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A NOTE ON COMPLEX L_1-PREDUAL SPACES

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ABSTRACT. Some characterizations of complex L_1 -predual spaces are proved.

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1. INTRODUCTION.

The aim of this note is to give some characterizations of complex L_1 -predual spaces. These are mostly complex analogous of the results proved by Lau [1]. Existing results that we need are given in §2 and the main results in §3.

Throughout the paper, we shall take V to be a complex Banach space, K its dual unit ball which being convex and compact in the w*-topology has a non-empty set of extreme points $\partial_e K$. For real valued bounded function f on K, $\hat{f}$ stands for its upper envelope. We shall write $\Gamma = \{z \in \mathbb{C}: |z| = 1\}$. By $A_o(K)$ we shall mean the set of continuous affine functions f on K which are Γ -homogeneous i.e. $f(\alpha x) = \alpha f(x)$ for all $x \in K$ and all $\alpha \in \Gamma$.

NOTATION. If f is a semi-continuous function on K, then we use the notation Sf to mean

$$Sf(x) = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos\theta \, f(xe^{i\theta})d\theta$$

2. SOME USEFUL RESULTS.

In what follows we need the following results.

THEOREM 2.1. For a complex Banach space V, the following are equivalent:

(i) V is L_1 -predual.

(ii) If g is l.s.c. concave function on K, such that

$$\sum_{k=1}^{n} g(\zeta_k x) > 0 \text{ whenever } \zeta_k \in \Gamma, \ (k = 1,2,3,\dots,n),$$

$\sum \zeta_k = 0$, then there is an $a \in A_o(K)$ such that $g > \mathrm{Re}\, a$ on K.

(iii) If h is a u.s.c. convex function on K, such that even $Sh(x) < 0$ for $x \in K$, then there is an $a \in A_o(K)$ such that $h < \mathrm{Re}\, a$ on K.

(iv) For u.s.c. convex function g on K,

$$\hat{g}(0) \le \sup\{ \sum_{k=1}^{n} \alpha_k g(\zeta_k x) : x \in K,\ n \in \mathbb{N},\ \alpha_k > 0, \sum_{k=1}^{n} \alpha_k = 1,\ \zeta_k \in \Gamma,\ \sum \alpha_k \zeta_k = 0\}.$$

The equivalence of (i) and (ii) is due to Olsen [2] while that of (i), (iii), (iv) is due to Das [3] and Roy [4]. The inequality in (iv) is in fact an equality since the reverse inequality follows from the fact the $g \le \hat{g}$ and that $\hat{g}$ is concave. The following result is due to Olsen [5].

THEOREM 2.2. For a complex Banach space V, the following are equivalent:

(i) V is L_1 -predual with $\partial_e K \cup \{0\}$ closed.

(ii) If f is a continuous Γ -homogeneous function on K, then there is a $v \in V$ such that $f|_{\partial_e K} = v|_{\partial_e K}$.

3. MAIN RESULTS.

This section contains the main results.

THEOREM 3.1. A complex Banach space V is L_1 -predual iff $\hat{f}(0) = \frac{1}{2} \sup \{Sf(x) + Sf(-x) : x \in K\}$. for all u.s.c. convex functions f on K.

PROOF. "If" - part.

Let us suppose that for u.s.c. convex functions f on K, $\hat{f}(0) = \frac{1}{2} \sup \{Sf(x) + Sf(-x) : x \in k\}$. We put

$$\alpha = \sup \{ \sum_{k=1}^{n} {}_k f(\zeta_k x) : x \in K,\ n \in \mathbb{N},\ \alpha_k > 0, \sum_{k=1}^{n} \alpha_k = 1,\ \zeta_k \in \Gamma,\ \Sigma \alpha_k \zeta_k = 0\}.$$

Then clearly $f(x) + f(-x) \le 2\alpha$ for $x \in K$. By linearity and canonical positivity of S, $Sf(x)+Sf(-x) \le 2\alpha$ for all $x \in K$. Then by the hypothesis $\hat{f}(0) \le \alpha$, so that by Theorem 2.1 (iv), v is L_1 -predual.

"Only if" -part.

Let V be L_1 -predual. Then by Theorem 2.1 (iv),

$$\hat{f}(0) = \sup \{ \Sigma_{k=1}^{n} \alpha_k f(\zeta_k x) : x \in K,\ n \in \mathbb{N},\ \alpha > 0 \sum_{k=1}^{n} \alpha_k = 1,\ \zeta_k \in \Gamma,\ \Sigma \alpha_k \zeta_k = 0\}.$$

We put $b = \frac{1}{2}$ sup $\{Sf(x) + Sf(-x) : x \in K\}$. Since f is u.s.c. convex and $Sf(x) + Sf(-x) \leq 2b$ for all $x \in K$, we apply Theorem 2.1 (iii), to the functions f-b to get $a_o \in A_o(k)$ such that $f-b \leq \text{Re } a_o$. But $\text{Re } a_o + b \in A(K)$, so that $\hat{f}(0) \leq b$. Now $\hat{f}(0)$ being real constant and S being linear and canonically positive

$$b \geq \hat{f}(0) \geq 1/2 \{f(x) + f(-x)\}$$

which yields $b \geq \hat{f}(0) \geq 1/2 \{ Sf(x) + Sf(-x) \}$.

Thus $b \geq \hat{f}(0) \geq \frac{1}{2}$ sup $\{Sf(x) + Sf(-x): x \in K\} = b$; the theorem is thus proved.

REMARK. The "if" part is proved by Roy [4] in a method quite different from ours, but he has failed to prove the converse and has kept the question open.

PROPOSITION 3.2. Let V be a complex L_1 -predual space. If $X \subset \partial_e K \cup \{0\}$ is closed such that $\alpha x \in X$ whenever $x \in X$, $\alpha \in \Gamma$, then every continuous $f: X \to \mathbb{C}$ with $f(\alpha x) = \alpha f(x)$ can be extended to an $\hat{f} \in A_o(K)$.

PROOF. As X is compact, Re f(x) attains infimum c(say) on X. Clearly $c \leq 0$, since $f(-x) = -f(x)$. We define a real-valued function F on K by

$$F(x) = \begin{cases} \text{Re } f(x), & x \in X, \\ c & , x \in K \backslash X. \end{cases}$$

Then F is u.s.c. and convex on K. Let us take $\zeta_k \in \Gamma$, $k=1,2,\ldots n$ such that $\Sigma \zeta_k = 0$.

If $\zeta_k = \exp(i\theta_k)$, $0 \leq \theta_k < 2\pi$, then $\sum_{k=1}^{n} \cos \theta_k = \sum_{k=1}^{n} \sin \theta_k = 0$. When $x \in K \backslash X$,

$\Sigma F(\zeta_k x) \leq 0$ and when $x \in X$, $\Sigma F(\zeta_k x) = \Sigma\{\cos \theta_k \text{Re } f(x) - \sin \theta_k \text{Im} f(x)\} = 0$. Thus for all $x \in K$, $\Sigma F(\zeta_k x) \leq 0$. Hence by Theorem 2.1(ii), there is an $\hat{f} \in A_o(k)$ such that $F \leq \text{Re } \hat{f}$. Let $x_o \in X$; then $\text{Re } f(x_o) \leq \text{Re } \hat{f}(x_o)$ and $\text{Re } f(-x_o) \leq \text{Re } \hat{f}(-x_o)$ which combined together give $\text{Re } f(x_o) = \text{Re } \hat{f}(x_o)$. Again $\text{Re } f(ix_o) \leq \text{Re } \hat{f}(ix)$ and $\text{Re } f(-ix_o) \leq \text{Re } \hat{f}(-ix_o)$ together give $\text{Im}\hat{f}(x_o) = \text{Im}\hat{f}(x_o)$. Thus $f(x_o) = \hat{f}(x_o)$. Hence $\hat{f}$ is the required extension.

THEOREM 3.3. A Banach space V is L_1-predual with $\partial_e K \cup \{0\}$ closed iff every continuous function $f: \partial_e K \cup \{0\} \to \mathbb{C}$ with $f(\alpha x) = \alpha f(x)$, $\alpha \in \Gamma$ can be extended to an $f \in A_o(K)$.

PROOF. "Only if" part.

Proof of this part is almost similar to that of Theorem 3.2 and is left out. In fact we can define an F as

$$F(x) = \begin{cases} \text{Re } f(x), & x \in \partial_e K \cup \{0\}, \\ \text{Inf } \{\text{Re } f(y): y \in \partial_e K \cup \{0\}, & x \in K \backslash \partial_e K \cup \{0\}, \end{cases}$$

which is u.s.c. convex and satisfies all the conditions of Theorem 2.1(ii).

"If" part.

Suppose that the extension property holds. To prove that V is L_1-predual with $\partial_e k \cup \{0\}$ closed, we shall show that Theorem 2.2(ii) holds.

So let h be a Γ -homogeneous continuous function on K and let $f=h|_{\partial_e K}$. Then $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \Gamma$ and for all $x \in \partial_e K$. So there is a $v \in V$ such that $v|_{\partial_e K} = f$, that is, $v|_{\partial_e k} = h|_{\partial_e K}$. . This completes the proof.

REMARK. This theorem is comparable with a characterizing result for Bauer simplex that every continuous function on $\partial_e K$ can be extended to a function in A(K).

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