A NOTE ON ONE-PARAMETER GROUPS OF AUTOMORPHISMS

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ABSTRACT. Let $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ be two commuting one-parameter groups of *-automorphisms of a von Neumann algebra M such that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$. The purpose of this note is to provide a simple and short proof of the central decomposition result: $\alpha_t = \beta_t$ on Mp and $\alpha_t = \beta_{-t}$ on M(1-p) for a central projection $p \in M$, without using the theory of spectral subspaces.

KEY WORDS AND PHRASES. Automorphism groups, central projections, invariant projections.

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1. INTRODUCTION

In [1], A. Van Daele, L. Vanheeswijck and one of the authors proved that if $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ are strongly continuous one-parameter groups of *-automorphisms of a von Neumann algebra M satisfying the equation: $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$, then M can be decomposed into subalgebras Mp and M(1-p) for a central projection p in M such that $\alpha_t = \beta_t$ on Mp and $\alpha_t = \beta_{-t}$ on M(1-p), $t \in R$. The proof of this result is very technical and fairly long. It depends on Arveson's theory of spectral subspaces ([2], [3]) and as such it lacks a proper emphasis on the decomposition itself. However, there are important situations where it is enough to consider the commuting automorphism groups (see, for instance [4], [5]). The purpose of this note is to provide a simple proof of this decomposition result for commuting automorphism groups without using the theory of spectral subspaces. We use simple algebraic techniques to obtain the proof. Of course, in doing so we lose the generality of the result but on the other hand we get a simple proof. For more details concerning the origin of this operator equation, its applications and related decomposition results, we refer to [6], [7], [8], [9], [10], [4] and [5].

2. MAIN RESULTS

The following is an important decomposition result for commuting automorphisms ([10], [11]).

PROPOSITION 2.1. Let α, β be commuting *-automorphisms of a von Neumann algebra M such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists a central projection p in M such that $\alpha = \beta$ on Mp and $\alpha = \beta^{-1}$ on M(1-p).

In this paper, however, we use another version (Proposition 2.2) of the above result for the sake of clarity. The essential idea is that when we consider the *-automorphism $\alpha\beta$ on M, then by [12], $N(\alpha\beta - 1) + R(\alpha\beta - 1)$ is σ- weakly dense in M where $N(\alpha\beta - 1)$ and $R(\alpha\beta - 1)$ denote, respectively, the null space and the range space of the operators under consideration. $R(\alpha\beta - 1) \subseteq N(\alpha - \beta)$ and the subalgebra L (say) generated by $R(\alpha\beta - 1)$ is a two-sided ideal in M and there exists a largest central projection p_0 in M such that $L = Mp_0$, $Mp_0 \subseteq N(\alpha - \beta)$ and p_0 is α, β- invariant (that is $\alpha(p_0) = \beta(p_0) = p_0$). In other words, $\alpha = \beta$ on Mp_0 (see for instance [10], [11]). Similarly, by considering the orthogonal ideal $L^{\perp}$ (note that $L^{\perp} \subseteq N(\alpha\beta - 1)$) , we get a largest α, β- invariant central projection q_0 in M such that $\alpha = \beta^{-1}$ on Mq_0 and the orthogonality relation implies that

$(1 - p_0)(1 - q_0) = 0$. For more details, we may refer to [10] and [11]. Thus we may have the following alternative version of Proposition 2.1.

Proposition 2.2. Let α, β be commuting *-automorphisms of a von Neumann algebra M such that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Among the projections $p \in M$ (respectively $q \in M$) such that $\alpha = \beta$ on Mp (respectively $\alpha = \beta^{-1}$ on Mq) there exists a largest central projection p_0 in M (respectively q_0 in M) such that p_0 and q_0 are α, β- invariant and $(1 - p_0)(1 - q_0) = 0$.

We now come to our main result.

THEOREM 2.3. Let $\{\alpha_t : t \in R\}$ and $\{\beta_t : t \in R\}$ be two commuting one-parameter groups of *-automorphisms of a von Neumann algebra M such that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in R$. Then there exists a central projection p in M such that

$\alpha_t(p) = \beta_t(p) = p$ for all $t \in R$ and $\alpha_t = \beta_t$ on Mp and $\alpha_t = \beta_{-t}$ on M(1 - p).

PROOF. By Proposition 2.2, let p_n, $n \in N$, be the largest $\alpha_{2^{-n}}$, $\beta_{2^{-n}}$ -invariant central projection in M such that

$$\alpha_{2^{-n}} = \beta_{2^{-n}} \text{ on } Mp_n$$

Similarly, we let q_n, $n \in N$, be the largest $\alpha_{2^{-n}}$, $\beta_{2^{-n}}$ -invariant central projection in M such that

$$\alpha_{2^{-n}} = \beta_{2^{-n}} \quad \text{on} \quad Mq_n$$

Then $(1 - p_n)(1 - q_n) = 0$ for any $n \in N$. The sequence $\{p_n\}$ is decreasing because

$\alpha_{2^{-(n+1)}} = \beta_{2^{-(n+1)}}$ on Mp_{n+1} implies that their squares $\alpha_{2^{-n}}$ and $\beta_{2^{-n}}$ coincide

on Mp_{n+1} and by maximality $p_{n+1} \leqslant p_n$. Similarly the sequence $\{q_n\}$ is decreasing.

Put $p = \lim_{n \to \infty} p_n$ and $q = \lim_{n \to \infty} q_n$. Then clearly $(1-p)(1-q) = 0$. Since p_n is

$\alpha_{2^{-m}}$, $\beta_{2^{-m}}$ -invariant for $n \geqslant m$, it follows that p is $\alpha_{2^{-m}}$, $\beta_{2^{-m}}$ -invariant for any

$m \in N$. The density of the set $\{k\,2^{-m}: k \in Z, m \in N\}$ in R and the continuity imply

that p is α_t, β_t -invariant for any $t \in R$. Similarly q is α_t, β_t -invariant. Also

$\alpha_{k2^{-m}}(x) = \beta_{k2^{-m}}(x)$ for $x \in Mp \subseteq Mp_m$. Again by the density of the set

$\{k2^{-m}: k \in Z, m \in N\}$, we conclude that $\alpha_t = \beta_t$ on Mp for any $t \in R$.

Similarly, $\alpha_t = \beta_{-t}$ on Mq. As $(1 - q)(1 - p) = 0$ then $M(1 - p) \subseteq Mq$ and this

completes the proof of the theorem.

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