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RESEARCH NOTES

NORMS IN FINITE GALOIS EXTENSIONS OF THE RATIONALS

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ABSTRACT. We show that under certain conditions a rational number is a norm in a given finite Galois extension of the rationals if and only if this number is a local norm at a certain finite number of places in a certain finite abelian extension of the rationals.

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1. INTRODUCTION.

Let k be a number field. L. Stern [1] has observed that two finite Galois extensions L, M of k coincide if and only if the corresponding norm subgroups $N_{L/k}L^*$, $N_{M/k}M^*$ of k^* coincide. So it seems worthwhile to determine the norm subgroups of k^* which is certainly a difficult task. We consider the case k = = $\mathbb{Q}$.

2. LOCAL CONTROL OF GLOBAL NORMS.

Let $K/\mathbb{Q}$ be a finite Galois extension of degree d and class number h. For a given finite set of places S of $\mathbb{Q}$ and a given positive integer m we say that the triple $(\mathbb{Q},m,S)$ is in the special case if $m = 2^t \cdot n$, $t \geq 1$, n odd, if $2 \in S$ and if the cyclotomic extension $\mathbb{Q}_2(\zeta_{2^t})/\mathbb{Q}_2$ is not cyclic; ζ_{2^t} denotes a primitive root of unity of order 2^t.

THEOREM. Let $\alpha \in \mathbb{Q}^*$ and let S denote the finite set of places of $\mathbb{Q}$ for which α is not a local unit and which are ramified in K. Assume that the triple $(\mathbb{Q},d\cdot h,S)$ is not in the special case. Then there is a finite abelian extension $E_S/\mathbb{Q}$ such that α is a norm in $K/\mathbb{Q}$ if and only if α is a norm locally in $E_S/\mathbb{Q}$ at all places in S. The degree $(E_S:\mathbb{Q})$ is bounded, in terms of d and h.

PROOF. Let H_K denote the Hilbert class field of K and let $C_K/\mathbb{Q}$ denote the maximal central extension of $K/\mathbb{Q}$ contained in $H_K/\mathbb{Q}$. It follows from [2], p. 216, Cor. III. 2.13, that α is a norm in $K/\mathbb{Q}$ if and only if α is a local norm in $C_K/\mathbb{Q}$ at all places in S. It is well known that a norm subgroup of a local extension coincides with the norm subgroup of its maximal abelian subextension. Therefore we see, [3], p. 95, (6.9), that there is a finite abelian extension $E_S/\mathbb{Q}$ such that the local extensions of $E_S/\mathbb{Q}$ at all places in S coincide with the maximal abelian subextensions of the corresponding local extensions of $C_K/\mathbb{Q}$ and such that $E_S/\mathbb{Q}$ has the asserted properties.

3. A PROBLEM

In connection with the theorem above the following problem arises. For a given finite Galois extension $K/\mathbb{Q}$ of degree d and class number h and a given finite set of places S of $\mathbb{Q}$ such that the triple $(\mathbb{Q}, d\cdot h, S)$ is not in the special case, determine the minimal conductor of an abelian extension $E/\mathbb{Q}$ such that the local extensions of $E/\mathbb{Q}$ at all places in S coincide with the maximal abelian subextensions of the local central Hilbert class field extensions of $K/\mathbb{Q}$ at all places in S.

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