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UNIVERSALLY CATENARIAN DOMAINS OF D + M TYPE, II

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ABSTRACT. Let T be a domain of the form K + M, where K is a field and M is a maximal ideal of T. Let D be a subring of K such that R = D + M is universally catenarian. Then D is universally catenarian and K is algebraic over k, the quotient field of D. If [K:k] $< \infty$, then T is universally catenarian. Consequently, T is universally catenarian if R is either Noetherian or a going-down domain. A key tool establishes that universally going-between holds for any domain which is module-finite over a universally catenarian domain.

KEY WORDS AND PHRASES. Universally catenarian, going between, altitude formula.
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1. INTRODUCTION.

All rings considered below are (commutative integral) domains. As in Bouvier et al [1], a ring A is said to be catenarian if, for each pair P $\subset$ Q of prime ideals of A, all saturated chains of primes from P to Q have a common finite length; and A is said to be universally catenarian if the polynomial rings $A[X_1,...,X_n]$ are catenarian for each positive integer n. Let T be a domain of the form K + M, where K is a field and M is a maximal ideal of T. Let D, with quotient field k, be a subring of K; put R = D + M. In order to develop a then-new class of universally catenarian rings, Anderson et al [2] proved that if K is algebraic over k and both D and T are universally catenarian, then R is universally catenarian [2, Corollary 2.3]. In [2, Corollary 2.4], they established the converse for a special case of the classical D + M construction (in the sense of Gilmer [3, Appendix II]) in which T is assumed to be a valuation domain. This sequel to [2] is devoted to a deeper study of that converse.

Specifically, we ask whether the universal catenarity of R implies that K is algebraic over k and both D and T are universally catenarian. Affirmative answers are given in case R is Noetherian (in Corollary 2.4) and in case R is a going-down domain (in the sense of Dobbs [4], in Corollary 2.5). The latter result generalizes [2, Corollary 2.4] and, i.a., includes the case of (Krull) dimension 1. Our general results may be summarized as follows. If R is universally catenarian, then K is algebraic over k and D is universally catenarian (Proposition 2.1); and if, in addition, [K:k] $< \infty$, then T is universally catenarian (Corollary 2.3).

Corollary 2.3 depends on an idea that was not anticipated in [2], namely that universally going-between holds for any domain which is module-finite over a universally catenarian domain (Proposition 2.2). As defined in section 2, "universally going-between" is a universalization of the "going-between" property introduced by Ratliff [5]. The study of this property began with the following question of Krull [6]. If $A \subset B$ is an integral extension of domains such that A is integrally closed, must each saturated chain of prime ideals of B contract to a saturated chain in A? This question was answered in the negative by Kaplansky [7].

Throughout, T,K,M,D,k and R retain the meanings assigned above.

2. RESULTS.

It was established in [1, Theorem 5.1(a)] that the class of universally catenarian domains is the largest class of catenarian domains with the following four properties: it is stable under factor domains and localizations, and each of its members A satisfies $\dim_v(A) = \dim(A)$ and the altitude formula. The first three of these properties figure in the proof of Proposition 2.1; and the fourth is central to the proof of Proposition 2.2.

PROPOSITION 2.1. Let R be universally catenarian. Then:

(a) D is universally catenarian.

(b) K is algebraic over k.

(c) In order to determine whether T is universally catenarian, one may suppose that D = k and T is quasilocal. (This reduction replaces D with k and T with a localization, thus possible changing M; K and k remain unchanged).

PROOF. (a) Since R/M $\cong$ D, this assertion follows from the fact that the class of universally catenarian domains is stable under factor domains [1, Corollary 3.3]. (b) and (c): We first establish the reductions announced in the statement of (c). Let S = D\ {0}. Evidently, $S^{-1}R = k + M$, and so we may assume that D = k without loss of generality. It follows that the canonical map Spec(T) $\rightarrow$ Spec(R) is a bijection. Indeed, it is a homeomorphism (for the Zariski topology), and hence an order-isomorphism. (This may be seen by viewing R as the pullback $T x_K k$ and applying [8, Theorem 1.4] of Fontana [8]).

Let Q be a maximal ideal of T other than M. (If no such Q exists, this paragraph and the next one may be omitted.) Let P be the corresponding maximal ideal of R. We claim that $T_Q = R_P$. This follows directly from [8, Theorem 11.4 (c)]. (Another instructive way to see this is to use the above order-isomorphism to show that the saturation in T of the multiplicatively closed set R\P is T\Q, and then conclude via [8, Proposition 1.9] that $R_P \cong T_Q x_o 0 \cong T_Q$. A similar proof shows $R_P = T_{R\backslash P}$ by direct calculation, and then invokes Gilmer [9, Corollary 5.2] to conclude that

($T_{R\backslash P} = T_Q$). Since being a universally catenarian domain is a local property, it follows from the above claim that T(resp., R) is universally catenarian if and only if T_M (resp., R_M) is. We show next that replacing $R \subset T$ with $R_M \subset T_M$ has no effect on k and K.

Consider the ring A = k + MR_M. This is a CPI-extension in the sense of Boisen-Sheldon [10]; namely, we have canonical isomorphisms

$$A = R + MR_M \cong R_M x_{R/M}(R_M/MR_M) \cong R_M x_k k \cong R_M$$

(This may also be seen computationally, as in several proofs in Dobbs [11]). Thus $R_M = k + MR_M$; and, similarly, $T_M = K + MT_M$. To complete the reduction (and the proof of (c)), it suffices to show that $MR_M = MT_M$. This follows by another application of [8, Proposition 1.9]. Indeed, we see as above that $T\backslash M$ is the saturation in T of $R\backslash M$, and so the cited result yields that $R_M \cong T_M x_K k$. Hence

$$MR_M = ker(R_M \to k) = ker(T_M \to K) = MT_M$$

It remains to establish (b). We have seen that T (and hence R = k + M) may be taken quasilocal. Now R, being (universally) catenarian, is locally finite-dimensional, hence finite-dimensional. Since dim(R) = $\dim_v$R by [1, Corollary 3.3], R is a Jaffard domain, in the sense of Bouvier and Kabbaj [12] and Anderson et al [13]. An application of [13, Proposition 2.5] now yields that K is algebraic over k, completing the proof.

It is convenient next to introduce a concept that was promised in the introduction. First, recall from [6] that an integral extension $A \subset B$ of rings satisfies going-between in case each saturated chain of prime ideals of B contracts to a saturated chain in A; that is, in case $ht(Q_2/Q_1) = 1$ for prime ideals $Q_1 \subset Q_2$ of B implies $ht(P_2/P_1) = 1$ where $P_i = Q_i \cap A$. (Of course, $P_1 \neq P_2$, by virtue of INC: cf. Kaplansky [14, Theorem 44].) In the spirit of [1], we can now make the following definition. An integral extension $A \subset B$ satisfies universally going-between if $A[X_1,\dots,X_n] \subset B[X_1,\dots,X_n]$ satisfies going-between for each positive integer n.

The next result provides a key step. It is in the spirit of an observation of Kaplansky [7, penultimate paragraph].

PROPOSITION 2.2. Let $A \subset B$ be a module-finite (hence integral) extension of domains. If A is universally catenarian, then $A \subset B$ satisfies universally going-between.

PROOF. Since $A[X_1,\dots,X_n] \subset B[X_1,\dots,X_n]$ inherits the assumptions on $A \subset B$, it suffices to show that $A \subset B$ satisfies going-between. Consider primes $Q_1 \subset Q_2$ of B such that $ht(Q_2/Q_1) = 1$; put $P_i = Q_i \cap A$. Suppose there exists P ε Spec(A) contained strictly between P_1 and P_2. Pass to the extension $D = A/P_1 \subset E = T/Q_1$. Of course, D inherits universal catenarity from A [1, Corollary 3.3]; thus, D is locally finite-dimensional and satisfies the altitude formula [1, Corollary 4.8]. Moreover, E is of finite type over D; and $q = Q_2/Q_1$ meets D in $p = P_2/P_1$. It follows from the altitude formula (as defined in [1, page 219]), that

$$ht(q) = ht(p) + t.d._D(E) - t.d._{D/p}(E/q).$$

However, the transcendence degree terms are each 0, because of integrality; $ht(q) = 1$ by assumption; and $ht(p) \geq 2$ since $0 \neq P/P_1 \neq p$. This contradiction shows that no such P exists, completing the proof.

We may now state our main result.

COROLLARY 2.3. Suppose that [K:k] $< \infty$. Then R is universally catenarian if and only if both D and T are universally catenarian.

PROOF. The "if" assertion is a special case of [2, Corollary 2.3] since K is algebraic over k. For the converse, Proposition 2.1 (a) takes care of the assertion about D. Next, observe (directly or via [8]) that [K:k] $< \infty$ implies (in fact, is equivalent to the fact) that T is module-finite over R. Hence, $B = T[X_1, ..., X_n]$ is module finite over the universally catenarian domain $A = R[X_1, ..., X_n]$. By Proposition 2.2, $A \subset B$ satisfies going-between. To show that T is universally catenarian, it suffices to show that $ht(Q_2) = ht(Q_1) + 1$ whenever $Q_1 \subset Q_2$ are adjacent primes of B, that is, whenever $ht(Q_2/Q_1) = 1$. Put $P_i = Q_i \cap A$. By going-between, P_1 and P_2 are adjacent. Since A is catenarian, it follows that $ht(P_2) = ht(P_1) + 1$. It therefore suffices to show that $ht(Q_i) = ht(P_i)$. This, in turn, follows via the altitude formula, as in the proof of Proposition 2.2. This completes the proof.

We next consider two cases of special interest.

COROLLARY 2.4. Suppose that R is Noetherian. Then R is universally catenarian if and only if both D and T are universally catenarian.

PROOF. By Corollary 2.3, it suffices to show that [K:k] $< \infty$. Moreover, k + M is Noetherian, since it is a ring of fractions of R. Thus, without loss of generality, D = k. Now, if T were quasilocal, we would have Spec(T) = Spec(R) as sets, whence [K:k] $< \infty$ (by Anderson and Dobbs [15, Corollary 3.29], for instance). However, we saw in the fourth paragraph of the proof of Proposition 2.1 that replacing $R \subset T$ with $R_M \subset T_M$ has no effect on $k \subset K$; moreover, R_M (resp., T_M) is universally catenarian if R(resp., T) is. Thus, without loss of generality, T is quasilocal, and the proof is complete.

COROLLARY 2.5. Suppose that R is a going-down domain. Then R is universally catenarian (if and) only if K is algebraic over k and both D and T are universally catenarian.

PROOF. Since R is a going-down domain, so is its ring of fractions k + M. In view of Proposition 2.1, we may assume D = k and T is quasilocal; it remains only to show that T is universally catenarian. Now, since R is a universally catenarian going-down domain, its integral closure R' is a (finite-dimensional) Prüfer domain, by [1, Theorem 6.2, (1) ==> (4)]. However, R' is also the integral closure of T (except in the trivial case M = O) since the algebraicity of K over k assures that T is an integral overring of R. Moreover, T is a (finite-dimensional) going-down domain because it has the same prime spectrum as the going-down domain R[15, Proposition B.2]. (In view of integrality, this also follows via Dobbs [16, Lemma 2.3].) Thus, by [1, Theorem 6.2, (4) ==> (1)], T is universally catenarian, completing the proof.

REMARK 2.6. (a) By easily adapting the above proof, one may obtain two variants of Corollary 2.5. Without changing the conclusion, these alter the hypothesis about R to either "T is a going-down domain" or "k + M is going-down domain." (b) We next sketch a proof of Corollary 2.5 which depends on Corollary 2.3. As before, we may take D = k and T quasilocal. View T' as the directed union of the rings (F + M)', where F is a finite-dimensional field extension of k inside K. As above, each F + M is a going-down domain; moreover, F + M is universally catenarian by Corollary 2.3. Hence, each (F + M)' is a Prüfer domain, and so is their directed union T'. (This follows from a classic fact [9, Proposition 22.6], which also admits a direct limit generalization;

three proofs of this generalization are given in Dobbs, et al [17].) As above, it suffices to show T is a going-down domain; this, in turn, follows via [15] or [16] as above, or via [17, Corollary 2.7]. (c) Despite (b), it need not be the case that a direct limit of universally catenarian domains is universally catenarian. This has been noticed by Kabbaj [18, Chapitre IV, Exemple 3.5], as an application of [1, Theorem 2.4 and Corollary 2.2], the pertinent directed union being $\cup Q[X_1,...,X_n]$.

In view of Proposition 2.1 and Corollary 2.3, the question whether the universal catenarity of R implies that of T may be studied with the assumptions D = k, T quasilocal, and K infinite-dimensional (and algebraic) over k. Our last result develops a new role for "universally going-between" in this context. Notice that a new proof for Corollary 2.3 is available by placing an appeal to Proposition 2.7 after the fifth sentence of the earlier proof.

PROPOSITION 2.7. Suppose that D = k. Then the following conditions are equivalent:

(1) R is universally catenarian and $R \subset T$ satisfies universally going-between;

(2) T is universally catenarian and K is algebraic over k.

PROOF. (1) ==> (2): Assume (1). By Proposition 2.1 (b), it only remains to show that $B = T[X_1,...,X_n]$ is universally catenarian, where n is any positive integer. It suffices to prove $ht(Q_2) = ht(Q_1) + 1$ if $Q_1 \subset Q_2$ are primes of B such that $ht(Q_2/Q_1) = 1$. Put $P_i = Q_i \cap A$, where $A = R[X_1,...,X_n]$. Since $R \subset T$ satisfies universally going-between, $ht(P_2/P_1) = 1$. Thus, since A is catenarian, $ht(P_2) = ht(P_1) + 1$. Hence, it suffices to show that $ht(Q_i) = ht(P_i)$.

If P_i does not contain $M[X_1,...X_n]$, the desired equality follows from the isomorphism $B_{Q_i} \cong A_{P_i}$ (obtained by applying [8, Theorem 1.4 (c)] to the pullback $A = B x_E D$, where $D = k[X_1,...,X_n]$ and $E = K[X_1,...,X_n]$). So we may suppose $M[X_1,...,X_n] \subset P_i$. Notice, via [8, Theorem 1.4], that $M[X_1,...,X_n]$ has the same height (call it h) in A as in B. Moreover, $ht_E(Q_i/M[X_1,...,X_n]) = ht_D(P_i/M[X_1,...,X_n])$: call this H; indeed, this follows since $D \subset E$ satisfies incomparability and going-down (cf. [9, Corollary 12.11]). As $ht(Q_i) \geq H + h$ trivially and $ht(P_i) = H + h$ by the catenarity of A, we have $ht(Q_i) \geq ht(P_i)$. But the reverse inequality also holds since $A \subset B$ satisfies incomparability. Thus, (1) ==> (2).

(2) ==> (1): By [2, Corollary 2.3], R is universally catenarian. Let A,B,D and E be as in the proof that (1) ==> (2). Since T is integral over R, it is enough to prove that if $Q_1 \subset Q_2$ are adjacent primes of B, then $P_i = Q_i \cap A$ must satisfy $ht(P_2/P_1) = 1$. Suppose not. Then some $P \in$ Spec(A) lies properly between P_1 and P_2. By going-up, one finds primes $Q \subset Q_3$ in B which contain Q_1 and satisfy $Q \cap A$ = P and $Q_3 \cap A = P_2$. Since Q_2 and Q_3 each lie over P_2, it follows via incomparability and going-down that $ht(Q_2) = ht(P_2) = ht(Q_3)$. However, since B is catenarian, $ht(Q_i) = ht(Q_i/Q_1) + ht(Q_1)$. Thus, since the existence of Q assures that $ht(Q_3/Q_1) \geq 2$,

$$1 + ht(Q_1) = ht(Q_2) = ht(Q_3) \geq 2 + ht(Q_1).$$

All these heights are finite since T is locally finite-dimensional. So we have the desired contradiction, completing the proof.

We close with the following observation. In view of Propositions 2.1 (b) and 2.7 and Corollary 2.3, it would be of interest to find sufficient conditions for direct limit to preserve (universally) going-between.

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