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ABELIAN THEOREM FOR THE STIELTJES TRANSFORM OF DISTRIBUTIONS

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ABSTRACT. First, a definition is given of the Stieltjes transform of distributions which contains some well-known. Then, a structural theorem for distributions having S--asymptotic is proved. This makes possible to prove two theorems of the Abelian type valued for Stieltjes transforms of distributions given in different ways.

KEYS WORDS AND PHRASES. Stieltjes transform, S-asymptotic, Abelian type theorem, distribution, generalized function.

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1. INTRODUCTION.

It is possible to define the Stieltjes transform of distributions in various ways. One of them is the so-called direct approach in which we construct a basic space $A \supset \mathcal{D}$ of smooth functions to which the set $\{(s + t)^{-r-1}, \operatorname{Im} s \neq 0, r \geq 0\}$ belongs. Then, we define a transform for T, belonging to the dual space A', by the expresion $< T(t), (s + t)^{-r-1} >$. In this paper we shall deal only with such definitions. If the constructed space A' is such that $T \in A'$ has the support belonging to $[0,\infty)$, we have the classical Stieltjes transform, extended to generalized functions in one dimension.

We shall give a definition of the Stieltjes transform in meny-dimensional case not only to have a new generalisation of the Stieltjes transform, but to prove Abelian type theorems valued for the Stieltjes transforms given in different ways.

2. NOTATIONS AND DEFINITIONS.

N is the set of natural numbers, $N_0 = N \cup \{0\}$. If $a,b \in R^n$, $z \in C^n$, $z = (x_1 + iy_1,\dots ,x_n + iy_n)$ and $e = (1,\dots,1)$, then: $(a\cdot b) = \sum_{i=1}^{n} a_i b_i$; $z^a = \prod_{i=1}^{n}(x_i + iy_i)^{a_i}$; $|z^a| = \prod_{i=1}^{n} |x_i + iy_i|^{a_i}$; $z^e = \prod_{i=1}^{n} z_i$; $\|a\|^2 = (a\cdot a)$; $a \geq 0$ means $a_i \geq 0$ for all $i = 1,\dots,n$.

Γ will be a convex acute cone with vertex at zero; $B(0,r)$ the closed ball with the center at zero and with radius $r > 0$; $S(0,1)$ the unit sphere, all in R^n. $\overset{\circ}{\Gamma}$ is the interior of Γ.

DEFINITION 1. Let $h_1, h_2 \in \Gamma$. We say that $h_1 \geq h_2$ in Γ if $h_1 \in h_2 + \Gamma$. For a complex valued function $G(h)$, $h \in \Gamma$, $\lim G(h) = A \in C$, $h \in \Gamma$, $h \to \infty$ if for every $\varepsilon > 0$ there exists $h(\varepsilon) \in \Gamma$ such that $|G(h) - A| < \varepsilon$ when $h \geq h(\varepsilon)$ in Γ.

If $\overset{\circ}{\Gamma} \neq \emptyset$, we can prove the following Lemma:

LEMMA 1. Suppose $\overset{\circ}{\Gamma} \neq \emptyset$ and $B(a,r) \subset \overset{\circ}{\Gamma}$. We denote by $\Gamma_1 = \bigcup_{\lambda>0} \lambda B(a,r)$. If $\lim_{h\in\Gamma, h\to\infty} G(h) = A$, then $\lim_{h\in\Gamma_1, h\to\infty} G(x+h) = A$ for every $x \in R^n$.

PROOF. By Definition 1 for every $\varepsilon > 0$ there exists $h(\varepsilon)$ such that $|G(h) - A| < \varepsilon$, $h \in h(\varepsilon) + \Gamma$. Now $x - h(\varepsilon) + \beta_0 a \in B(\beta_0 a, \beta_0 r)$ if $\beta_0 \geq \|x - h(\varepsilon)\|/r$, β_0 depends on ε. Hence $x - h(\varepsilon) + \beta_0 a \in \Gamma_1 \subset \Gamma$. Since Γ is a convex cone, $x - h/(\varepsilon) + \beta_0 a + \Gamma_1 \subset \Gamma + \Gamma_1 \subset \Gamma$ and $x + \beta_0 a + \Gamma_1 \subset h(\varepsilon) + \Gamma$. It follows that $|G(x+h) - A| < \varepsilon$ for $h \in \beta_0 a + \Gamma_1$.-

REMARK. In Lemma 1 we can choose the ball $B(a,r)$ in such a way that the distance between $B(a,r)$ and any coordinate axis X_i, $i = 1,\dots,n$, in R^n is positive, $d(B(a,r),X_i) \geq \alpha > 0$. Then $|a_i| - r \geq \alpha > 0$, $i = 1,\dots,n$. We denote by $\Gamma_2 = \bigcup_{\lambda>0} \lambda B(a,r)$, for such constructed ball $B(a,r)$. If $u \in h_0 + \Gamma_2$ for a $h_0 \in \Gamma_2$, then $u = \lambda_0 a + \lambda_0 z + \lambda a + \lambda y$, where $z,y \in B(0,r)$; λ_0, $\lambda > 0$. It is easy to see that $|u_i| \geq (\lambda + \lambda_0)(|a_i| - r) \geq (\lambda + \lambda_0)\alpha$ and for any $M > 0$ we can find h_0 with the property $|u_i| \geq M$. Hence, if $h \to \infty$, $h \in \Gamma_2$, then $|h_i| \to \infty$, $i = 1,\dots,n$.

By P we denote the set of all the real and positive functions c. Notations for the spaces of distributions are as in the book of L. Schwartz [1].

DEFINITION 2. A distribution $T \in \mathcal{D}'$ has the S-asymptotic in Γ related to the function $c \in P$ and with the limit $U \in \mathcal{D}'$, if the following limit exists

$$\lim_{h\in\Gamma, h\to\infty} \langle T(x+h)/c(h), \phi(x) \rangle = \langle U, \phi \rangle, \ \phi \in \mathcal{D}. \tag{2.1}$$

Then we write $T(x+h) \overset{s}{\sim} c(h)\cdot U(x)$, $h \in \Gamma$ and we say that T has the S-asymptotic in $\mathcal{D}'$ (Pilipović and Stanković [2] and Stanković [3]).

We shall use the same definition for a $T \in B' \equiv \mathcal{D}'(L^\infty)$ and $\phi \in \mathcal{D}(L^1)$ stressing that T has the S-asymptotic in B'.

If the interior of the cone Γ is not empty, $\overset{\circ}{\Gamma} \neq \emptyset$, then we can give the analytical expression of the limit distribution U:

Suppose that $T \in \mathcal{D}'$ and has the S-asymptotic with the limit $U \neq 0$. Then there exists a $\phi_0 \in \mathcal{D}$ such that $\langle U, \phi_0 \rangle \neq 0$. For this ϕ_0 and $t \in R^n$, using Lemma 1, we have

$$\lim_{h\in\Gamma_2, h\to\infty} \frac{c(h+t)}{c(h)} \langle \frac{T(x+(h+t))}{c(h+t)}, \phi_0(x) \rangle = \lim_{h\in\Gamma_2, h\to\infty} \langle \frac{T((x+t)+h)}{c(h)}, \phi_0(x) \rangle. \tag{2.2}$$

From trhis relation it follows the existence of the following limit

$$\lim_{h\in\Gamma_2, h\to\infty} c(h+t)/c(h) = d(t), \ t \in R^n \tag{2.3}$$

and that U satisfies the equation

$$d(t) \langle U, \phi_0 \rangle = \langle U(x+t), \phi_0(x) \rangle, \ d(0) = 1, \ t \in R^n. \tag{2.4}$$

Since U, as a distribution, has all the derivatives, it follows from relation (2.4) that:

$$\begin{aligned}[d(t+\Delta t_1) - d(t)] \langle U, \phi_0 \rangle &= \langle U(x+t+\Delta t_1) - U(x+t), \phi_0(x) \rangle \\ &= [d(\Delta t_1) - d(0)] d(t) \langle U, \phi_0 \rangle, \ d(0) = 1.\end{aligned} \tag{2.5}$$

It is now easy to prove that (Pilipović and Stanković [2]):

$$d(t) = \exp(a\cdot t) \quad \text{and} \quad U(t) = C\exp(a\cdot t),\ t\in R^n. \tag{2.6}$$

3. STIELTJES TRANSFORM OF DISTRIBUTIONS

In the following we shall use the well-known function $\eta_\varepsilon \in C^\infty$, $\omega > 0$ (Vladimirov [4]):

$$\eta_\varepsilon(x) = \int_{B(0,2\omega)} q_\omega(x-t)dt,\ x\in R^n \tag{3.1}$$

where

$$q_\omega(x) = \begin{cases} D\omega^{-n}\exp\left(-\frac{\omega^2}{\omega^2-\|x\|^2}\right), & \|x\| < \omega \\ 0 & \|x\| \geq \omega \end{cases}\ ;\quad D\int_{R^n} q_1(t)dt = 1. \tag{3.2}$$

The function η_ω, has the properties: $0 \leq \eta_\omega(x) \leq 1$, $x\in R^n$; $\eta_\omega(x) = 1, x\in B(0,\omega)$; $\eta_\omega(x) = 0$, $\|x\| > 3\omega$; $|D^k\eta_\omega(x)| \leq C_k\omega^{-(k\cdot e)}$, $x\in R^n$. The constants C_k do not depend on ω.

DEFINITION 3. The Stieltjes transform of a distribution $T\in \mathcal{D}'$ (S_ρ-transform) is defined by the limit

$$\lim_{\omega\to\infty} \langle T(x), \eta_\omega(x)(s+x)^{-(\rho+e)}\rangle = S_\rho(T)(s),\ s\in (C\setminus R)^n, \tag{3.3}$$

if it exists for a $\rho\in R^n$. By e we denoted $e = (1,\dots,1)$.

REMARK. s can belong to a larger set, as well. This depends on the support of T. So $S_\rho(\delta)(s) = s^{-(\rho+e)}$, $s \neq 0$.

We shall give a relation between Definition 3 of the Stieltjes transform and definitions of some other authors frequently used in many papers. All of them are in one dimension. Let us start with the classical definition.

If T is defined by the function f, $\operatorname{supp} f \subset [0,\infty)$, Definition 3 gives the classical Stieltjes transform, if it exists. Let $s\in (C\setminus(-\infty,0])$, then

$$S_\rho(f)(s) = \lim_{\omega\to\infty}\int_0^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt = \lim_{\omega\to\infty}\int_0^{\omega} f(t)(s+t)^{-(\rho+1)}dt + \tag{3.4}$$
$$+ \lim_{\omega\to\infty}\int_\omega^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt.$$

We have only to prove that:

$$\lim_{\omega\to\infty}\int_\omega^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt = 0,\quad s\in (C\setminus(-\infty,0]) \tag{3.5}$$

when the classical Stieltjes transform exists.

Since for $\omega \leq x \leq 3\omega$

$$\eta_\omega(x) = \int_{-2\omega}^{2\omega} q_\omega(x-t)dt = \int_{(x-2\omega)/\omega}^{1} q_1(y)dy; \tag{3.6}$$

the function $\eta_\omega(x)$ is positive and monotone decreasing in this interval. By the mean value theorem, there exists a ξ, $0 < \xi < 2$ such that

$$\int_\omega^{3\omega} \eta_\omega(t)\mathrm{Re}[f(t)(s+t)^{-(\rho+1)}]dt = \eta_\omega(\omega)\int_\omega^{\omega+\xi\omega} \mathrm{Re}[f(t)(s+t)^{-(\rho+1)}]dt\,. \tag{3.7}$$

The last integral tends to zero when $\omega\to\infty$, because we suposed that the Stieltjes transform of function f exists. We have the same situation with the imaginary part of the integral from relation (3.5).

Lavoine and Misra [5-6] defined the Stieltjes transform of distributions belonging to a subset $\mathcal{J}'(r)$, Re r>-1 of $\mathcal{D}'$, which is used in many papers. A distribution T belongs to $\mathcal{J}'(r)$ if and only if there exist $m \in N$ such that $T = D^m G$, where G is a locally integrable function having a support in $[0,\infty)$ and $G(x) = o(x^{r+m-\alpha})$, $\alpha > 0$. The Stieltjes transform of $T \in \mathcal{J}'(r)$ is, by these two authors:

$$S_r(T) = (r+1)\dots(r+m)\int_0^\infty G(t)(s+t)^{-r-m-1}dt, \quad s \in C\backslash(-\infty,0]. \tag{3.8}$$

For a $T \in \mathcal{J}'(r)$, $r = \rho$, relation (3.3) gives

$$S_\rho(T)(s) = \lim_{\omega\to\infty} < T(x), \eta_\omega(x)(s+x)^{-(\rho+1)} > = \tag{3.9}$$
$$= (-1)^m \sum_{k=1}^m \binom{m}{k}(\rho+1)\dots(\rho+m) \lim_{\omega\to\infty}\int_0^\infty \frac{G(x)}{(s+x)^{\rho+k+1}} D^{m-k}\eta_\omega(x)dx\,.$$

We have two types of integrals

$$\int_0^\infty \frac{G(x)}{(s+x)^{\rho+m+1}}\eta_\omega(x)dx \quad \text{and} \quad \int_\omega^{3\omega}\frac{G(x)}{(s+x)^{\rho+k+1}} D^{m-k}\eta_\omega(x)dx,\ m-k \geq 1. \tag{3.10}$$

For the first one we proceed as in the case of the classical Stieltjes integral. The second one tends to zero when $\omega \to \infty$. We have only to start from

$$\int_\omega^{3\omega}\frac{G(x)}{(s+x)^{\rho+m+1}} (s+x)^{m-k} D^{m-k}\eta_\omega(x)dx \tag{3.11}$$

and to use the integration by parts.

In the next cases the authors followed the construction of the Stieltjes transform as we mentioned in the Introduction. The basic spaces A are topological vector spaces of complex valued smooth functions. The space A contains $\mathcal{D}$ and the topology of $\mathcal{D}$ is stronger then that induced on it by A. The retstriction T of an element $\bar{T} \in A'$ to $\mathcal{D}$ is in $\mathcal{D}'$. For a $\rho \geq 0$ and s belonging to a subset of C, $(s+t)^{-(\rho+e)}$ is in A. The Stieltjes transform of a $\bar{T} \in A'$ is defined by $< \bar{T}(t),(s+t)^{-(\rho+e)} >$.

In all the cases, we shall list, the family $\{(s+t)^{-(\rho+e)}\eta_\omega(t),\ \omega > 0\}$ converges in A to $(s+t)^{-(\rho+e)}$ when $\omega \to \infty$. Then for a $\bar{T} \in A'$ and $T \in \mathcal{D}'$, we have:

$$S_\rho(T)(s) = \lim_{\omega\to\infty} < \bar{T}(x),\eta_\omega(x)\ (s+x)^{-(\rho+e)} > = < \bar{T}(x),(s+x)^{-(\rho+e)} > \tag{3.12}$$

By Zemanian [7] it is $\rho = 0$ and the basic space A is $\mathcal{J}_{c,d} = \{\psi \in C^\infty_{(0,\infty)}, P_{c,d,k}(\psi) = \sup_{0<t<\infty}\chi_{c,d}(t)|(tD)^k\sqrt{t}\psi(t)| < \infty,\ k \in N_0\}$, where $\chi_{c,d}(t) = t^c$, $1 \leq t < \infty$; $\chi_{c,d}(t) = t^d$, $0 < t < 1$ and $c < 1/2$, $d > -1/2$. The topology of $\mathcal{J}_{c,d}$ is defined by the seminorms $P_{c,d,k}$, $k \in N_0$.

By Pandey [8] the basic space A is $S_\alpha = \{\phi \in C^\infty_{(0,\infty)}, \gamma_k(\phi) = \sup (1+x^\alpha)|(xD)^k\phi(x))| < \infty,\ k \in N_0\}$. The topology in S_α is defined by the seminorms γ_k, $k \in N_0$. The function $(s+t)^{-(\rho+1)} \in S_\alpha$ for $\alpha \leq \rho+1$, but the family $\{(s+t)^{-(\rho+1)}\eta_\omega(t),\ \omega > 0\}$ tends to $(s+t)^{-(\rho+1)}$ in S_α for $\alpha < \rho+1$ when $\omega \to \infty$.

Bremermann [9] introduced the basic space $0_\alpha = \{\zeta \in \mathcal{E}(R), \zeta^{(k)}(t) = 0(t^\alpha),\ k \in N_0\}$. The topology of 0_α is that induced by $\mathcal{E}$. He treated the case $\rho = 0$, Im $s \neq 0$ and $\alpha \geq -1$. For the Stieltjes transform of distributions in many dimensional case see also [10].

4. DISTRIBUTIONS HAVING THE S-ASYMPTOTIC

We shall prove a structural theorem for the distributions having the S-asymptotic in a cone Γ with the nonempty interior, $\overset{\circ}{\Gamma} \neq \emptyset$.

THEOREM 1. Suppose $T_0 \in \mathcal{B}'$ and $T_0(x+h) \overset{s}{\sim} 1\cdot U(x)$, $h \in \Gamma$ in $\mathcal{D}'$, then

a) $U = C$;

b) $T_0 = \sum_{i=0}^{2} \Delta^{ik} F_i$, where F_i are continuous functions belonging to L^∞;

c) For every $0 \leq i \leq 2$ functions $F_i(x+h)$ converge uniformly to a constant when x belongs to a compact set K and $h \in \Gamma$, $h \to \infty$;

d) T_0 has the S-asymptotic in $\mathcal{B}'$, as well, related to c = 1 and with the limit U = = C in the cone Γ.

PROOF. a) By relation (2.6) U has to be a constant because c = 1.

b) From the fact $T_0 \in \mathcal{B}'$ it follows that $(T_0*\hat{\zeta}) \in L^\infty$ for every $\zeta \in \mathcal{D}$ (Schwartz [1], II, p. 57) and the set of distributions $\mathcal{Q} = \{T_h \equiv T_0(x+h),\ h \in R^n\}$ is weakly bounded and bounded in $\mathcal{D}'$.

In addition to $\mathcal{Q}$, we shall construct an other bounded set of distributions. We denote by $S = \{\psi \in \mathcal{D}, \|\psi\|_{L^1} \leq 1\}$. We have seen that for a fixed $\zeta \in \mathcal{D}$, $(T_0*\hat{\zeta}) \in L^\infty$. Now, for every $\psi \in S$:

$$|< T_0*\hat{\psi},\zeta >| = |< T_0*\hat{\zeta},\psi >| = |\int_R (T_0*\hat{\zeta})(t)\psi(t)dt| \leq \tag{4.1}$$
$$\leq \|T_0*\hat{\zeta}\|_{L^\infty}\|\psi\|_{L^1}.$$

Hence the set of regular distributions, defined by the set of continuous functions $H = \{U_\psi \equiv T_0*\psi,\ \psi \in S\}$ is weakly bounded and bounded in $\mathcal{D}'$.

A set $W' \in \mathcal{D}'$ is bounded if and only if for every $\alpha \in \mathcal{D}$ the set of functions $\{T*\alpha, T \in W'\}$ is bounded on every compact set M belonging to R^n (Schwartz [1], II, p. 50). Hence $\{T*\alpha, T \in W'\}$ defines a bounded set of regular distributions. In such a way $\{T_h*\zeta, T_h \in \mathcal{Q}\}$ and $\{U_\psi*\zeta, U_\psi \in H\}$ give two bounded sets of regular distributions. Now, for these two sets we can repeat twice a part of the proof of Theorem XXII from Schvartz [1], II, p. 51.

We denote by Ω an open neighbourhood of zero in R^n which is relatively compact in R^n, $\mathrm{cl}\Omega = K$ a compact set. Then, by the mentioned part of the proof by Schwartz [1], there exist $m_1 \geq 0$ and $m_2 \geq 0$, such that the mappings $(\alpha,\beta) \to U_\psi*(\alpha*\beta)$ or $(\alpha,\beta) \to T_h*$ $*(\alpha*\beta)$are equicontinuous and map $\mathcal{D}_\Omega^{m_1} x \mathcal{D}_\Omega^{m_1}$ or $\mathcal{D}_\Omega^{m_2} x \mathcal{D}_\Omega^{m_2}$ into L_B^∞; B is the ball B(0,r) where r is a positive number. Hence, for every $x \in B$ and $h \in R^n$ the function $(T_h*\alpha*\beta)(x) = (T_0*\alpha*\beta)(x+h)$ is continous.

Let $Z(0,\rho)$ be a ball in L_B^∞, then there exists a neighbourhood $V_1(m_1,\varepsilon_1 K_1)$ in $\mathcal{D}_\Omega^{m_1}$, such that $U_\psi*(\alpha*\beta) \in Z(0,\rho)$ for $\alpha,\beta \in V_1(m_1,\varepsilon_1,K_1)$, $U_\psi \in H$ and a neighbourhood $V_2(m_2,\varepsilon_2,K_2) \subset \mathcal{D}_\Omega^{m_2}$, such that $T_h*(\alpha*\beta) \in Z(0,\rho)$ for $\alpha,\beta \in V_2(m_2,\varepsilon_2,K_2)$, $T_h \in \mathcal{Q}$. Let $K_0 = K_1 \cap K_2$, $\varepsilon_0 = \min(\varepsilon_1,\varepsilon_2)$ and $m = \max(m_1,m_2)$. We shall now use relation (VI, 6; 23) from Schwartz [1], II:

$$T_0 = \Delta^{2k}*(\gamma E*\gamma E*T) - 2\Delta^k*(\gamma E*\xi*T) + (\xi*\xi*T), \tag{4.2}$$

where E is a solution of the iterated Laplace equation; $\Delta^k E = \delta$; $\gamma,\xi \in \mathcal{D}_\Omega$, supp$\gamma$ and suppξ belonging to $K_0 = K_1 \cap K_2$. We have only to choose the number k large enough so that $\gamma E \in \mathcal{D}^m_\Omega$. Now, we can take: $F_2 = \gamma E*\gamma E*T_0$; $F_1 = \gamma E*\xi*T_0$ and $F_0 = \xi*\xi*T_0$. All of these functions are of the from: $F_i = T_0*\alpha_i*\beta_i$; $\alpha_i,\beta_i \in V(m,\varepsilon'_0,K_0)$, $\varepsilon'_0 > 0$.

We have to prove that F_i have the properties given in Theorem 1. For $\alpha_i,\beta_i \subset V(m,\varepsilon'_0,K_0)$ and $\psi \in S$

$$|< (T_0*\alpha_i*\beta_i),\psi >| = |[(T_0*\hat{\psi})*(\hat{\alpha}_i*\hat{\beta}_i)](0)| \le \rho(\varepsilon'_0/\varepsilon_0)^2 \equiv M \tag{4.3}$$

Now let $\mu \neq 0$ be any element from L^1, then $\mu/\|\mu\|_{L^1} \in S$ and $|< (T_0*(\alpha_i*\beta_i)),\mu >| \le \le M\|\mu\|_{L^1}$ which proves that $T_0*(\alpha_i*\beta_i)$ belong to L^∞. Since $F_i = T_0*(\alpha_i*\beta_i)$, $\alpha_i,\beta_i \in V(m,\varepsilon'_0,K_0)$, F_i are continuous and belong to L^∞.

c)We shall continue with investigations of the properties of F_i. By the properties of the convolution we have that $F_i(x+h) = F_i*\tau_{-h} = T_0*(\alpha_i*\beta_i)*\tau_{-h} = T_h*(\alpha_i*\beta_i)$ for α_i, $\beta_i \in \mathcal{D}^m_\Omega$, where τ_{-h} is the translation operator.

We have proved that the mappings $(\alpha,\beta) \to T_h*\alpha*\beta$, $T_h \in Q$, are equicontinuous and map $\mathcal{D}^m_\Omega x \mathcal{D}^m_\Omega$ into L^∞_B. $\mathcal{D}$ is a dense subset od $\mathcal{D}^m$, $m \ge 0$. We can construct a subset of $\mathcal{D}_K$, $\mathrm{cl}\Omega = K$, which is dense in $\mathcal{D}^m_\Omega$. Since $T_h*(\zeta*\psi) \to C*\zeta*\psi$ for $\zeta*\psi \in \mathcal{D}_\Omega x \mathcal{D}_\Omega$, then $T_h*\alpha_i*\beta_i$ converges to $C*\alpha_i*\beta_i$, as well (Schwartz [1], II, p.53).

d) there remains to prove the last part of Theorem 1. For $\mu \in \mathcal{D}(L^1)$ and $T \in B'$, noting that $F_i \in L^\infty$, we have:

$$|< T_0(x+h),\mu(x) >| \le \sum_{i=0}^{2} \int_{R^n} |F_i(x+h)\Delta^{ik}\mu(x)|dx \le \sum_{i=0}^{2} M_i \int_{R^n} |\Delta^{ik}\mu(x)|dx, \tag{4.4}$$

where $M_i = \sup|F_i(x)|$, $x \in R^n$. Hence the set $\{T_0(x+h),\ h \in R^n\}$, is weakly bounded in B'. Since $\mathcal{D}$ is dense in $\mathcal{D}(L^1)$, by the Banach-Steinhaus theorem the limit:

$$\lim_{h\in\Gamma, h\to\infty} < T_0(x+h),\mu(x) >,\ \mu \in \mathcal{D}(L^1), \tag{4.5}$$

exists, as well, and equals $< C,\mu >$.

5. ABELIAN THEOREMS FOR THE STIELTJES TRANSFORM

THEOREM 2. Suppose that $T \in \mathcal{D}'$, the cone Γ is with the nonempty interior and

i) $T(x+h) \overset{R}{\sim} c(h)\cdot U(x)$, $h \in \Gamma$ in $\mathcal{D}'$;

ii) For a $r > 0$ and $s_0 \in (C\backslash R)^n$ the distribution $T(x)/(s_0+x)^r$ belongs to B';

iii) For the same r and s_0, $c(h)/(s_0+x+h)^r$ converges to $C_1 \neq 0$ when $h \in \Gamma$, $h \to \infty$ and x belongs to any compact set in R^n.

Then T has the S_ρ-transform for all $\rho > r$, $S_\rho(T)(s) = < T(x)/(s_0+x)^r, (s_0+x)^r/(s+x)^{\rho+e} >$ and

$$\lim_{h\in\Gamma, h\to\infty} S_\rho(T)(s-x)/c(h) = 0,\ \rho > r. \tag{5.1}$$

PROOF. From supposition iii) it follows that the limit distribution U = C. Namely, for a $y \in R^n$, x belonging to a compact set in R^n and $s_0 \in (C\backslash R)^n$ from the relation:

$$\frac{c(h+y)}{c(h)} = \frac{c(h+y)}{(s_0+x+h+y)^r}\,\frac{(s_0+x+h)^r}{c(h)}\,\frac{(s_0+x+h+y)^r}{(s_0+x+h)^r} \tag{5.2}$$

and from the Lemma 1 with the Remark after Lemma 1, we have $\lim_{h\in\Gamma_2, h\to\infty} c(h+y)/c(h) = 1$. Now, relation (2.6) gives U = C.

By Theorem X from [1], I, p. 72 (see also Pilipović and Stanković [2]), for a $\zeta \in \mathcal{D}$ it follows the existance of the limit:

$$\lim_{h\in\Gamma, h\to\infty} < T(x+h)/(s_0+x+h)^r, \zeta(x) > = \lim_{h\in\Gamma, h\to\infty} < \frac{T(x+h)}{c(h)}, \frac{c(h)}{(s_0+x+h)^r}\zeta(x) > = \qquad (5.3)$$
$$= < CC_1, \zeta(x) >.$$

In such a way we proved that the distribution $T(x)/(s_0+x)^r$, $s_0 \in (C\backslash R)^n$, has the same properties as the distribution T_0 from Theorem 1 and we can use the assertion of this theorem.

We shall prove, now, the existence of the S_ρ-transform for $\rho > r$.

Since $|D^k\eta_\omega(x)| \le C_k\omega^{(-k\cdot e)}$, $k \ge 0$, where C_k do not depend on ω, the set of functions $\eta_\omega(x)(s+x)^{r-\rho-e}$ converges to $(s+x)^{r-\rho-e}$, $\rho > r$, in $\mathcal{D}(L^1)$, when $\omega \to \infty$. Moreover, $(s_0+x)^r/(s+x)^r$, for $s_0, s \in (C\backslash R)^n$ belongs to $\mathcal{D}(L^\infty)$; consequently

$$S_\rho(T)(s) = \lim_{\omega\to\infty} < T(x), \eta_\omega(x)(s+x)^{-\rho-e} > = \lim_{\omega\to\infty} < \frac{T(x)}{(s_0+x)^r}\frac{(s_0+x)^r}{(s+x)^r}, \eta_\omega(x)(s+x)^{r-\rho-e} > = \qquad (5.4)$$
$$= < T(x)/(s_0+x)^r, (s_0+x)^r(s+x)^{-\rho-e} >, \ \rho > r.$$

We have seen that the distribution $T(x)/(s_0+x)^r$ satisfies the conditions of Theorem 1, therefore

$$\frac{1}{c(h)} S_\rho(T)(s-h) = \frac{1}{c(h)} < \frac{T(x+h)}{(s_0+x+h)^r}, (s_0+x+h)^r(s+x)^{-\rho-e} > = \qquad (5.5)$$
$$= \sum_{i=0}^{2} \frac{(-1)^{i(k\cdot e)}}{c(h)} \int_{R^n} F_i(x+h)\Delta^{ik} \frac{(s_0+x+h)^r}{(s+x)^{\rho+e}}\, dx.$$

The expression $\Delta^{ik}[(s_0+x+h)^r(s+x)^{-\rho-e}]$ is given by the finite sum of elements which have the following form:

$$H_{j,p} = C_{j,p}(s_0+x+h)^{r-j+p}(s+x)^{-\rho-e-p}, \ j \ge p \ge 0. \qquad (5.6)$$

We shall analyse $H_{j,p}/c(h)$ when $x \in R^n$ and $h \in \Gamma$.

First we prove two inequalities:

$$|(s_0+x+h)^r| = \prod_{i=1}^n |s_{0,i}+x_i+h_i|^{r_i} \le \prod_{i=1}^n (|s_{0,i}+h_i| + 1)^{r_i}(|x_i|+1)^{r_i} \le \qquad (5.7)$$
$$\le \prod_{i=1}^n |s_{0,i}+h_i|^{r_i}\left(1 + \frac{1}{|\mathrm{Im}\, s_{0,i}|}\right)^{r_i} \prod_{i=1}^n (|x_i| + 1)^{r_i} \le$$
$$\le C_r|(s_0+h)^r/c(h)|\prod_{i=1}^n (|x_i|+1)^{r_i};$$

$$|(s_0+x+h)^{p-j}| = \prod_{i=1}^n |s_{0,i}+x_i+h_i|^{p_i-j_i} \le \prod_{i=1}^n |\mathrm{Im}\, s_{0,i}|^{p_i-j_i} = C'_{p,j}. \qquad (5.8)$$

Now

$$|H_{j,p}|/c(h) \le C_{j,p}C'_{j,p}C_r\left|\frac{(s_0+h)^r}{c(h)}\right| \prod_{i=1}^n \frac{(|x_i|+1)^{r_i}}{(s_i+x_i)^{\rho_i+p_i+1}}, \ p \ge 0. \qquad (5.9)$$

This inequality shows that $|H_{j,p}|/c(h)$ is bounded by a function which belongs to L^1, when $h \in \Gamma$. Since $F_i(x+h)$ are bounded, as well, when $x \in R^n$ and $h \in \Gamma$, we can use the Lebesgue theorem for the integral in (5.5) to obtain that $S_\rho(T)(s-h)/c(h)$ tends to zero when $h \in \Gamma$, $h \to \infty$.

The next theorem presents more precisely how $S_\rho(T)(s-h)$ tends to zero when $h \in \Gamma_2$, $h \to \infty$. Γ_2 is defined in the Remark after Lemma 1. We have seen that $|h_i| \to \infty$, $i = 1,\ldots$ $\ldots,n$, if $h \to \infty$, $h \in \Gamma_2$.

THEOREM 3. Let $T \in \mathcal{D}'$ and $s_0 \equiv (s_{0,1}\ldots s_{0,n}) \in (C\backslash R)^n$. We suppose:

i) $(s_{0,j}+x_j+h_j)T(x+h) \overset{s}{\sim} 1\cdot C$, $h \in \Gamma$ in $\mathcal{D}'$; $1 \leq j \leq n$

ii) $(s_{0,j}+x_j)T(x) \in \mathcal{B}'$.

Then, for $\rho > 0$ and Γ convex cone, $\overset{\circ}{\Gamma} \neq \emptyset$

$$\lim_{h\in\Gamma_2, h\to\infty} h_j S_\rho(T)(s-h) = 0 \tag{5.10}$$

PROOF. We can write:

$$S_\rho((s_{0,j}+x_j)T(x))(s-h) = \langle (s_{0,j}+x_j+h_j)T(x+h), (s+x)^{-\rho-e} \rangle = \tag{5.11}$$

$$= \langle T(x+h), (s+x)^{-\rho-e'} \rangle + (s_{0,j}-s_j+h_j) \langle T(x+h)(s+x)^{-\rho-e} \rangle; \; e' = (e'_1\ldots e'_n),$$

where $e'_i = 1$, $i \neq j$ and $e'_j = 0$. Hence

$$S_\rho(T)(s-h) = (s_{0,j}-s_j+h_j)^{-1}[S_\rho((s_{0,j}+x_j)T(x))(s-h) - \langle T(x+h), (s+x)^{-\rho-e'} \rangle] \tag{5.12}$$

First, we shall prove that $S_\rho((s_{0,j}+x_j)T(x))(s-h) \to 0$ when $h \in \Gamma_2$, $h \to \infty$. By Theorem 1 it follows:

$$\lim_{h\in\Gamma_2, h\to\infty} S_\rho((s_{0,j}+x_j)T(x))(s-h) = \langle C, (s+x)^{-\rho-e} \rangle = 0 \tag{5.13}$$

It remains to prove that $\langle T(x+h), (s+x)^{-\rho-e'} \rangle \to 0$ when $h \in \Gamma_2$, $h \to \infty$, as well. By Theorem 1 we know that our distribution T has the form

$$T(x) = (s_{0,j}+x_j)^{-1} \sum_{i=0}^{2} \Delta^{ik} F_i(x) \tag{5.14}$$

where $F_i(x)$ are bounded functions when $x \in R^n$. Using this form of T, we have:

$$\langle T(x+h), (s+x)^{-\rho-e'} \rangle = \sum_{i=0}^{2} (-1)^{i(k\cdot e)} \langle F_i(x+h), \Delta^{ik}(s_{0,j}+x_j+h_j)^{-1}(s+x)^{-\rho-e'} \rangle \tag{5.15}$$

The second part of relation (5.15) consists in fact of a sum of integrals

$$\int_{R^n} F_i(x+h)(s_{0,j}+x_j+h_j)^{-m}(s+x)^{-\rho-\alpha}dx \tag{5.16}$$

where $m \geq 1$; $\alpha_j \geq 0$ and $\alpha_i \geq 1$, $i \neq j$. All of these integrals tend to zero when $h \in \Gamma_2$, $h \to \infty$. We shall prove only the case: $m = 1$, $\alpha_j = 0$. In other cases it is trivial.

Let us consider the integral

$$\int_{-\infty}^{\infty} F_i(x+h)(s_{0,j}+x_j+h_j)^{-1}(s_j+x_j)^{-\rho_j}dx_j \tag{5.17}$$

for $x_k \in R$, $k \neq j$ and $h \in \Gamma_2$. The function in x_j $F_i(x+h)(s_{0,j}+x_j+h_j)^{-1}$ belongs to $\mathcal{D}'(L^v)$ for every $v > 1$. Let p be such that $p\rho_j > 1$ and $\frac{1}{p} - \frac{1}{v} - 1 \geq 0$. Then, for $x_i \in R$, $i \neq j$, integral (5.17) is bounded by a function in h belonging to $\mathcal{D}(L^u)$, $\frac{1}{u} = \frac{1}{v} + \frac{1}{p} - 1$ (Schwartz [1], II, p. 60).

Now, we have to prove that we can find p, satisfying our conditions, and $v > 1$, such that the number u remains in the interval $1 \leq u < \infty$. Then, we have only to use the property of a $\zeta \in \mathcal{D}(L^u)$, $1 \le u < \infty$; namely: $\zeta(x) \to 0$, $\|x\| \to \infty$ (Schwartz [1], II, p. 55) and it will follow that integral (5.16) tend to zero when $h \in \Gamma_2$, $h \to \infty$. Therefore we shall analyse two cases.

In case $\rho_j \ge 1$, we can take $\frac{1}{p} = 1 - \frac{1}{2v}$, hence $u = 2v$.

Case $0 < \rho_j < 1$. Since $\frac{1}{v}+\frac{1}{p} - 1 \ge 0$, we have $1 > \frac{1}{v} \ge 1-\frac{1}{p} > 1-\rho_j$, or $1 < v < (1-\rho_j)^{-1}$. For such an v the number $1/u$ is strictly positive, hence $1 \le u < \infty$.

The next example shows that Theorem 3 cannot be proved for $\rho = 0$

$$\int_0^\infty \frac{dx}{(x+a)(s+x)} = (a-s)^{-1}\ln\left(\frac{a}{s}\right), \quad a,s > 0. \tag{5.18}$$

There arises an another question: If we know that $(s_{0,j}+x_j)^r T(x) \in \mathcal{B}'$ for a $r > 1$, is it true that $S_\rho(T)(s-h) = o(h_j^{-r})$, $h \in \Gamma_2$, $h \to \infty$? The answer is in general negative. This shows the following integral:

$$\int_0^\infty e^{-t} \frac{dt}{(s+t)^\nu} = s^{-\nu/2}\cdot\exp(s/2)W_{-\nu/2,(1-\nu)/2}(s) \sim s^{-\nu}, \quad s > 0,\ s \to \infty, \tag{5.19}$$

where $W_{\nu,\mu}$ is the Whittaker function.

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