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AN APPLICATION OF MILLER AND MOCANU'S RESULT

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ABSTRACT. The object of the present paper is to give an application of Miller and Mocanu's result for a certain integral operator.

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1. INTRODUCTION.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. For a function f(z) belonging to the class A, we define the integral operator J_c, by [1, p. 126, Equation (2.1)]

$$J_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \qquad (c > -1). \tag{1.2}$$

The operator J_c, when $c \in N = \{1,2,3,\dots\}$, was introduced by Bernardi [2]. In particular, the operator J_1 was studied earlier by Libera [3] and Livingston [4]. Recently, Owa and Srivastava [1] have proved a property of the operator J_c $(c > -1)$.

With the above operator J_c, we now introduce

DEFINITION. Let H_c be the set of complex-valued functions h(r,s,t);

$$h(r,s,t): C^3 \to C \quad (C \text{ is the complex plane})$$

such that

(i) $h(r,s,t)$ is continuous in $D \subset C^3$;

(ii) $(0,0,0) \in D$ and $|h(0,0,0)| < 1$;

(iii) $\left|h(e^{i\theta}, \frac{m+c}{c+1} e^{i\theta}, me^{i\theta} + L\right| > 1$ whenever

$(e^{i\theta}, \frac{m+c}{c+1} e^{i\theta}, me^{i\theta} + L) \in D$ with $\mathrm{Re}(e^{-i\theta}L) \geq \frac{m(m-1)}{c+1}$ for real θ and real $m \geq 1$.

2. AN APPLICATION OF MILLER AND MOCANU'S RESULT.

We begin with the statement of the following lemma due to Miller and Mocanu [5].

LEMMA. Let $w(z) \in A$ with $w(z) \not\equiv 0$ in U. If $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)|$, then

$$z_0 w'(z_0) = m w(z_0) \tag{2.1}$$

and

$$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq m, \tag{2.2}$$

where m is real and $m \geq 1$.

Applying the above lemma, we derive the following

THEOREM. Let $h(r,s,t) \in H_c$, and let $f(z) \in A$ satisfy

$$(J_c(f(z)), f(z), zf'(z)) \in D \subset C^3$$

and

$$|h(J_c(f(z)), f(z), zf'(z))| < 1 \tag{2.3}$$

for $c > -1$ and $z \in U$. Then we have

$$|J_c(f(z))| < 1 \qquad (z \in U) \tag{2.4}$$

where $J_c(f(z))$ is defined by (1.2).

PROOF. Letting $J_c(f(z)) = w(z)$ for $f(z) \in A$, we have $w(z) \in A$ and $w(z) \not\equiv 0$ $(z \in U)$. Since

$$z(J_c(f(z)))' = (c + 1)f(z) - cJ_c(f(z)),$$

we have

$$f(z) = \frac{c}{c+1} w(z) + \frac{1}{c+1} zw'(z) \tag{2.5}$$

and

$$zf'(z) = zw'(z) + \frac{1}{c+1} z^2 w''(z).$$

Suppose that there exists a point $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ such that

$$|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = 1.$$

Then, using the above lemma, we have

$$J_c(f(z_0)) = e^{i\theta_0},\ f(z_0) = \frac{m+c}{c+1} e^{i\theta_0},\ z_0 f'(z_0) = me^{i\theta_0} + L,$$

where $L = z_0^2 w''(z_0)/(c+1)$. Further, applying the lemma, we see that

$$\mathrm{Re}\{\frac{z_0 w''(z_0)}{w'(z_0)}\} = \mathrm{Re}\{\frac{z_0^2 w''(z_0)}{me^{i\theta_0}}\} \geq m - 1,$$

that is,

$$\mathrm{Re}(e^{-i\theta} L) \geq \frac{m(m-1)}{c+1}. \qquad (2.7)$$

Therefore, the condition $h(r,s,t) \in H_c$ implies that

$$|h(J_c(f(z_0)), f(z_0), z_0 f'(z_0))| = |h(e^{i\theta_0}, \frac{m+c}{c+1} e^{i\theta}, me^{i\theta_0} + L)| > 1. \qquad (2.8)$$

This contradicts our condition (2.3). Consequently, we conclude that $|w(z)| = |J_c(f(z))| < 1$ for all $z \in U$. Thus we complete the proof of the assertion of the theorem.

Taking $c = 0$ in the theorem, we have the following

COROLLARY. Let $h(r,s,t) \in H_0$, and let $f(z) \in A$ satisfy $(F(z), zF'(z), z(zF'(z))') \in D \subset C^3$ and

$$|h(F(z), zF'(z), z(zF'(z))')| < 1 \qquad (z \in U). \qquad (2.9)$$

Then $|F(z)| < 1$ $(z \in U)$, where

$$F(z) = J_0(f(z)) = \int_0^z \frac{f(t)}{t}\, dt.$$

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