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A NOTE ON SOME SPACES L_γ OF DISTRIBUTIONS WITH LAPLACE TRANSFORM

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ABSTRACT. In this paper we calculate the dual of the spaces of distributions L_γ introduced in [1]. Then we prove that L_γ is the dual of a subspace of $C^\infty(\mathbb{R})$.

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1. INTRODUCTION

Let $\mathcal{D}'$ and S' be the classical Schwartz's spaces of distributions in $\mathbb{R}$ and denote by L the Laplace transformation. In (Pérez-Esteva [1]) were introduced spaces $L^a_{p\gamma}$ as follows:

$L^a_{o\gamma}$ is the subspace of $L^1_{loc}(\mathbb{R})$ of functions f with supp $f \subset [a,\infty)$ and $e_{-\gamma} f \in L^2(\mathbb{R})$, where $e_{-\gamma}(x)=e^{-\gamma x}$. $L^a_{o\gamma}$ is a Hilbert space with the inner product

$$(f,g) = \int_{\mathbb{R}} e_{-2\gamma} f \bar{g}\, dx$$

then we define $L^a_{p\gamma} = D^p L^a_{o\gamma}$ where D^p is the distributional derivative of order p. Since $D^p: L^a_{o\gamma} \to L^a_{p\gamma}$ is bijective, we can copy the Hilbert space structure of $L^a_{o\gamma}$ on $L^a_{p\gamma}$. We have the continuous inclusions

$$L^a_{p\gamma} \subset L^b_{p\gamma}, \quad \text{for } a > b$$

$$L^a_{p\gamma} \subset L^a_{q\gamma}, \quad \text{if } p \leqslant q$$

Hence for $p = \{0,1,\dots\}$ the strict inductive limit

$$L_{p\gamma} = \operatorname*{ind\,lim}_{a \to -\infty} L^a_{p\gamma}$$

makes sense. Then

$$L_\gamma = \operatorname*{ind\,lim}_{p \to \infty} L_{p\gamma} = \operatorname*{ind\,lim}_{p \to \infty} L^{-p}_{p\gamma}$$

is also well defined.

In [1] it was studied the spaces of distributions g for which the convolution

$$f \to f*g: L_\gamma \to L_\gamma$$

is continuous.

Here we describe the strong dual of L_γ, which turns out to be a subspace S_γ of $C^\infty(\mathbb{R})$. Then we prove the reflexivity of S_γ and conclude that $(S_\gamma)' = L_\gamma$, which is the main result of the paper. $\|\cdot\|_2$ will denote the norm of $L^2(\mathbb{R})$, γ will be assumed to be a positive constant, and N will be the set of nonegative integers.

2. THE DUAL OF L_γ

DEFINITION 1. Let L_γ be the space of all complex measurable functions g in $\mathbb{R}$ such that $\chi_{[a,\infty)} e_{-\gamma} g \in L^2(\mathbb{R})$ for every $a \in \mathbb{R}$, where $\chi_{[a,\infty)}$ stands for the characteristic function of $[a,\infty)$. We provide L_γ with the topology given by the seminorms

$$P_a(g) = \|\chi_{[a,\infty)} e_{-\gamma} g\|_2 , \quad a \in \mathbb{R}.$$

Next we denote by S_γ the subspace of L_γ such that $D^n f \in L_\gamma$ for every $n \in N$. Define the topology of S_γ by the system of seminorms

$$P_{an}(g) = \|\chi_{[a,\infty)} e_{-\gamma} D^n g\|_2 \quad a \in \mathbb{R}, \quad n \in N$$

It is clear that L_γ and S_γ are Frechet spaces and since $D^n g \in L^1_{loc}(\mathbb{R})$ for any $n \in \mathbf{N}$ and $g \in S_\gamma$, we have that $S_\gamma \subset C^\infty(\mathbb{R})$.

LEMMA 1. Let $\Phi \in L'_\gamma$, then for every $p \in N$, there exists $g_p \in L_\gamma$ such that

$$\Phi(D^p f) = \int_{\mathbb{R}} e_{-2\gamma} f \, g_p dx, \quad f \in L_{o\gamma}$$

The sequence $\{g_p\}_{p\in N}$ satisfies

$$g_{p+1} = -Dg_p + 2\gamma g_p, \quad p \in N \tag{2.1}$$

Hence Φ is determined by $g_o \in S_\gamma$.

PROOF. Fix $a \in \mathbb{R}$ and $p \in N$. Then $\Phi \in (L^a_{p\gamma})'$, and there exists $g_{pa} \in L^a_{o\gamma}$ such that

$$(D^p f) = \int_{\mathbb{R}} e_{-2\gamma} f \, g_{pa} \, dx, \qquad D^p f \in L^a_{p\gamma}$$

If $a < b$, we have $L^b_{p\gamma} \subset L^a_{p\gamma}$, then

$$\Phi(D^p f) = \int_{\mathbb{R}} e_{-2\gamma} f \, g_{pb} \, dx = \int_{\mathbb{R}} e_{-2\gamma} f \, \chi_{[b,\infty)} g_{pa} \, dx$$

for $D^p f \in L^b_{p\gamma}$, which shows that

$$g_{pb} = \chi_{[b,\infty)} g_{pa}$$

If $\tilde{g}_{pa}$ is the restriction of g_{pa} to $[a,\infty)$, then $g_p = \cup_a \tilde{g}_{pa}$ is well defined, belongs to L_γ and

$$\Phi(D^p f) = \int_{\mathbb{R}} e_{-2\gamma} f \, g_p dx, \; D^p f \in L_{p\gamma}$$

Let $\varphi \in \mathcal{D}$. Since $D^{p+1}\varphi \in L_{p+1\gamma} \cap L_{p\gamma}$, we have

$$\Phi(D^{p+1}\varphi) = \int_{\mathbb{R}} e_{-2\gamma}\varphi g_{p+1}\, dx = \int_{\mathbb{R}} e_{-2\gamma} D\varphi g_p\, dx$$
$$= \int_{\mathbb{R}} \{D(e_{-2\gamma}\varphi) + 2\gamma\, e_{-2\gamma}\varphi\} g_p\, dx$$
$$= \langle -e_{-2\gamma} Dg_p + 2\gamma\, e_{-2\gamma} g_p, \varphi \rangle$$

where $\langle \cdot , \cdot \rangle$ represents the duality between $\mathcal{D}$ and $\mathcal{D}'$. It follows that

$$g_{p+1} = -Dg_p + 2\gamma g_p$$

or

$$e_{-2\gamma}\, g_{p+1} = -D(e_{-2\gamma}\, g_p)$$

Hence, every g_p belongs to S_γ.

LEMMA 2. Let $g \in S_\gamma$ and H be the differential operator defined by $H = -D + 2\gamma I$. Then the functional

$$\Phi(D^p f) = \int_{\mathbb{R}} e_{-2\gamma} f\, H^{(p)} g\, dx, \quad f \in L_{o\gamma}$$

is well defined in L_γ and is continuous.

PROOF. Let $f \in L^a_{o\gamma}$ be such that $f = Dh$ with $h \in L_{o\gamma}$. There exists a sequence $\{f_n\}_{n\in N} \subset \mathcal{D}$ converging to f in $L^b_{o\gamma}$ if $b < a$.

Let

$$\varphi_n(x) = \int_{-\infty}^{x} f_n\, dy$$

Then $f_n \in L^b_{o\gamma}$, $D(\varphi_n - h) = f_n - f$, and since the inclusion $L^b_{o\gamma} \subset L^b_{1\gamma}$ is continuous, we have that $\{\varphi_n\}_{n\in N}$ converges to h in $L_{o\gamma}$. If follows that

$$\int_{\mathbb{R}} e_{-2\gamma} h\, H(g) dx = \operatorname*{Lim}_{n\to\infty} \int_{\mathbb{R}} e_{-2\gamma}\varphi_n\, H(g) dx \qquad (2.2)$$

and

$$\int_{\mathbb{R}} e_{-2\gamma} f\, g\, dx = \operatorname*{Lim}_{n\to\infty} \int_{\mathbb{R}} e_{-2\gamma} f_n g\, dx \qquad (2.3)$$

On the other hand

$$\int_{-\infty}^{B} e_{-2\gamma}\varphi_n\, H(g) dx = -\int_{-\infty}^{B} \varphi_n\, D(e_{-2\gamma} g)\, dx$$
$$= -\varphi_n(B) e_{-2\gamma}(B) g(B) + \int_{b}^{B} f_n\, e_{-2\gamma} g\, dx \qquad (2.4)$$

But we have the estimate

$$|g(x)| \leq |g(b)| + e_\gamma(x)\, \|\chi_{[b,\infty)} e_{-\gamma}(Dg - \gamma g)\|_2 (x-b)^{1/2} \qquad \text{for } x > b$$

Hence

$$\int_{\mathbb{R}} e_{-2\gamma}\varphi_n\, H(g) dx = \int_{\mathbb{R}} e_{-2\gamma} f_n\, g\, dx$$

From (2.2) and (2.3) it follows that

$$\int_{\mathbb{R}} e_{-2\gamma} f\, g\, dx = \int_{\mathbb{R}} e_{-2\gamma} h\, H(g) dx \qquad (2.5)$$

By induction we obtain

$$\int_{\mathbb{R}} e_{-2\gamma} f\, g\, dx = \int_{\mathbb{R}} e_{-2\gamma} h\, H^{(p)}(g) dx \tag{2.6}$$

if $f = D^p h$ and $f,h \in L_{o\gamma}$.

Finally, if $D^p f = D^q h$ with $f,h \in L_{o\gamma}$ and $q \geqslant p$, then $f = D^{q-p}h$, hence by (2.6) we have

$$\int_{\mathbb{R}} e_{-2\gamma} f\, H^{(p)}(g) dx = \int_{\mathbb{R}} e_{-2\gamma} h\, H^{(q)}(g) dx$$

Thus Φ is well defined and it is clearly continuous.

THEOREM 1. The strong dual of L_γ is S_γ.

PROOF. By lemmas 1 and 2 we know that $L_\gamma' = S_\gamma$. It remains to prove that the strong topology $\beta(L_\gamma', L_\gamma)$ coincides with the topology τ of S_γ. First notice that τ is defined by the system of seminorms

$$q_{ap}(g) = \| \chi_{[a,\infty)} e_{-\gamma} H^{(p)}(g) \|_2, \qquad a \in \mathbb{R}, \qquad p \in N$$

Fix $a \in \mathbb{R}$ and $p \in N$. Let $V = \{g \in S_\gamma : q_{ap}(g) \leqslant 1\}$. Denote by U the unit ball in $L^a_{o\gamma}$, then the set $B = D^p U$ is bounded in $L_{p\gamma}$ and hence in L_γ. If $g \in B^o$ (the polar of B), then for every $f \in U$ we have

$$\left|\int e_{-2\gamma} f\, H^{(p)}(g) dx\right| = |\langle D^p f, g\rangle| \leqslant 1$$

Thus

$$\| e_{-\gamma} \chi_{[a,\infty)} H^{(p)}(g) \|_2 \leqslant 1$$

It follows that $B^o \subset V$ and $\tau \subset \beta(L_\gamma', L_\gamma)$. Now, let B be a bounded set in L_γ. Then for some $p \in N$, $B \subset L^{-p}_{p\gamma}$ and is bounded there (see Kucera, McKennon [2]). Hence $B \subset \varepsilon D^p U$ for some $\varepsilon > 0$, where U is the unit ball in $L^{-p}_{o\gamma}$. Let $V = \{g \in S_\gamma : q_{-p\, p}(g) \leqslant \varepsilon^{-1}\}$, then $g \in V$ implies for $f \in \varepsilon U$ that

$$\langle D^p f, g\rangle = \left|\int_{\mathbb{R}} e_{-2\gamma} f\, H^{(p)}(g) dx\right| \leqslant 1$$

Then $g \in B^o$, so we proved that $V \subset B^o$. This completes the proof.

COROLLARY 1. L_γ is the strong dual of S_γ.

PROOF. By (Kucera, McKennon [2], Theorem 4) we know that L_γ is reflexive. Hence the corollary follows from Theorem 1.

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