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INEQUALITIES FOR WALSH LIKE RANDOM VARIABLES

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ABSTRACT. Let $(X_n)_{n \geq 1}$ be a sequence of mean zero independent random variables. Let $W_k = \{\prod_{j=1}^{k} X_{i_j} \,|\, 1 \leq i_1 < i_2 \ldots < i_k\}$, $Y_k = \bigcup_{j \leq k} W_j$ and let $[Y_k]$ be the linear span of Y_k. Assume $\delta \leq |X_n| \leq K$ for some $\delta > 0$ and $K > 0$ and let

$C(p,m) = 16(5\sqrt{2}\, \frac{p^2}{p-1})^{m-1} \frac{p}{\log p} (\frac{K}{\delta})^m$ for $1 < p < \infty$. We show that for $f \in [Y_m]$ the following inequalities hold:

$$||f||_2 \leq ||f||_p \leq C(p,m)||f||_2 \qquad \text{for } 2 < p < \infty$$

$$||f||_2 \leq C(q,m)||f||_p \leq C(q,m)||f||_2 \qquad \text{for } 1 < p < 2,\ \frac{1}{p} + \frac{1}{q} = 1$$

and $||f||_2 \leq C(4,m)^2||f||_1 \leq C(4,m)^2||f||_2$. These generalize various well known inequalities on Walsh functions.

KEY WORDS AND PHRASES. Walsh Functions, Martingales, Square Function.
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1. INTRODUCTION.

Let $(X_n)_{n \geq 1}$ be a sequence of independent mean zero random variables. Let W_k be products of length k of the $(X_n)_{n \geq 1}$ i.e.

$$W_k = \{X_{i_2} X_{i_2} \ldots X_{i_k} \,|\, i_1 < i_2 < \ldots < i_k\},$$

let $Y_k = \bigcup_{j \leq k} W_j$ and let $[Y_k]$ be the linear span of functions in Y_k. The object of this note is to show that for functions in $[Y_k]$ the p'th mean is of the same order as the second moment. As such this generalizes classical inequalities such as Khinchin's inequality in Zygmund [1] as well as more recent inequalities on Walsh functions such as those of H. Rosenthal [2] and A. Bonami [3]. Precisely we prove the following:

THEOREM 2.4. Let $(X_n)_{n \geq 1}$ be a sequence of independent mean zero random variables on a probability space (Ω, μ). Suppose there exist $\delta > 0$ and $K > 0$ such that $\delta \leq |X_n| \leq K$ for all n. For $f \in [Y_n]$ we have

$$||f_2|| < ||f||_p < C(p,n)||f||_2, \text{ for } 2 < p < \infty \tag{1.1}$$

$$||f||_2 < C(q,n)||f||_p < C(q,n)||f||_2, \text{ for } 1 < p < 2, \frac{1}{p} + \frac{1}{q} = 1. \tag{1.2}$$

$$||f||_2 < C(4,n)^2 \; ||f||_1 < C(4,n)^2 \; ||f||_2 \tag{1.3}$$

where $C(p,n) = 16 \; (5\sqrt{2} \, \frac{p^2}{p-1})^{n-1} \, \frac{p}{\log p} \, (\frac{K}{\delta})^n$.

We assume of course that the $(X_n)_{n\geq 1}$ belong to some $L^p(\Omega)$. Recall that for $1 < p < \infty$, $L^p(\Omega)$ is the space of all measurable functions f such that $\int|f(\omega)|^p d\mu < \infty$ and the norm of f is $||f||_p = (\int|f(\omega)|^p d\mu)^{1/p}$. We assume the reader is familiar with martingales and refer to Garsia [2] for unexplained notation.

2. PROOF OF THE INEQUALITIES.

We require three preliminary facts in order to prove theorem 2.4. We denote by E(X), the expectation of a random variable X.

THEOREM 2.1. [1]. Let $r_n(t)$ be the Rademacher functions on [0,1].

Then $\int_0^1 |\sum_{k\leq n} a_k r_k(t)|dt \geq \frac{1}{\sqrt{2}} (\sum_{k\leq n} |a_k|^2)^{1/2}$ for any complex numbers $(a_k)_{k=1}^n \subseteq \mathbb{C}$.

THEOREM 2.2. (Johnson, Schechtman, and Zinn [5]) Let $(X_n)_{n\geq 1}$ be a sequence of independent mean zero variables and let $(a_k)_{k=1}^n \in \mathbb{C}^n$. Then for $p > 2$

$$||\sum_{k=n} a_k X_k||_p < \frac{16p}{\log p} \max(||\sum_{k\leq n} a_k X_k||_2, \; (\sum_{k\leq n} |a_k|^p \; E|X_k|^p)^{1/p}) \; .$$

Recall that for a martingale $f = (f_n)_{n\geq 1}$, its difference sequence is $d_n = f_n - f_{n-1}$ and its square function is $S(f) = (\sum_n d_n^2)^{1/2}$. The last fact that we need is:

THEOREM 2.3 [4]. For a martingale $f = (f_n)$, we have

$$||f_n||_p < \frac{5p^2}{p-1} \; || \; (\sum_{k\leq n} d_k^2)^{1/2}||_p \text{ for } 1 < p < \infty.$$

We may now prove Theorem 2.4 quite easily.

THEOREM 2.4. Let $(X_n)_{n\geq 1}$ be a sequence of independent mean zero random variables on a probability space (Ω,μ). Suppose there exist $\delta > 0$ and $K > 0$ such that $\delta < |X_n| < K$ for all n. For $f \in [Y_m]$ we have

$$||f||_2 < ||f||_p < C(p,m)||f||_2 \text{ for } 2 < p < \infty \tag{2.1}$$

$$||f||_2 < C(q,m)||f||_p < C(q,m)||f||_2 \text{ for } 1 < p < 2 \tag{2.2}$$

and $1/p + 1/q = 1$.

$$||f||_2 < C(4,m)^2||f||_1 < C(4,m)^2||f||_2 \tag{2.3}$$

where $C(p,m) = 16\,(5\sqrt{2}\,\frac{p^2}{p-1})^{m-1}\,\frac{p}{\log p}\,(\frac{K}{\delta})^m$.

PROOF. The proof is by induction on m. We will first consider the case $p > 2$. Suppose $m = 1$ and $f \in [Y_1]$. Then $f = \sum_{k\le n} a_k X_k$ for some $a_k \in \mathbf{C}$, $k = 1,\ldots,n$. By Theorem 2.2 we have,

$$\|f\|_p = \|\sum_{k\le n} a_k X_k\|_p \le 16\,\frac{p}{\log p}\max(\|\sum_{k\le n} a_k X_k\|_2,\ (\sum_{k\le n}|a_k|^p E|X_k|^p)^{1/p})$$

$$= \frac{16p}{\log p}\max\left((\sum|a_k|^2 E|X_k|^2)^{1/2},\ (\sum|a_k|^p E|X_k|^p)^{1/p}\right)$$

(Since $EX_k=0$ and the X_k are independent)

$$\le \frac{16\,pK}{\log p}\max\left((\sum_{k\le n}|a_k|^2)^{1/2},\ (\sum_{k\le n}|a_k|^p)^{1/p}\right)$$

$$= \frac{16pK}{\log p}(\sum_{k\le n}|a_k|^2)^{1/2}. \qquad (2.4)$$

However $\|f\|_2 = (\sum_{k\le n}|a_k|^2 E|X_k|^2)^{1/2} \ge \delta(\sum_{k\le n}|a_k|^2)^{1/2}$ and so by (2.4) the result follows for $m = 1$.

We assume the result is valid for $f \in [Y_m]$. Let $f \in [Y_{m+1}]$. Note that we may write f as $f = \sum_{n\ge 1} f_n X_n$ where $f_n \in [Y_m]$ and f_n only depends on the random variables X_j, $1 \le j < n$. It is clear then that f is a sum of a martingale difference sequence. Applying Theorem 2.3 we have

$$\|f\|_p \le 5\,\frac{p^2}{p-1}\,\|S(f)\|_p$$

$$= 5\,\frac{p^2}{p-1}\,\|(\sum_{n\ge 1} f_n^2 X_n^2)^{1/2}\|_p$$

$$\le 5\,\frac{p^2}{p-1}\,K\|(\sum_{n\ge 1} f_n^2)^{1/2}\|_p \quad (\text{since } |X_n| \le K)$$

$$\le 5\,\frac{\sqrt{2}\,p^2}{p-1}\,K\,\|\int_0^1|\sum_{n\ge 1} r_n(t)f_n|dt\|_p \quad (\text{by Theorem 2.1})$$

$$\le 5\sqrt{2}\,\frac{p^2}{p-1}\,K\int_0^1\|\sum_{k\ge 1} r_n(t)f_n\|_p dt$$

$$\le 5\sqrt{2}\,\frac{p^2}{p-1}\,K\,C(p,m)\int_0^1\|\sum_{n\ge 1} r_n(t)f_n\|_2 dt \quad (\text{by induction})$$

$$\le 5\sqrt{2}\,\frac{p^2}{p-1}\,K\,C(p,m)\left(\int_0^1\|\sum_{n\ge 1} r_n(t)f_n\|_2^2\right)^{1/2}$$

$$= 5\sqrt{2}\,\frac{p^2}{p-1}\,K\,C(p,m)\left(\int_0^1\int_\Omega|\sum_{n\ge 1} r_n(t)f_n(\omega)|^2 d\mu\,dt\right)^{1/2}$$

$$= 5\sqrt{2}\,\frac{p^2}{p-1}\,K\,C(p,m)\left(\int_\Omega\int_0^1|\sum_{n\ge 1} r_n(t)f_n(\omega)|^2 dt\,d\mu\right)^{1/2} \quad (\text{by Fubini's Theorem})$$

$$= 5\sqrt{2}\,\frac{p^2}{p-1}\,K\,C(p,m)\left(\int_\Omega\sum_{n\ge 1} f_n^2(\Omega)d\mu\right)^{1/2} \quad (\text{since the Rademacher's are orthogonal})$$

$$< 5\sqrt{2}\ \frac{p^2}{p-1}\ \frac{K}{\delta}\ C(p,m)\ \left(\int_\Omega \sum_{n\geq 1} f_n^2(\omega)X_n^2(\omega)d\mu\right)^{1/2} (\text{since } |X_n| > \delta)$$

$$= 5\sqrt{2}\ \frac{p^2}{p-1}\ \frac{K}{\delta}\ C(p,m)\,||S(f)||_f$$

$$= 5\sqrt{2}\ \frac{p^2}{p-1}\ \frac{K}{\delta}\ C(p,m)\ ||f||_2 \ (\text{since } f \text{ is a martingle})$$

$$= C(p,m+1)\ ||f||_2$$

Hence $||f||_p < C(p,m+1)\ ||f||_2$ for $f \in [Y_{m+1}]$ proving the result for $p > 2$. For $1 < p < 2$ we employ the classical trick of Hölder. Let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then $\frac{1}{2} = \frac{1-\theta}{q} + \frac{\theta}{p}$ for $\theta = \frac{1}{2}$. Let $f \in [Y_m]$.

Then, $||f||_2 < ||f||_q^{1/2}||f||_p^{1/2}$ (by Hölder's inequality)

$$< C(q,m)^{1/2}||f||_2^{1/2}||f||_p^{1/2} \ (\text{since } q > 2).$$

So $||f||_2 < C(q,m)||f||_p$ while $||f||_p < ||f||_2$ is obvious, proving (2.2). Finally to see (2.3) note that $\frac{1}{2} = \frac{1-\theta}{4} + \frac{\theta}{1}$ for $\theta = 1/3$, so again by Hölder's inequality,

$$||f||_2 < ||f||_4^{2/3}||f||_1^{1/3} < ||f||_2^{2/3}\ C(4,m)^{2/3}||f||_1^{1/3} \ (\text{for } f \in [Y_m]).$$

So $||f||_2 < C(4,m)^2||f||_1$, while $||f||_2 > ||f||_1$ is automatic, proving (2.3).

REFERENCES

1. ZYGMUND, A., Trigonometric Series, Cambridge University Press, 1959.
2. ROSENTHAL, H., Biased Coin Convolution Operators, The Altgeld Book, University of Illinois at Urbana, 1976.
3. BONAMI, A., Etude des Coefficients de Fourier des Fonctions de L^p, Ann. Inst. Fourier 20 (1970), 335-402.
4. GARSIA, A., Martingale Inequalities, Seminar Notes on Recent Progress, Math. Lecture Note Series, W.A. Benjamin Inc., 1973.
5. JOHNSON, W., SCHECHTMAN, G. and ZINN, J., Best Constants in Moment Inequalities For Linear Combinations of Independent and Exchangeable Randon Variables, Ann. of Probability 13 (1985), 234-253.