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ON CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. The class $S^*(b)$ of starlike functions of complex order b was introduced and studied by M.K. Aouf and M.A. Nasr. The authors using the Ruscheweyh derivatives introduce the class K(b) of functions close-to-convex of complex order b, $b \neq 0$ and its generalization, the classes $K_n(b)$ where n is a nonnegative integer. Here $S^*(b) \subset K(b) = K_0(b)$. Sharp coefficient bounds are determined for $K_n(b)$ as well as several sufficient conditions for functions to belong to $K_n(b)$. The authors also obtain some distortion and covering theorems for $K_n(b)$ and determine the radius of the largest disk in which every $f \in K_n(b)$ belongs to $K_n(1)$. All results are sharp.

KEY WORDS AND PHRASES. Starlike functions, close-to-convex functions of complex order, Ruscheweyh derivatives, Hadamard product.
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1. INTRODUCTION.

Let A denote the class of functions f(z) analytic in the unit disk $E = \{z: |z| < 1\}$ having the power series

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in E. \tag{1.1}$$

Aouf and Nasr [1] introduced the class $S^*(b)$ of starlike functions of order b, where b is a nonzero complex number, as follows:

$$S^*(b) = \{ f: f \in A \text{ and } \operatorname{Re}\left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0, \ z \in E\}.$$

We define the class K(b) of close-to-convex functions of complex order b as follows: $f \in K(b)$ if and only if $f \in A$ and

$$\operatorname{Re} \{1 + \frac{1}{b} \left(\frac{zf'(z)}{g(z)} - 1\right)\} > 0, \quad z \in E, \tag{1.2}$$

for some starlike function g.

The classes R_n, $n \in N_0$ and where N_0 is the set of nonnegative integers, were introduced by Singh and Singh [2], $f \in R_n$ if and only if $f \in A$ and

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > 0, \quad z \in E, \tag{1.3}$$

where

$$D^n f(z) = f(z) * \frac{z}{(1 - z)^{n+1}}, \tag{1.4}$$

and (*) stands for the Hadamard product of power series, i.e., if

$$f(z) = \sum_0^\infty a_n z^n, \quad g(z) = \sum_0^\infty b_n z^n \text{ then } f(z) * g(z) = \sum_0^\infty a_n b_n z^n.$$

The operator D^n is referred to in Al-Amiri [3] as the Ruscheweyh derivative of order n. Note that R_0 is the familiar class of starlike functions, S^*. More, it is known [2] that $R_{n+1} \subset R_n$, $n \in N_0$, and consequently R_n consists of functions starlike in E.

Let $K_n(b)$, $n \in N_0$, b is a nonzero complex number, denote the class of functions $f \in A$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n g(z)} - 1 \right] \right\} > 0, \quad z \in E, \tag{1.5}$$

for some $g \in R_n$. Here $K_0(b) = K(b)$.

Many authors have studied various classes of univalent and multivalent functions using the Ruscheweyh derivatives D^n, $n \in N_0$. In particular one can look at the work of Ruscheweyh [4].

Section 2 determines coefficient estimates of functions in $K_n(b)$, $n \in N_0$. In section 3, we obtain some distortion and covering theorems for $K_n(b)$ and several sufficient conditions for functions to be in $K_n(b)$. The radius of close-to-convexity for the class of close-to-convex of complex order b is also determined in section 3.

2. COEFFICIENT ESTIMATES.

In this section, sharp estimates for the coefficients of functions in $K_n(b)$ are determined in Theorem 2.1. First, we need the following lemmas.

LEMMA 2.1. For $n \in N_0$, let

$$(D^n f(z))' = \frac{1 + (2b - 1)z}{(1 - z)^3}. \tag{2.1}$$

Then $f \in K_n(b)$.

PROOF. Let $g \in A$ be defined so that

$$D^n g(z) = \frac{z}{(1 - z)^2}. \tag{2.2}$$

The definition of R_n implies $g \in R_n$. A brief computation gives

$$1 + \frac{1}{b}\left[\frac{z(D^n f(z))'}{D^n g(z)} - 1\right] = \frac{1+z}{1-z}, \quad z \in E.$$

This proves that $f \in K_n(b)$.

REMARK 2.1. The function f as defined in (2.1) has the power series representation in E

$$f(z) = z + \sum_{m=2}^{\infty} \frac{n!\,(M-1)!}{(n+m-1)!}\,[(m-1)b + 1]z^m. \tag{2.3}$$

LEMMA 2.2. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m \in R_n$ where $n \in N_0$.

Then $|c_m| \leq \frac{n!\,m!}{(n+m-1)!}$.

PROOF. A brief computation gives

$$D^n\, g(z) = z + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{n!\,(m-1)!}\, c_m z^m.$$

Since $g \in R_n$, $D^n\, g(z) \in S^*$. Thus, using the well known coefficient estimates for starlike functions one gets

$$\frac{(n+m-1)!}{n!\,(m-1)!}\,|c_m| \leq m, \quad m \geq 2,$$

and the proof is complete.

LEMMA 2.3. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in K_n(b)$, $n \in N_0$, then

$$|ma_m - c_m|^2 \leq 4\left[\frac{(m-1)!}{(n+m-1)!}\right]^2 |b|$$

$$\cdot \left\{n!^2|b| + \sum_{k=2}^{m-1}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2 \left[|ka_k - c_k|\,|c_k| + |b|\,|c_k|^2\right]\right\} \tag{2.4}$$

PROOF. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $K_n(b)$. Then (1.5) implies

$$1 + \frac{1}{b}\left[\frac{z(D^n\, f(z))'}{D^n\, g(z)} - 1\right] = \frac{1 + w(z)}{1 - w(z)}, \quad z \in E, \tag{2.5}$$

for some $g \in R_n$ and where $w \in A$ such that $w(0) = 0$, $w(z) \neq 1$ and $|w(z)| \leq 1$ for

$z \in E$. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$. Then (2.5) and the Definition 1.4 imply

$$w(z)\left\{n!\,2bz + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{(k-1)!}[ka_k + (2b-1)c_k]z^k\right\} = \sum_{k=2}^{\infty} \frac{(n+k-1)!}{(k-1)!}(ka_k - c_k)z^k. \qquad (2.6)$$

Using Clunie's method, that is to examine the bracketed quantity of the left-hand side in (2.6) and keep only those terms that involve z^k for $k \leq m - 1$ for some fixed m, moving the other terms to the right side, one obtains

$$w(z)\left\{n!2bz + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{(k-1)!}[ka_k + (2b-1)c_k]z^k\right\}$$

$$= \sum_{k=2}^{m} \frac{(n+k-1)!}{(k-1)!}(ka_k - c_k)z^k + \sum_{k=m+1}^{\infty} A_k z^k.$$

Let

$$\varphi(z) = w(z)\left\{n!2bz + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{(k-1)!}[ka_k + (2b-1)c_k]z^k\right\}$$

$$= \sum_{k=2}^{m} \frac{(n+k-1)!}{(k-1)!}(ka_k - c_k)z^k + \sum_{k=m+1}^{\infty} A_k z^k. \qquad (2.7)$$

Let $z = re^{i\theta}$, $0 < r < 1$. Computing $\frac{1}{2\pi}\int_0^{2\pi} \varphi(z)\overline{\varphi(z)}\,dz$ for both expressions of $\varphi(z)$ in (2.7) and using $|w(z)| < 1$ we get

$$\sum_{k=2}^{m}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2 |ka_k - c_k|^2 r^{2k}$$

$$\leq n!^2\, 4|b|^2 r^2 + \sum_{k=2}^{m-1}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2 |ka_k + (2b-1)c_k|^2 r^{2k}.$$

Upon letting $r \to 1^-$ and after some easy computations we obtain

$$|ma_m - c_m|^2 \leq \left[\frac{(m-1)!}{(n+m-1)!}\right]^2 4|b|\left\{n!^2|b| + \sum_{k=2}^{m-1}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2\right.$$

$$\left.\cdot\,[|ka_k - c_k||c_k| + |b|\,|c_k|^2]\right\}.$$

In particular, when $m = 2$ we have

$$|2a_2 - c_2| \leq \frac{1}{n+1}\,2|b|. \qquad (2.8)$$

The proof of the lemma is complete.

THEOREM 2.1. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in K_n(b)$ where $n \in N_0$, then

$$|a_m| \le \frac{n!\,(m-1)!}{(n+m-1)!}\,[(m-1)|b| + 1].$$

This result is sharp. An extremal function is given by (2.3).

PROOF. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $K_n(b)$. Let the associate function of f, $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$. We claim that for $m \geq 2$ and $n \in N_0$,

$$|ma_m - c_m| \le \frac{n!(m-1)!}{(n+m-1)}\, 2|b| \left[1 + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{n!\,(k-1)!}\,|c_k|\right]. \tag{2.9}$$

We use the second principle of finite induction on m to prove (2.9).

For $m = 2$, $|2a_2 - c_2| \le \frac{n!}{(n+1)!} \cdot 2|b| = \frac{2(b)}{(n+1)}$ is true as shown in (2.8). Now assume (2.9) is true for all $m \le p$. Taking $m = p + 1$ in (2.4), we get

$$|(p+1)a_{p+1} - c_{p+1}|^2 \le 4\left[\frac{p!}{(n+p)!}\right]^2 |b| \cdot \left\{n!^2|b| + \left[\sum_{k=2}^{p} \frac{(n+k-1)!}{(k-1)!}\right]^2 |ka_k - c_k|\,|c_k| + |b|\,|c_k|^2\right\}$$

$$= 4\left[\frac{p!}{(n+p)!}\right]^2 |b| \left\{n!^2|b| + \sum_{k=2}^{p}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2 |ka_k - c_k|\,|c_k| + |b|\sum_{k=2}^{p}\left[\frac{(n+k-1)!}{(k-1)!}\right]^2 |c_k|^2\right\}.$$

Now using (2.9) since $k \le p$, the above yields

$$|(p+1)a_{p+1} - c_{p+1}|^2 \le 4\left[\frac{n!\,p!}{(n+p)!}\right]^2 |b|^2 \left\{1 + 2\sum_{k=2}^{p} \frac{(n+k-1)!}{n!\,(k-1)!}\,|c_k| \cdot \left[1 + \sum_{\ell=2}^{k-1} \frac{(n+\ell-1)!}{n!(\ell-1)!}\,|c_\ell|\right] + \sum_{k=2}^{p}\left[\frac{(n+k-1)!}{n!\,(k-1)!}\right]^2 |c_k|^2\right\}$$

$$= 4\left[\frac{n!\,p!}{(n+p)!}\right]^2 |b|^2 \left\{1 + 2\sum_{k=2}^{p} \frac{(n+k-1)!}{n!\,(k-1)!}\,|c_k| + 2\sum_{k=2}^{p} \frac{(n+k-1)!}{n!\,(k-1)!}\left[|c_k| \sum_{\ell=2}^{k-1} \frac{(n+\ell-1)!}{n!\,(\ell-1)!}\,|c_\ell|\right] + \sum_{k=2}^{p}\left[\frac{(n+k-1)!}{n!\,(k-1)!}\right]^2 |c_k^2|\right\}.$$

Applying the principle of mathematical induction on p, it is easily seen that the sum of the last two terms appearing in the bracketed expression in the right hand side of the above is equal to $\left[\sum_{k=2}^{p} \frac{(n+k-1)!}{n!\,(k-1)!} |c_k|\right]^2$. Consequently it follows that

$$|(p+1)a_{p+1} - c_{p+1}|^2 \leq 4\left[\frac{n!\,p!}{(n+p)!}\right]^2 |b|^2 \left[1 + \sum_{k=2}^{p} \frac{(n+k-1)!}{n!\,(k-1)!} |c_k|\right]^2 .$$

This shows that (2.9) is valid for $m = p + 1$. Hence, by the second principle of finite induction, the claim is correct. From Lemma 2.2 and 2.9 it follows that

$$|ma_m - c_m| \leq \frac{n!\,m!(m-1)}{(n+m-1)!} |b|, \ m \geq 2. \tag{2.10}$$

Finally from Lemma 2.2 and 2.10 we deduce that

$$|a_m| \leq \frac{n!\,(m-1)!}{(n+m-1)!} [(m-1)|b| + 1], \ m \geq 2.$$

Hence the proof of the Theorem 2.1 is complete.

Putting $n = 0$ in Theorem 2.1 we have the following corollary.

COROLLARY 2.1. If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is a close-to-convex function of complex order b, then $|a_m| \leq (m-1)|b| + 1$. This result is sharp.

REMARK 2.2. For $b = 1$, Corollary 2.1 is reduced to the well known coefficient bounds for the close-to-convex functions due to Reade [5].

Next we have two theorems that provide sufficient conditions for a function to be in $K_n(b)$.

THEOREM 2.2. Let $f \in A$ and $n \in N_0$. If any of the following conditions is satisfied in E, then $f \in K_n(b)$.

(i) $\mathrm{Re}\left\{1 + \frac{1}{b}[(D^n f(z))' - 1]\right\} > 0,$

(ii) $\mathrm{Re}\left\{1 + \frac{1}{b}[(1-z)(D^n f(z))' - 1]\right\} > 0,$

(iii) $\mathrm{Re}\left\{1 + \frac{1}{b}[(1-z^2)(D^n f(z))' - 1]\right\} > 0,$

(iv) $\mathrm{Re}\left\{1 + \frac{1}{b}[(1-z)^2 (D^n f(z))' - 1]\right\} > 0.$

PROOF. The proofs follow by choosing g as below:

(i) $g(z) = z,$

(ii) $g(z) = z + \sum_{m=2}^{\infty} \frac{n!\,(m-1)!}{(n+m-1)!} z^m,$

(iii) $g(z) = z + \sum_{m=2}^{\infty} \frac{n!\,(2m-2)!}{(n+2m-2)!} z^{2m-1},$ and

(iv) $g(z) = z + \sum_{m=2}^{\infty} \frac{n!\,m!}{(n+m-1)!} z^m$ respectively.

THEOREM 2.3. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. For $n \in N_0$, each of the following conditions is sufficient for f to be in $K_n(b)$.

(i) $\sum_{m=2}^{\infty} \frac{(n+m-1)!}{n!\,(m-1)!} m\,|a_m| < |b|.$

(ii) $\sum_{m=2}^{\infty} \frac{(n+m-1)!}{n!\,(m-1)!} \left|ma_m - \frac{(n+m)(m+1)}{m} a_{m+1}\right| < |b|,$

(iii) $2(n+1)|a_2| + \sum_{m=2}^{\infty} \frac{(n+m-2)!}{n!\,(m-2)!} \left|(m-1)a_{m-1} - \frac{(n+m)(n+m-1)(m+1)}{m(m-1)} \cdot a_{m+1}\right| < |b|,$ where $a_1 = 1,$

(iv) $2|(n+1)a_2 - a_1| + \sum_{m=2}^{\infty} \frac{(n+m-2)!}{n!\,(m-2)!} \left|(m-1)a_{m-1} - \frac{2(n+m-1)}{m-1} a_m + \frac{(n+m)(n+m-1)(m+1)}{m(m-1)} a_{m+1}\right| < |b|,$ where $a_1 = 1.$

PROOF. We prove the sufficiency of part (i) since the proofs of the remaining parts are similar to the proof of (i).

From (i) of Theorem 2.2, $f \in K_n(b)$ if f satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left[(D^n f(z))' - 1\right]\right\} > 0, \quad z \in E. \tag{2.11}$$

Condition (2.11) would be satisfied if

$$\left|\frac{1}{b}\left[(D^n f(z))' - 1\right] - 1\right| < 2, \quad z \in E \tag{2.12}$$

is true. However upon substituting

$$(D^n f(z))' = 1 + \sum_{m=2}^{\infty} \frac{m(n+m-1)}{n!\,(m-1)!} a_m z^{m-1}$$

in (2.12) one needs only show

$$\left|\frac{1}{b}\sum_{m=2}^{\infty}\frac{m(n+m-1)!}{n!\,(m-1)!}\,a_m z^{m-1}-1\right|<2,\quad z\in E. \tag{2.13}$$

Assuming (i) of this theorem we have

$$\left|\frac{1}{b}\sum_{m=2}^{\infty}\frac{m(n+m-1)!}{n!\,(m-1)!}\,a_m z^{m-1}-1\right|<\frac{1}{|b|}\sum_{m=2}^{\infty}\frac{m(n+m-1)!}{n!\,(m-1)!}\,|a_m|+1<2.$$

Thus (2.13) is established and the proof of the sufficiency of part (i) is complete.

REMARK 2.3. For $n = 0$ and $b = 1$, Theorems 2.2 and 2.3 are reduced to theorems of Ozaki [6].

3. DISTORTION THEOREMS.

The objective of this section is to obtain some distortion theorems for the class $K_n(b)$. The radius of the largest disk $E(r) = \{z/|z| < r\}$, $0 < r \leq 1$ such that if $f \in K_n(b)$ then $f \in K_n(1)$ can be determined as a consequence of one of those results.

THEOREM 3.1. Let $f \in K_n(b)$, $n \in N_0$. Then for $|z| = r < 1$, and $|2b - 1| \leq 1$

$$\frac{1-|2b-1|r}{(1+r)^3}\leq|(D^n f(z))'|\leq\frac{1+|2b-1|r}{(1-r)^3}. \tag{3.1}$$

This result is sharp. An extremal function f is given by (2.1).

PROOF. Let $f \in K_n(b)$. Then (1.5) implies for some $g \in R_n$

$$\frac{z(D^n f(z))'}{D^n g(z)}=\frac{1+(2b-1)\,w(z)}{1-w(z)},\quad z\in E,$$

where $w \in A$ and $|w(z)| \leq |z|$ in E. This gives for $|z| = r < 1$

$$\frac{1-|2b-1|r}{1+r}\leq\left|\frac{z(D^n f(z))'}{D^n g(z)}\right|\leq\frac{1+|2b-1|r}{1-r}. \tag{3.2}$$

The definition of R_n implies $D^n g(z)$ is a starlike function. Hence by the well known bounds on functions which are starlike in E, we get for $|z| = r < 1$

$$\frac{r}{(1+r)^2}\leq|D^n g(z)|\leq\frac{r}{(1-r)^2}. \tag{3.3}$$

Using (3.2) together with (3.3) one can get (3.1) and the proof of the Theorem 3.1 is complete.

Taking (i) $n = 0$, and (ii) $n = 0$, $b = 1$ in Theorem 3.1, one can immediately obtain the following corollaries, respectively.

COROLLARY 3.1. If f is a close-to-convex function of complex order b where $|2b - 1| \leq 1$, then for $|z| = r < 1$

$$\frac{1-|2b-1|r}{(1+r)^3}\leq|f'(z)|\leq\frac{1+|2b-1|r}{(1-r)^3}.$$

COROLLARY 3.2. If f is a close-to-convex function then for $|z| = r < 1$,

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

For the proof of Theorem 3.2, we need the following well known result [7; p. 84] concerning the class P of functions p(z) which are regular in E such that p(0) = 1 and Re p(z) > 0, $z \in E$.

LEMMA 3.1. Let $p \in P$. Then for $|z| = r < 1$,

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}. \tag{3.4}$$

This result is sharp.

THEOREM 3.2. Let $f \in K_n(b)$, $n \in N_0$. Then for some $g \in R_n$ and for $|z| = r < 1$,

$$\left|\frac{z(D^n f(z))'}{D^n g(z)} - \frac{1+(2b-1)r^2}{1-r^2}\right| \le \frac{2|b|r}{1-r^2}. \tag{3.5}$$

This result is sharp. An extremal function is given in (2.1).

PROOF. $f \in K_n(b)$ implies that for some $g \in R_n$

$$1 + \frac{1}{b}\left[\frac{z(D^n f(z))'}{D^n g(z)} - 1\right] = p(z), \quad z \in E,$$

where $p \in P$. Hence (3.5) can be obtained by substituting p(z) in (3.4).

It is interesting to note that the result in Theorem 3.2 does not depend on the value of n. Also, it can be used to solve the problem concerning the radii of $K_n(b)$ in $K_n(1)$.

THEOREM 3.3. Let $n \in N_0$. If $f \in K_n(b)$, then $f \in K_n(1)$ for $|z| < r'$ where

$$r' = \frac{1}{|b| + \sqrt{|b|^2 - 2\text{Re } b + 1}}.$$

This result is also sharp. An extremal function is given in (2.1).

PROOF. Let $f \in K_n(b)$. Then according to Theorem 3.2 there is some $g \in R_n$ such that for $|z| = r < 1$, $\frac{z(D^n f(z))'}{D^n g(z)}$ lies in the closed disk with center at $\frac{1+(2b-1)r^2}{1-r^2}$ and radius $\frac{2|b|r}{1-r^2}$. It can be shown that this disk lies in the right half plane if $r < r'$. This completes the proof of Theorem 3.3.

REMARK 3.1. Taking n = 0 in Theorem 3.3, one can see that, r' is the sharp radius of close-to-convexity for close-to-convex functions of complex order b.

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