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RING HOMOMORPHISMS ON H(G)

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ABSTRACT. It is shown that a ring homomorphism on H(G), the algebra of analytic functions on a regular region G in the complex plane, is either linear or conjugate linear provided that the ring homomorphism takes the identity function into a nonconstant function.

KEY WORDS AND PHRASES. Ring homomorphism, Algebra of analytic function, Linear, Conjugate linear.
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1. INTRODUCTION.

An operator M on a commutative algebra A is called a **ring homomorphism** if for all $x,y \in A$, $M(x+y) = M(x) + M(y)$ and $M(xy) = M(x)M(y)$. Throughout this paper G denotes a region, i.e., a connected open set in the complex plane, H(G) denotes the algebra of analytic functions on a region G in the complex plane equipped with the topology of uniform convergence on compact subsets of G, M denotes a nontrivial ring homomorphism on H(G), and I denotes the identity function on G. A region G in C is called **regular** if G = interior (closure G). The rationals, reals and complex numbers are denoted by Q, R, and C respectively.

If N is a maximal ideal in H(G) then the quotient algebra H(G)/N is isomorphic (as an algebra) to C if and only if N is the kernel of a linear homomorphism. Henriksen [1] has shown that H(G)/N is isomorphic (as a ring) to C where G=C and the maximal ideal M is not closed. This implies that there exist discontinuous homomorphisms from the ring of entire functions onto C.

In this paper we show that if G is a regular region in C and a ring homomorphism M on H(G) takes the identity function I to a non-constant function, then M is necessarily continuous. Essentially we prove that homomorphisms under consideration preserve constants. However, the results of this fact can be obtained by the techniques used in [2] and [3] except in the case G = C.

If M is a ring homomorphism on H(G) then the following assertions are equivalent:

1) M is continuous,
2) either $M(k) = k$ for all $k \in C$ or $M(k) = \bar{k}$ for all $k \in C$,
3) M is either linear or conjugate linear,
4) there exists $h \in H(G)$ with $h(G) \subset G$ such that $M(f) = foh$ for all $f \in H(G)$ or there exist $h \in H(G)$ with $\overline{h(G)} \subset G$ such that $M(f) = \overline{fo\bar{h}}$ for all $f \in H(G)$.

The implications 4 ==> 1) ==> 2) ==> 3) are trivial or easy to prove; 3) ==> 4) is the content of Lemma 2.1.

We show that a ring homomorphism M on H(G) which takes the identity function to a non-constant function is necessarily linear or conjugate linear using Nienhuys-Thiemann's theorem [4] which states that given any two countable dense subsets A and B of R there exists an entire function which is real valued and increasing on the real line R such that $f(A) = B$. In Section 2 we give some lemmas and a theorem of Nienhuys and Thiemann. In Section 3 we prove the following main theorem.

THEOREM 1.1. Let G be a regular region in C and let M be a ring homomorphism on H(G) such that M(I) is not a constant function where I is the identity function. Then $M(i) = \pm i$. Further

a) if $M(i) = i$ then M is linear,
b) if $M(i) = -i$ then M is conjugate linear,

2. LEMMAS.

The following lemma is well known and we give the proof for the sake of completeness.

LEMMA 2.1. Let M be a ring homomorphism on H(G). If M is linear then there exists a $h \in H(G)$ with $h(G) \subset G$ such that $M(f) = foh$ for all $f \in H(G)$.

PROOF. Let $M(I) = h$ and $z_0 \in G$. We claim that $h(z_0) \in G$. Suppose not, then

$$(I - h(z_0)) \left(\frac{1}{I - h(z_0)}\right) = 1 .$$

Applying M on both sides and evaluating at z_0 with the observation that $M(h(z_0)) = h(z_0)$ we obtain

$$0 = (M(I)(z_0) - h(z_0)) M\left(\frac{1}{I - h(z_0)}\right) (z_0)$$

$$= M(I - h(z_0))(z_0) M\left(\frac{1}{I - h(z_0)}\right) (z_0)$$

$$= M(1)(z_0)$$

$$= 1.$$

which is a contradiction. Since z_0 is arbitary we have $h(G) \subset G$.

Since $h(z_0) \in G$ we have $\frac{f - f(h(z_0))}{I - h(z_0)} \in H(G)$ and

$$f - f(h(z_0)0 = (I - h(z_0)) \left(\frac{f - f(h(z_0))}{I - h(z_0)}\right) .$$

Appling M on both sides and evaluating at z_0 we obtain

$$M(f)(z_0) = M(f(h(z_0)))(z_0) = f(h(z_0)) .$$

Since z_0 is arbitrary the result follows.

LEMMA 2.2. Let G be a regular region in C and M be a ring homomorphism on H(G) with $M(i) = i$. If $M(I) = h$ is not a constant function then $h(G) \subset G$.

PROOF. Since M is a nontrivial ring homomorphism it is easy to show that $M(\alpha) = \alpha$ for all $\alpha \in Q$. Since $M(i) = i$ we have $M(\alpha + i\beta) = \alpha + i\beta$ where α, $\beta \in Q$. Let $z_0 \in G$ such that $h(z_0) \in Q + iQ$. Just as in the above lemma it is easy to show that $h(z_0) \in G$. Since h is not a constant function we have $h(z_0) \in G$ for a dense set of z_0 in G and since h(G) is open we have $h(G) \subset \text{interior}(\text{closure } G) = G$.

Let $K \in Q$. Denote by H_k the set of all entire functions which map $Q + ik$ into Q except possibly for one point of $Q + ik$ and also denote by EM the class of entire functions whose restriction to R is a real monotonically increasing function. The proof of Lemma 2.3 follows the proof of the following theorem of Nienhuys & Thiemann [4].

THEOREM 2.1. Let S and T be countable everywhere dense subsets of R, let p be a continuous positive real function such that $\lim_{t \to \infty} t^{-n}p(t) = \infty$ for all $n \in N$ and let $f_0 \in EM$.

Then there exists a function $f \in EM$ such that

i) f is strictly increasing on R and $f(S) = T$,

ii) $|f(z) - f_0(z)| < p(|z|)$ for all $z \in C$.

LEMMA 2.3. Let $k \in Q$, $\beta \in R$ and $\alpha \in Q + ik$. Then there exists an entire function $f \in H_k$ such that $f(\alpha) = \beta$ and $f(Q + ik) = \{\beta\} \cup Q$.

PROOF. In Nienhuys and Thiemann's Theorem [4] take $S = Q$ and $T = \{\beta\} \cup Q$. Let $x_1, x_2, \ldots,$ be an enumeration of Q. Then as in the proof of this theorem there exists an entire function g such that $g(x_1) = \beta$ and $g(Q) = \{\beta\} \cup Q$. Let $x_1 = \alpha - ik$ and $h(z) = z - ik$. Then $f = goh$ is the desired function.

3. PROOF OF THE MAIN THEOREM.

It is easy to see that M is linear over the field of rational numbers and hence we have $-1 = M(-1) = M(i^2) = M(i)^2$ which implies $M(i) = \pm i$. We prove here only Part a) of the theorem and the proof of Part b) follows similarly.

Since $h = M(I)$ is a nonconstant analytic function on G, h(G) is a nonempty open set in C and by Lemma 2.2, $h(G) \subset G$. Hence there exists $k \in Q$ such that $S = (R + ik) \cap h(G)$ has an interval parallel to real axis. Let $f \in H(G)$ and $h(z_0) \in (Q + ik) \cap G$. Then applying M on both sides and evaluating at z_0 in the following

$$f - f(h(z_0)) = (I - h(z_0)) \left(\frac{f - f(h(z_0))}{I - h(z_0)}\right)$$

we obtain

$$M(f - f(h(z_0)))(z_0) = 0$$

for all z_0 in G such that $h(z_0) \in Q + ik$. Thus we have

$$M(f)(z_0) = M(f(h(z_0)))(z_0) \text{ , for all } f \in H(G) \text{ and for all } h(z_0) \in Q + ik. \tag{1}$$

Since a function f in H_k takes Q + ik into rationals except for one point of Q + ik, we obtain $M(f(h(z_0))) = f(h(z_0))$ whenever $h(z_o)$ is in $(Q + ik) \cap G$. Since f is analytic we obtain

$$M(f) = foh, \text{ for all } f \in H_k. \tag{2}$$

For a given $\beta \in R$ and each $h(z_0)$ in Q + ik, by Lemma 2.3 there exists an entire function in H_k such that $f(h(z_0)) = \beta$. Substituting this in (1) on the one hand we obtain

$$M(f)(z_0) = M(\beta)(z_0)$$

and evaluating (2) at z_0 on the other we obtain

$$M(f)(z_0) = foh(z_0) = f(h(z_0)) = \beta$$

Thus we obtain from the above two relations that

$$M(\beta)(z_0) = \beta \text{ for all } z_0 \in h^{-1} (Q + ik) \cap G.$$

Since $M(\beta)$ is analytic we have $M(\beta) = \beta$. Thus we have $M(k) = k$ for all $k \in C$. This implies M is linear.

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