_1(E^1) \tag{3.13}$$

We denote by $\mathcal{E}^1(U_1)$ a basis for $[\mathcal{E}(U_1)]$ which contains $\mathcal{B}(\pi_1(E^1))$, a basis for $\pi_1(E^1)$. By Lemma 3.2 we know that $L(\mathcal{E}^1(U_1))$ is a basis for W_1. Since $\pi_1(E^1)$ is a proper subspace of $[\mathcal{E}(U_1)]$, it follows that $\mathcal{B}(\pi_1(E^1))$ must be a proper subset of $\mathcal{E}^1(U_1)$. For if $\mathcal{B}(\pi_1(E^1))$ were equal to $\mathcal{E}_1^1(U_1)$, then we would have

$$\pi_1(E^1) = [\mathcal{B}(\pi_1(E^1)] = [\mathcal{E}^1(U_1)] = [\mathcal{E}(U_1)] \tag{3.14}$$

which contradicts the supposition that $\pi_1(E^1)$ is a proper subspace of $[\mathcal{E}(U_1)]$. Thus, (3.14) and the definition of π_1 imply that

$$L(E^1) = L(U_1) \oplus L(\pi_1(E^1)), \tag{3.15}$$

and consequently that $L(E^1)$ is a proper subspace $L(U_1) \oplus W_1 = U_1$.

Hence, $L(E^1)$ does not contain U_1. We use transfinite induction to construct for every ordinal α less than $\delta + 1$, where δ is the ordinality of a basis for V, a set E_α which is minimal with respect to the property that

$$U_\alpha = W \oplus (\cup\{E_\gamma : \gamma < \alpha\} \subset L(E_\alpha) \tag{3.16}$$

and

$$\cup\{E_\gamma : \gamma < \alpha\} \subset E_\alpha \tag{3.17}$$

in the sense that if

$$W_\alpha \subset E^1 \subsetneq E_\alpha \tag{3.18}$$

then $L(E^1)$ does not contain U_α. We have constructed E_α for $\alpha = 1$. Thus, we suppose that E_α has been constructed for all $\beta < \alpha$. Then define U_α as in (3.16). It is clear that U_α is a vector space, since W and each E_γ is a vector space and $\gamma_1 < \gamma_2$ implies $E_{\gamma_1} \subset E_{\gamma_2}$. Let W_α be a subspace of V such that

$$U_\alpha = L(U_\alpha) \oplus W_\alpha . \tag{3.19}$$

Let $\mathcal{B}(W_\alpha)$ be a basis for W_α. Let $\mathcal{E}(U_\alpha)$ be defined to be the set obtained by taking one element from each set in the family

$$\{L^{-1}(w) : w \in \mathcal{B}(W_\alpha)\} . \tag{3.20}$$

We let E_α be the vector space generated by U_α and $\mathcal{E}(U_\alpha)$. Then the Lemmas 3.1, 3.2 and the previous argument show that if E^1 satisfies (3.18), then $L(E^1) \neq E^1$. Since $L(V) = V$, it is clear that there is some ordinal α_0 such that $L(E_{\alpha_0}) = E_{\alpha_0}$. But the ordinals are well ordered. Thus, we may suppose

$$\beta = \min\{\gamma : L(E_\gamma) = E_\gamma\} \tag{3.21}$$

Indeed, it is easy to see that β is the first infinite ordinal.

The following lemma gives important information about the space E_β.

<u>Lemma 3.4</u>. <u>If</u> β <u>is defined by</u> (3.21), <u>then</u> $U_\beta = E_\beta$.

<u>Proof of Lemma 3.4</u>. Suppose $U_\beta \neq E_\beta$. Then $L(U_\beta)$ would be a proper subspace of U_β. It is easy to show by transfinite induction that $L(U_\alpha) \subset U_\alpha$ for every ordinal α. Then we can write

$$U_\beta = L(U_\beta) \oplus W_\beta \tag{3.22}$$

Then we would let $\mathcal{B}(W_\beta)$ be a basis for W_β and let $\mathcal{E}(U_\beta)$ be the set consisting of one member from each set in the family,

$$\{L^{-1}(w) : w \varepsilon \mathcal{B}(W_\beta)\} \tag{3.23}$$

Then E_β is the vector space generated U_β and $\mathcal{E}(U_\beta)$ and clearly $L(E_\beta) \subset U_\beta$. This contradicts the supposition that $L(E_\beta) = E_\beta$.

Lemma 3.5. *For every ordinal* α *less than* $\delta + 1$ let

$$U_\alpha = \{E_\gamma : \gamma < \alpha\} , \tag{3.24}$$

let W_α be a subspace of V such that

$$U_\alpha = L(U_\alpha) \oplus W_\alpha , \tag{3.25}$$

let $\mathcal{B}(W_\alpha)$ *be a basis for* W_α, *let* $\mathcal{E}(U_\alpha)$ *be a set consisting of precisely one element from each set in the family*

$$\{L^{-1}(w) : w \varepsilon \mathcal{B}(W_\alpha)\} , \tag{3.26}$$

and let

$$\pi_\alpha : V \to [\mathcal{E}(U_\alpha)] \tag{3.27}$$

be a projector of V *onto the vector space* $[\mathcal{E}(U_\alpha)]$ *generated by* $\mathcal{E}(U_\alpha)$. *Then*

$$[\mathcal{E}(U_\alpha)] \cap U_\alpha = \{0\} \tag{3.28}$$

and for every vector v *in* V

$$v = w + \sum \pi_\alpha(v) \; (1 \le \alpha < \delta + 1) , \tag{3.29}$$

where w *is a member of* W.

Proof of Lemma 3.5. To prove (3.28) note that if $v \varepsilon [\mathcal{E}(U_\alpha)] \cap U_\alpha$ then $L(v) \varepsilon L(U_\alpha) \cap W_\alpha = \{0\}$. Thus, $v = c_1 v_1 + \ldots + c_m v_m$ for some elements $v_1, \ldots, v_m$ in $\mathcal{E}(U_\alpha)$ and some scalars $c_1, \ldots, c_m$. But $\{L(v_1), \ldots, L(v_m)\}$ is an m-element subset of $\mathcal{B}(W_\alpha)$, a basis for W_α, and is, therefore, a linearly independent set. Hence, $L(v) = 0$ implies $v = 0$. In other words (3.28) is valid. Next we show that $\alpha < \beta$ implies $\pi_\alpha(V) \cap \pi_\beta(V) = \{0\}$. Now

$$[\mathcal{E}(U_\alpha)] \subset E_\alpha \subset U_\beta \tag{3.30}$$

and by (3.28)

$$U_\beta \cap [\mathcal{E}(U_\beta)] = \{0\} \tag{3.31}$$

Combining (3.30) and (3.31) we deduce that $[\mathcal{E}(U_\alpha)] \cap [\mathcal{E}(U_\beta)] = \{0\}$ if α and β are ordinals and $\alpha < \beta$. Then (3.29) is an immediate consequence of the fact that

$$V = W \oplus (\oplus\sum [\mathcal{E}\ (U_\alpha)](1 \leq \alpha < \delta + 1)) \tag{3.32}$$

<u>Proof of Theorem 3.1</u>. Let E^1 be a proper subset of E_β which contains W, where β is defined by (3.21). Then $E_\beta \cap E^1 = E^1 \neq E_\beta$. Thus, we deduce that

$$\{\gamma : E^1 \cap E_\gamma \neq E_\gamma\} \neq \phi \tag{3.33}$$

The well ordering property and (3.33) enable us to define

$$\alpha = \min\{\gamma : E^1 \cap E_\gamma \neq E_\gamma\} . \tag{3.34}$$

From the fact that $\gamma < \alpha$ implies $E^1 \cap E_\gamma = E_\gamma$ we deduce that

$$U_\alpha \cap E^1 = U_\alpha \tag{3.35}$$

or that U_α is a subspace of E^1 if α is defined by (3.34). Thus, Lemma 3.5 implies that

$$(\oplus\sum_{\gamma\geq\alpha} \pi_\gamma(E^1)) = (I-\tilde{\pi}_\alpha)(E^1)$$

where $\tilde{\pi}_\alpha$ is the projection of V onto U_α. Furthermore,

$$E^1 \subset U_\alpha \oplus (I-\tilde{\pi}_\alpha)(E^1) \tag{3.36}$$

But $e^1 \in E^1$ implies there is a $u_\alpha \in U_\alpha$ and a $v_\alpha \in (I-\tilde{\pi}_\alpha)(E^1)$ such that

$$e^1 = u_\alpha + v_\alpha$$

But $e^1 - u_\alpha \in E^1$ implies $v_\alpha \in E^1$. Thus, from (3.36) we deduce that

$$E^1 = U_\alpha \oplus ((I-\tilde{\pi}_\alpha)(E^1) \cap E^1) . \tag{3.37}$$

In order that $L(E^1) = E^1$ we must have in particular that $U_\alpha \subset L(E^1)$. But

$$U_\alpha = L(U_\alpha) \oplus L([\mathcal{E}(U_\alpha)]) \tag{3.38}$$

Now $(I-\tilde{\pi}_\alpha)(E^1)) \cap E^1$ is a proper subspace of $(I-\tilde{\pi}_\alpha)(V)$. We want to show that $L(E^1)$ could not possibly contain U_α. Suppose $L(E^1)$ did contain U_α. Then $L(E^1) \cap U_\alpha = L(U_\alpha) \oplus \tilde{W}_\alpha = U_\alpha$. Let $\mathcal{B}(\tilde{W}_\alpha)$ be a basis for $\tilde{W}_\alpha$. Then $\mathcal{B}(\tilde{W}_\alpha) \subset L(E^1)$. Then each set in the family

$$\{L^{-1}(w) \cap E^1 : w \in \mathcal{B}(\tilde{W}_\alpha)\} \tag{3.39}$$

must be nonempty. Let $E^1(U_\alpha)$ be a set consisting of one member from each set in the family (3.39). Then $[E^1(U_\alpha)] \subset E^1 \subset E_\beta \subset V$. Let γ be an ordinal number larger than α. Therefore $L([E^1(U_\alpha)]) \subset U_\alpha$ and $L([E(U_\gamma)]) \cap U_\alpha = \{0\}$. Thus, $[E(U_\gamma)] \cap [E^1(U_\alpha)] = \{0\}$ for all ordinals $\gamma > \alpha$. Thus, $E^1 \subset E_\beta$ implies that

$$[E^1(U_\alpha)] \subset E^1 \subset E_\beta = E_\alpha \oplus (\oplus\sum[E\ (U_\gamma)]\ \alpha < \gamma < \beta) \tag{3.40}$$

Thus, $v \in [E^1(U_\alpha)]$ implies

$$v = v_\alpha + (\sum v_\gamma,\ \alpha < \gamma < \beta)\ . \tag{3.41}$$

$L(v-v_\alpha) \in U_\alpha$ and

$$L(\oplus\sum v_\gamma, \alpha < \gamma < \beta) = 0 \tag{3.42}$$

or else

$$L(\oplus\sum v_\gamma, \alpha < \gamma < \beta) \notin U_\alpha \tag{3.43}$$

But since $L(v-v_\alpha) \in U_\alpha$, it follows that (3.43) is impossible. Hence, $L(v-v_\alpha) \in U_\alpha$. But L is one-to-one on each of the subspaces $[E\ (U_\gamma)]$ and is consequently one-to-one on their direct sum. Thus, $v_\gamma = 0$ for all ordinals γ such that $\alpha < \gamma < \beta$, where the v_γ are defined by (3.41). Hence, $v = v_\alpha$. This implies that $[E^1(U_\alpha)]$ is a subspace of E_α and since $[E^1(U_\alpha)] \cap U_\alpha = \{0\}$ is consequently a subspace of $[E(U_\alpha)]$. But this would imply that

$$U_\alpha \oplus [E^1(U_\alpha)] = E^1 \cap E_\alpha$$

is a subspace of E_α with the property that $U_\alpha \subset L(U_\alpha \oplus [E^1(U_\alpha)])$. This implies by Lemma 3.2 that

$$U_\alpha \oplus [E^1(U_\alpha)] = U_\alpha \oplus [E(U_\alpha)]\ .$$

Hence, $E^1 \cap E_\alpha = E_\alpha$, which contradicts (3.34).

4. NONUNIQUENESS OF THE MSS PROBLEM FOR THE CATEGORY OF VECTOR SPACES AND LINEAR TRANSFORMATIONS

In this section we prove that if V is a vector space with a subspace W and L is an epimorphism of V which maps W into a proper subspace of itself, then there does not necessarily exist a unique subspace E of V containing W such that L(E) = E and $L(E^1) \neq E^1$ for all proper subspaces E^1 of E containing W. We also define the semigroup $S(V, L, W)$ of endomorphisms of V which commute with L and leave W fixed.

Theorem 4.1. Let F be a field. Then there exists a pair of vector spaces V and W such that both V and W have countable dimension, with W being a subspace of V, such that there exists a linear transformation L of V onto itself which maps W into but not onto itself. Furthermore, V, W, and L described above can be constructed in such a way that there exists a sequence of spaces $\{V^{(m)} : m \in \mathcal{N}\}$ such that $V = V^{(0)} \underset{\neq}{\supset} V^{(1)} \underset{\neq}{\supset} V^{(2)} \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} W$, L maps each $V^{(m)}$ linearly onto itself, and

$$\bigcap_{m=0}^{\infty} V^{(m)} = W \tag{4.1}$$

In addition there exists an uncountable family of subspaces $\{E_\alpha : \alpha \in A\}$ of V containing W such that L maps each E_α linearly onto itself and such that if E_α^1 is a subspace of V satisfying

$$W \subset E_\alpha^1 \underset{\neq}{\subset} E_\alpha , \tag{4.2}$$

then $L(E_\alpha^1)$ is not equal to E_α^1. Furthermore, we may construct the minimal spaces of surjectivity E_α so that if $\alpha_1 \neq \alpha_2$, then

$$E_{\alpha_1} \cap E_{\alpha_2} = W \tag{4.3}$$

if we ask only that A be countable.

Proof. Let V_i denote the set of all functions from $\mathcal{N} = \{0,1,2,\ldots\}$ into F which vanish for almost all members of $\mathcal{N}$ for $i = 1,2,3,4,\ldots$

Let

$$V = V_1 \times V_2 \times V_3 \times \ldots \times V_i \times \ldots \tag{4.4}$$

Let

$$W = V_1 \times \{\underline{0}\} \times \ldots \times \{\underline{0}\} \times \ldots \tag{4.5}$$

where $\underline{0}$ is the zero element of V_i for $i = 1,2,\ldots$ Let

$$\gamma : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \tag{4.6}$$

be a bijection. Define $L_1 : V_1 \to V$ by the rule,

$$\begin{aligned} L_1(v) &= (\chi_{2\mathcal{N}} v, \underline{0}, \underline{0}, \ldots) \\ &= ((v(0),0,v(2),0,v(4),0,\ldots),\underline{0},\underline{0},\ldots) \end{aligned} \tag{4.7}$$

Define

$L_2 : V_2 \to V$ by the rule,

$$L_2(v) = ((0, \sum_{n=0}^{\infty} v(\gamma(0,n)), 0, \sum_{n=0}^{\infty} v(\gamma(1,n)), 0, \ldots), \underline{0}, \underline{0}, \ldots) \tag{4.8}$$

where γ is defined by (4.6). Define $L_3 : V_3 \to V$ by the rule

$$L_3(v) = (\underline{0}, v, \underline{0}, \underline{0}, \ldots) . \tag{4.9}$$

If m is an integer larger than three, define $L_m : V_m \to V$ by the rule,

$$L_m:(V) = (\underline{0}, \ldots, \underset{\substack{\uparrow \\ (m-2)\text{nd} \\ \text{position}}}{\underline{0}}, v, \underset{\substack{\uparrow \\ m\text{th} \\ \text{position}}}{\underline{0}}, \underline{0}, \ldots) \tag{4.10}$$

Define $L : V \to V$ by the rule,

$$L(v^{(1)}, v^{(2)}, \ldots) = \sum_{i=1}^{\infty} L_i v^{(i)} \tag{4.11}$$

Then L maps V linearly onto itself and L maps W linearly into, but not onto, itself. Define

$$V^{(m)} = V_1 \times V_2^{(m)} \times V_3^{(m)} \times \ldots \tag{4.12}$$

where for each $i \in \{1,2,3,\ldots\}$,

$$V_i^{(m)} = \{ v : \mathbb{N} \to F \mid v(n) = 0 \quad \text{for} \quad n < m \} , \tag{4.13}$$

We first show that $L : V^{(m)} \to V^{(m)}$ is a linear mapping of $V^{(m)}$ onto itself. Let

$$((v(0), v(1), v(2), \ldots), v^{(2)}, v^{(3)}, \ldots) \tag{4.14}$$

denote an arbitrary member of $V^{(m)}$.

Let $P_n = \gamma(n, m_n)$, where

$$m_n = \inf\{ k \in \mathbb{N} : \gamma(n,k) > m \} \tag{4.15}$$

Define

$$w^{(1)}(2n) = v(n) \text{ for } n \in \mathbb{N}$$

and define

$$w^{(1)}(2n+1) = 0$$

for $n \in \mathbb{N}$. Define

$$w^{(2)}(P_n) = v(2n+1) \text{ for } n \in \mathbb{N}$$

and define

$$w^{(2)}(k) = 0 \text{ if } k \notin \{P_0, P_1, P_2, \ldots\} .$$

Define

$$w^{(3)} = v^{(2)}$$

and

$$w^{(k+1)} = v^{(k)} \quad \text{for} \quad k = 2,3,\ldots$$

Then

$$L(w^{(1)},w^{(2)},w^{(3)},\ldots) = (v,v^{(2)},v^{(3)},\ldots) \tag{4.16}$$

which shows that L maps $V^{(m)}$ linearly onto itself. Let $\pi_k : V \to V_k$ be the natural projection.

Lemma 4.1. We may write

$$E = \pi_1 E \times \pi_2 E \times \pi_3 E \times \ldots$$

The proof of Lemma 4.1 is immediate. Let $E_k = \pi_k E$ for $k = 1,2,\ldots$ Note that E_2 and hence every E_k must be infinite dimensional, since by hypothesis $E_1 = V_1$. The proof we have just repeated shows that if we just require that $\text{supp}(E_2)$, the support of the functions in E_2, satisfy $\text{supp}(E_2) \cap \{\gamma(n,0), \gamma(n,1), \gamma(n,2), \ldots\} \neq \emptyset$ for every n, and take $E_k = E_2$ for all $k = 3,4,5,\ldots$ then L will map E linearly onto itself. Also E will be minimal provided that

$$\text{supp}E_2 \cap \{\gamma(n,0), \gamma(n,1), \gamma(n,2), \ldots\} \tag{4.17}$$

has just one element in it for every $n \in N$. There are clearly an uncountable number of ways of choosing E_2 so that the set (4.17) has just one element in it for every $n \in N$. Also, the number of ways of choosing E_2 so that condition (4.2) satisfied is at most countable since for each integer n it must be true that

$$\text{supp}(\pi_2 E_{\alpha_1}) \cap \{\gamma(n,0), \gamma(n,1),\ldots\}$$

is not equal to

$$\text{supp}(\pi_2 E_{\alpha_2}) \cap \{\gamma(n,0), \gamma(n,1),\ldots\} .$$

Definition 4.1. Let L be an epimorphism of a vector space V which has a subspace W with the property that L(W) is a proper subspace of itself. Let $S(V,L,W)$ be the semi-group of all endomorphisms of V which commute with L and leave elements of W fixed.

Theorem 4.2. If L, V, and W satisfy the conditions of Definition 4.1 then $S(V,L,W)$ contains only one element if and only if E is a proper subspace of V containing W implies $L(E) \neq E$.

Proof. Suppose E were a subspace of V containing W. Let us write $V = \text{Ker}(L) \oplus F$ and write $E = \text{Ker}(L) \cap E \oplus \tilde{F}$. Let $B(\tilde{F})$ be a basis for $\tilde{F}$, and let $B(F)$ be a basis for F containing $B(\tilde{F})$. Let $B(\text{Ker}(L))$ be a basis for Ker(L) containing $B(\text{Ker}(L) \cap E)$, a basis for $\text{Ker}(L) \cap E$. Then each v in V may be written as

$$v = a_1\tilde{w}_1 + \dots + a_p\tilde{w}_p + b_1w_1 + \dots + b_qw_q$$
$$+ c_1\tilde{u}_1 + \dots + c_r\tilde{u}_r + d_1u_1 + \dots + d_su_s \tag{4.18}$$

where

$$\{\tilde{w}_1,\dots,\tilde{w}_p\} \subset \mathcal{B}(\mathrm{Ker}(L) \cap E) , \tag{4.19}$$

$$\{w_1,\dots,w_q\} \subset \mathcal{B}(\mathrm{Ker}(L)) - \mathcal{B}(\mathrm{Ker}(L) \cap E) , \tag{4.20}$$

$$\{\tilde{u}_1,\dots,\tilde{u}_r\} \subset \mathcal{B}(\tilde{F}) , \tag{4.21}$$

and

$$\{u_1,\dots,u_s\} \subset \mathcal{B}(F) - \mathcal{B}(\tilde{F}) \tag{4.22}$$

Then define

$$\pi(v) = a_1\tilde{w}_1 + \dots + a_p\tilde{w}_p + c_1\tilde{u}_1 + \dots + c_r\tilde{u}_r \tag{4.23}$$

We deduce from the definition (4.23) of π and the fact that $L(E) = E$, that

$$L(\pi(v)) = c_1L(\tilde{u}_1) + \dots + c_rL(\tilde{u}_r) = \pi(L(v)) \tag{4.24}$$

Thus, the projector π we have constructed belongs to $\mathcal{S}(V,L,W)$. Conversely, if ψ belongs to $\mathcal{S}(V,L,W)$ then $E = \psi(V)$ satisfies $L(E) = L(\psi(V)) = \psi(L(V)) = E$. It is clear that the identity transformation I belongs to $\mathcal{S}(V,L,W)$. From our construction it is clear that if E is a proper subspace of V such that $L(E) = E$, then the projector π we have constructed is distinct from the identity. This completes the proof of the Theorem.

5. SURJECTIVITY OF DIFFERENTIAL OPERATORS ON LOCALLY CONVEX SPACES CONTAINING THE MEROMORPHIC FUNCTIONS

Let $\mathcal{R}(C^n)$ denote the meromorphic functions of n complex variables. We construct a special locally-convex space $E^{(n)}$ containing $\mathcal{R}(C^n)$ such that every nontrivial linear partial differential operator with n independent variables and constant coefficients maps $E^{(n)}$ continuously onto itself.

For each $j \in \{1,\dots,n\}$ and for each $u(X) \in C[[X_1,\dots,X_n]]$ define

$$T_ju(X) = u(X_1,\dots,X_{j-1},1-X_j,X_{j+1},\dots,X_n) . \tag{5.1}$$

Let $F^{(n)} = \sum_{\gamma} E_{\gamma}^{(n)}$ (exterior direct sum), where γ runs through the set $\mathcal{I}$, where

$$\mathcal{I} = \{0,1\} \times \dots \times \{0,1\} \quad \text{(n factors)}, \tag{5.2}$$

where

$$E_0^{(n)} = C[[X_1,\ldots,X_n]] \;,$$

$$E_\gamma^{(n)} = T_1^{\gamma_1} \ldots T_n^{\gamma_n} C[[X_1,\ldots,X_n]] \;, \tag{5.3}$$

and we agree that $T_j^0 = I$, the identity map on $C[[X_1,\ldots,X_n]]$. These spaces of formal power series are equipped with the usual locally-convex topology of simple convergence of coefficients.

Proposition 1. The dual space of $E_\gamma^{(n)}$ is isomorphic to the space $C[T_1^{\gamma_1}X_1,\ldots,T_n^{\gamma_n}X_n]$ of polynomials. Furthermore, $E_\gamma^{(n)}$ is a reflexive Frechet space for all γ in I, where I is given by (5.2).

Proof. It is well-known (e.g. Trèves [2], page 266) that $E_0^{(n)}$ and $C[X_1,\ldots,X_n]$ are duals of one another. But if $E' \subset E$, where E is a topological vector space, and $J : E \to F$ is an isomorphism then $F' = J(E')$. Furthermore, if E is a Frechet space, F is a locally convex space and $J : E \to F$ is a topological isomporphism, then F is also a Frechet space.

Definition 1. The duality bracket between a polynomial P in $C[X_1,\ldots,X_n]$ and a formal power series u in $C[[X_1,\ldots,X_n]]$ is given by

$$\langle P,u\rangle = \sum_{\alpha\in N^n} \left(\frac{1}{\alpha!}\right) [(\partial/\partial X)^\alpha P(X)]_{X=0}[(\partial/\partial X)^\alpha u(X)]_{X=0} \;. \tag{5.4}$$

The duality bracket between a polynomial Q_γ in the dual of $E_\gamma^{(n)}$ and a formal power series v_γ in $E_\gamma^{(n)}$ is given by

$$\langle Q_\gamma,v_\gamma\rangle = \sum_{\alpha\in N^n} \left(\frac{1}{\alpha!}\right) [(\partial/\partial X)^\alpha Q_\gamma]_{X=\gamma} [(\partial/\partial X)^\alpha v_\gamma]_{X=\gamma} \;. \tag{5.5}$$

It is easy to see that if

$$P(\partial/\partial X) = \sum_{\alpha\in N^n} (1/\alpha!)[(\partial/\partial X)^\alpha P(X)]_{X=\gamma}(\partial/\partial X)^\alpha \tag{5.6}$$

and $u \in E_0^{(n)}$, then

$$\langle P,u\rangle = [P(\partial/\partial X)u]_{X_i=\gamma_i} \qquad i \in \{1,\ldots,n\} \tag{5.7}$$

It is similarly easy to check that if

$$Q_\gamma(\partial/\partial X) = \sum_{\alpha\in N^n} (1/\alpha!)[(\partial/\partial X)^\alpha T^\gamma Q_\gamma(X)]_{X=\gamma}(\partial/\partial X)^\alpha \tag{5.8}$$

and $v_\gamma \in E_\gamma^{(n)}$, then

$$\langle Q_\gamma,v_\gamma\rangle = [Q_\gamma(\partial/\partial X)v_\gamma]_{X_i=\gamma_i} \qquad i \in \{1,\ldots,n\} \tag{5.9}$$

By E. Borel's Theorem (e.g. Trèves [3], Theorem 18.1) if u belongs to $E_0^{(n)}$, there is a ϕ in $C_0^\infty(R^n)$ such that the coefficients of the Taylor series expansion for ϕ about

$X^{(0)} = (0,\dots,0)$ are identical to the coefficients of u. Thus, we may rewrite (7) as

$$<P,u> = <P(-\partial/\partial X)\delta,\phi> \ , \tag{5.10}$$

where δ is the Dirac delta function. Again by E. Borel's Theorem (e.g. Trèves [3], Theorem 18.1) there is for every v_γ in $E_\gamma^{(n)}$ a ϕ_γ in $C_0^\infty(R^n)$ such that the coefficients of the Taylor series expansion of $\phi^{(j)}$ about γ are equal to the corresponding coefficients of v_j, where $X^{(\gamma)} = \gamma$. Thus, we may rewrite (5.9) as

$$<Q_\gamma,v_\gamma> = <Q_\gamma(-\partial/\partial X)\delta(X-X^{(\gamma)}),\phi_\gamma> \ . \tag{5.11}$$

Let $L = L(\partial/\partial X)$ denote a linear partial differential operator with constant coefficients. We can use m (5.10) and (5.11) to determine the action to ${}^tL$ on $E_0^{(n)\prime}$ and $E_\gamma^{(j)\prime}$ for all γ in I, where I is given by (5.2). It is well known that

$$<P,L(\partial/\partial X)u> = <P(-\partial/\partial X)\delta,L(\partial/\partial X)\phi> \tag{5.12}$$

implies that the transpose of L is one-to-one, since it follows that

$$^tLP(X) = L(X)P(X) \ . \tag{5.13}$$

Similarly, if $Q_\gamma \in E_\gamma^{(n)\prime}$ and $v_\gamma \in E_\gamma^{(n)}$, then

$$<Q_\gamma,L(\partial/\partial X)u> = <L(-\partial/\partial X)P(-\partial/\partial X)\delta,\phi_\gamma> \ . \tag{5.14}$$

Thus, for every polynomial Q_γ in $E_\gamma^{(n)\prime}$

$$^tLQ_\gamma = L(T_1^{\gamma_1}X_1,\dots,T_n^{\gamma_n}X_n)Q_\gamma \tag{5.15}$$

for all γ in I. By Theorem 28.1 of Trèves [3] it follows that $L(X)E_\gamma^{(n)\prime}$ is a closed subspace of $E_\gamma^{(n)\prime}$ for every γ in I. This, in view of a classical theorem due essentially to S. Banach, which states that a continuous linear map L of one Frechet space E into another Frechet space F is an epimorphism if and only if its transpose is one-to-one and weakly closed, implies that $L : E_\gamma^{(n)} \to E_\gamma^{(n)}$ is an isomorphism for all γ in I.

Let $V_\gamma^{(n)}$ be the subspace of $E_\gamma^{(n)}$ consisting of all members of $E_\gamma^{(n)}$ which may be identified with a member of $\sum_{\gamma'\neq\gamma} E_{\gamma'}^{(n)}$ (exterior direct sum). Then L is still an epimorphism of $E_\gamma^{(n)}/V_\gamma^{(n)}$ since $L(V_\gamma^{(n)}) \subset V_\gamma^{(n)}$. Note that $V_\gamma^{(n)}$ is closed. The space $R(C^n)$ can be identified with a subspace of

$$E^{(n)} = E_n^{(n)}/V_n^{(n)} \oplus \dots \oplus E_1^{(n)}/V_1^{(n)} \oplus E_0^{(n)} \ . \tag{5.16}$$

Thus, $R(C^n)$ can, when regarded as a vector space, be given a locally convex topology in a natural way, namely the one induced by $E^{(n)}$.

Let $f(z_1,\dots,z_n)/g(z_1,\dots,z_n)$ be a member of $R(C^n)$. Let $\pi_j:C^n \to C^{n-1}$ be defined by the rule,

$$\pi_j(z) = (z_1,\dots,z_{j-1},z_{j+1},\dots,z_n) \ . \tag{5.17}$$

Let us write

$$g(z_1,\dots,z_n) = g_0(\pi_j(z)) + z_j g_1(\pi_j(z)) + \dots + z_j^r g_r(\pi_j(z)) + \dots \ .$$

Let r be the smallest positive integer such that $g_r(\pi_j(z)) \neq 0$. Then

$$\frac{1}{g(z)} = \frac{1}{z_j^r} \sum_{k=0}^{\infty} h_k(\pi_j(z)) z_j^k$$

$$= \frac{h_0(\pi_j(z))}{z_j^r} + \dots + \frac{h_{r-1}(\pi_j(z))}{z_j} + \sum_{k=r}^{\infty} h_k(\pi_j(z)) z_j^{k-r} \ , \tag{5.18}$$

where the functions $h_k(\pi_j(z))$, are members of $R(C^{n-1})$. Proceeding in this manner we deduce that each representative of a member of $R(C^n)$ is contained in

$\sum_{\gamma\varepsilon} T^\gamma C[[X_1,\dots,X_n]]$ (exterior direct sum) or eliminating redundancy that

$$R(C^n) \subset C[[X]] \oplus \sum_{\substack{\gamma\varepsilon \\ \gamma\neq 0}} T^\gamma C[[X]]J/V_\gamma^{(n)} \ . \tag{5.19}$$

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