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BEST APPROXIMATION IN ORLICZ SPACES

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ABSTRACT. Let X be a real Banach space and (Ω,μ) be a finite measure space and ϕ be a strictly increasing convex continuous function on $[0,\infty)$ with $\phi(0) = 0$. The space $L_\phi(\mu,X)$ is the set of all measurable functions f with values in X such that

$\int_\Omega \phi(||c^{-1}f\ (t)||)d\mu(t) < \infty$ for some $c > 0$. One of the main results of this paper is:

"For a closed subspace Y of X, $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$ if and only if $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$". As a result if Y is reflexive subspace of X, then $L_\phi(\phi,Y)$ is proximinal in $L_\phi(\mu,X)$. Other results on proximinality of subspaces of $L_\phi(\mu,X)$ are proved.

1. INTRODUCTION.

Let ϕ be a convex Orlicz function, i.e. ϕ is a continuous, strictly increasing convex function defined on $[0,\infty)$ with $\phi(0) = 0$ and let (Ω,μ) be a finite measure. For a real Banach space X, let

$$L_\phi(\mu,X) = \{\text{measurable function } f: \Omega \to X: \int_\Omega \phi(||c^{-1}f(t)||)d\mu(t) < \infty$$

for some $c > 0$. Define a norm on $L_\phi(\mu,X)$ by

$$||f||_\phi = \inf\ \{c > 0: \int_\Omega \phi(||c^{-1}f(t)||)d\mu(t) \le 1\}.$$

A subspace Y in a Banach space X is called proximinal if for each $x \in X$ there is at least one $y \in Y$ such that $||x - y|| = d(x,y)=\inf\ \{||x-h||,\ h \in Y\}$. The element y is called best approximant of x in Y. Set $P(x,Y) = \{y \in Y: d(x,y) = ||x-y||\}$.

In this paper we prove that for a closed subspace Y of a Banach space X, $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$ if and only if $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$. In [1] Deeb and Khalil, have shown the same result for the linear metric space $L_\phi(\mu,X)$ with ϕ modulus function and some Banach space X. As a consequence, if Y is a reflexive subspace of a Banach space X then $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$.

The proximinality of some closed subspaces in X are discussed. Throughout this paper Ω will be the unit interval [0,1], ϕ convex, strictly increasing with $\phi(0) = 0$, $\phi(1) = 1$ and X is a Banach space. See Deeb and Khalil [1,2,3], Light and Cheney [4], and Khalil [5] for more details about proximinality and related topics.

2. PROXIMINALITY IN $L_\phi(\mu,X)$.

LEMMA 2.1. If ϕ is convex, then $L_\phi(\mu,X) \subseteq L^1(\mu,X)$.

PROOF. Let $f \in L_\phi(\mu,X)$, then

$$\int_0^1 \phi(||c^{-1}f(t)||)d\mu(t) < M \text{ for some } c \text{ and some } M$$

By Jensen's Inequality, [6]

$$\phi\left(\int_0^1 ||c^{-1}f(t)||d\mu(t)\right) \leq \int_0^1 \phi(||c^{-1}f(t)||)d\mu(t) < M$$

or

$$\phi\left(\int_0^1 ||c^{-1}f(t)||\, d\mu(t)\right) < M.$$

Hence

$$\int_0^1 ||c^{-1}f(t)||\, d\mu(t) < \phi^{-1}(M).$$

Therefore

$$\int_0^1 ||f(t)||d\mu(t) < c\, \phi^{-1}(M) < \infty.$$

Hence $f \in L^1(\mu,X)$.

LEMMA 2.2. Let Y be a subspace of X, then for each $f \in L_\phi(\mu,X)$

$$\text{dist}(f, L_\phi(\mu,Y)) = \inf\{c > 0: \int_0^1 \phi|\, c^{-1}\text{dist}(f(t),Y)|d\mu(t) \leq 1\}.$$

PROOF. For any $g \in L_\phi(\mu,Y)$ we have,

$$||f-g||_\phi = \inf\{c > 0: \int_0^1 (||c^{-1}(f(t) - g(t))||)d\mu(t) \leq 1\}$$

$$\geq \inf\{c > 0: \int_0^1 \phi(|c^{-1}\text{dist}(f(t),Y)|)d\mu(t) \leq 1\}.$$

By taking the infimum over $g \in L_\phi(\mu,Y)$ we get

$$\text{dist}(f,L_\phi(\mu,Y)) \geq \inf\{c > 0: \int_0^1 \phi(|c^{-1}\text{dist}(f(t),Y)|)d\phi(t) \leq 1\}.$$

Conversely, let $\varepsilon > 0$ and let f' be a simple function in $L_\phi(\mu,X)$, such that

$||f-f'||_\phi < \varepsilon$. Write $f' = \sum_{i=1}^{n} \chi_i x_i$, where $x_i \in X$ and χ_i are the characteristic

functions on A_i which are disjoint measurable sets in $[0,1]$. It is clear that $f' \in L_\phi(\mu,X)$. Select $h_i \in Y$ such that

$$\phi(c^{-1}||x_i - h_i||) < \phi[c^{-1}\mathrm{dist}(x_i,Y) + \varepsilon], \quad \text{for some } c > 0.$$

Let $g = \sum_{i=1}^{n} \chi_i h_i$, then

$$\int_0^1 \phi(||c^{-1}g(t)||)d\mu(t) = \sum_{i=1}^{n} \phi(||c^{-1}h_i||)\mu(A_i) < \infty.$$

Hence $g \in L_\phi(\mu,Y)$, then

$$||f-g||_\phi = ||f-f' + f' - g||_\phi \leq \varepsilon + ||f' - g||_\phi$$

But $\mathrm{dist}(f, L_\phi(\mu,Y)) \leq ||f-g||_\phi$

$$\leq \varepsilon + \inf\ \{c > 0: \int_0^1 \phi(c^{-1}||f'(t)-g(t)||)d\mu(t) \leq 1\}$$

$$= \varepsilon + \inf\ \{c > 0: \sum_{i=1}^{n} \int_0^1 \phi(c^{-1}||x_i-h_i||)d\mu(t) \leq 1\}$$

$$= \phi + \inf\{c > 0: \sum_{i=1}^{n} \phi(c^{-1}||x_i-h_i||)\ \mu(A_i) \leq 1\}$$

$$\leq \varepsilon + \inf\ \{c > 0: \sum_{i=1}^{n} \phi[c^{-1}\mathrm{dist}(x_i,Y) + \varepsilon]\ \mu(A_i) \leq 1\}$$

$$= \varepsilon + \inf\{c > 0: \int_0^1 \phi[c^{-1}\mathrm{dist}(f'(t),Y) + \varepsilon]d\mu(t) \leq 1\}$$

$$\leq \varepsilon + \inf\{c > 0: \int_0^1 \phi(c^{-1}\mathrm{dist}(f(t),Y) + ||f(t) - f'(t)||+\varepsilon)d\mu(t)\leq 1\}.$$

$$\leq \varepsilon + \inf[c > 0: \int_0^1 \phi(c^{-1}\mathrm{dist}(f(t),Y) + 2\varepsilon)d\mu(t) \leq 1\}.$$

Since ε is arbitrary, we have

$$\mathrm{dist}(f,L_\phi(\mu,Y)) \leq \inf\{c > 0: \int_0^1 \phi(c^{-1}\mathrm{dist}(f(t),Y))d\mu(t) \leq 1\}.$$

REMARK 2.1. For $f \in L_\phi(\mu,X)$,

$$||f||_\phi = \inf\{c > 0: \int_0^1 \phi(\frac{||f(t)||}{c})d\mu(t) \leq 1\} = C_o$$

such that $\int_0^1 \phi(\frac{||f(t)||}{C_o})d\mu(t) = 1.$

COROLLARY 2.1. Let Y be a closed subspace of X. To an element f of $L_\phi(\mu,X)$, g of $L_\phi(\mu,Y)$ is a best approximant of f in $L_\phi(\mu,Y)$ if and only if g(t) is a best approximant of f(t) in Y.

PROOF. Let g(t) be a best appoximant of f(t) in Y. This means that

$$||f(t) - g(t)|| \leq ||f(t) - y|| \text{ for all } t \text{ and for all } y \in Y.$$

It follows that for any $h \in L_\phi(\mu,Y)$

$$||f(t) - g(t)|| \leq ||f(t) - h(t)|| \text{ for all } t.$$

Since ϕ is increasing, we have

$$\phi(c^{-1}||f(t) - g(t)||) \leq \phi(c^{-1}||f(t) - h(t)||) \text{ for any } c > 0.$$

Then

$$\int_0^1 \phi(c^{-1}||f(t) - g(t)||)d\mu(t) \leq \int_0^1 \phi(c^{-1}||f(t) - h(t)||)d\mu(t).$$

Therefore

$$\inf\{c > 0: \int_0^1 \phi(c^{-1}||f(t) - g(t)||)\, d\mu(t) \leq 1\} \leq \inf\{c > 0: \int_0^1 \phi(c^{-1}||f(t) - h(t)||\, d\mu(t) \leq 1\}$$

or

$$||f - g||_\phi \leq ||f - h||_\phi \quad \text{for a ll } h \in L_\phi(\mu,Y).$$

Conversely, let g be a best approximant of f in $L_\phi(\mu,Y)$, then $\text{dist}(f,L_\phi(\mu,Y)) = ||f-g||_\phi$. By Lemma 2.2 and the previous remark, we have

$$||f-g||_\phi = \inf\{c > 0: \int_0^1 \phi(c^{-1}\text{dist}(f(t),Y))d\mu(t) \leq 1\} = c_o \text{ such that}$$

$$\int_0^1 \phi(\frac{||f(t)-g(t)||}{c_o})\, d\mu(t) = \int_0^1 \phi(\frac{(\text{dist}(f(t),Y)}{c_o})\, d\mu(t) = 1.$$

Hence $\int_0^1 [\phi(c_o^{-1}||f(t)-g(t)||) - \phi(c_o^{-1}\text{dist}(f(t),Y))]d\mu(t) = 0$

since ϕ is strictly increasing and $\phi(c_o^{-1}||f(t)-g(t)||) \geq \phi(c_o^{-1}\text{dist}(f(t),Y))$ then $||f(t)-g(t)|| = \text{dist}(f(t),Y)$.

Now we prove the main theorem of this paper.

THEOREM 2.1. Let Y be a closed subspace of X, then the following are equivalent:

(i) $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$

(ii) $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$.

PROOF. (ii) → (i). Let $f \in L_\phi(\mu,X)$, then by Lemma 2.1 $f \in L^1(\mu,X)$. By the assumption, there exists $g \in L^1(\mu,Y)$ such that

$$||f-g||_1 \leq ||f-h||_1 \text{ for every } h \in L^1(\mu,Y).$$

By lemma 2.10 [3], we have

$$||f(t) - g(t)|| \leq ||f(t) - y|| \text{ for all } t \text{ and for all } y \in Y.$$

Hence by Corollary 2.1 it follows that g is a best approximant of f in $L_\phi(\mu,Y)$.

Conversely: (i) → (ii). Define a map

$$J: L^1(\mu,X) \to L_\phi(\mu,X) \text{ by } J(f) = \hat{f} \text{ where } \hat{f}(t) = \frac{\phi^{-1}(||f(t)||)}{||f(t)||} f(t)$$

if $f(t) \neq 0$, and zero otherwise. Then for c = 1

$$\int_0^1 \phi(||c^{-1}\hat{f}(t)||)d\mu(t) = \int_0^1 \phi\left(\left|\left|\frac{(\phi^{-1}(||f(t)||)}{||f(t)||} f(t)\right|\right|\right)d\mu(t)$$
$$= \int_0^1 \phi(||f(t)||)d\mu(t) < \infty$$

for all $f \in L^1(\mu,X)$. Hence $J(f) \in L_\phi(\mu,X)$. Since ϕ is strictly increasing, it follows that J is (1-1). To show that J is onto, let $g \in L_\phi(\mu,X)$, then take

$$f(t) = \frac{\phi(||g(t)||)}{||g(t)||} g(t)$$

if $g(t) \neq 0$ and zero otherwise. Clearly $f \in L^1(\mu,X)$ and

$$J(f) = \frac{\phi^{-1}(||f(t)||)}{||f(t)||} f(t)$$
$$= \frac{\phi^{-1}(\phi(||g(t)||)}{\phi(||g(t)||)} \quad \frac{\phi(||g(t)||)}{||g(t)||} g(t)$$
$$= g(t).$$

Thus J is onto. Now let $f \in L^1(\mu,X)$, then $\hat{f} \in L_\phi(\mu,X)$. By assumption there exists $\hat{g} \in L_\phi(\mu,Y)$ such that

$$||\hat{f} - \hat{g}||_\phi \leq ||\hat{f} - \hat{h}||_\phi \text{ for all } \hat{h} \in L_\phi(\mu,Y),$$

then by Corollary 2.1 we have

$$||\hat{f}(t) - \hat{g}(t)|| \leq ||\hat{f}(t) - y|| \text{ for all } y \in Y \text{ or}$$

$$||f(t) - \frac{||f(t)||\phi^{-1}(||g(t)||)}{||g(t)||\phi^{-1}(||f(t)||)} g(t)|| \leq ||f(t) - \frac{y||f(t)||}{\phi^{-1}(||f(t)||)}|| \text{ for}$$

all $y \in Y$. Put $w(t) = \frac{||f(t)||\ \phi^{-1}(||g(t)||)}{||g(t)||\ \phi^{-1}(||f(t)||)}\ g(t)$.

Using the facts that $||g(t)|| \leq 2||f(t)||$ since $0 \in Y$ and $\phi^{-1}(2||f(t)||) \leq 2(\phi^{-1}(||f(t)||))$ we can show that $w \in L^1(\mu,Y)$ as follows

$$\begin{aligned} ||w(t)|| &= \frac{||f(t)||\ \phi^{-1}(||g(t)||)}{\phi^{-1}(||f(t)||)} \\ &\leq \frac{||f(t)||\ \phi^{-1}(2||f(t)||)}{\phi^{-1}(||f(t)||)} \\ &\leq \frac{2||f(t)||\ \phi^{-1}(||f(t)||)}{\phi^{-1}(||f(t)||)} \\ &= 2||f(t)||. \end{aligned}$$

Now take any $h \in L^1(\mu,Y)$ then

$$\frac{\phi^{-1}(||f(t)||)}{||f(t)||} h(t) \in Y \quad \text{for all } t.$$

Hence

$$||f(t) - w(t)|| \leq ||f(t) - \frac{||f(t)||}{\phi^{-1}(||f(t)||)} \frac{\phi^{-1}(||f(t)||)}{||f(t)||} h(t)\ ||$$

$= ||f(t) - h(t)||$ for all t and for all $h \in L^1(\mu,Y)$, so $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$.

As a corollary.

COROLLARY 2.2. If Y is a reflexive subspace of X, then $L_\phi(\mu,Y)$ is priximinal in $L_\phi(\mu,X)$.

PROOF. It follows from the main theorem and Theorem 1.2 in Kahalil [5].

THEOREM 2.2. Let Y be a proximinal subspace of X. Then for every simple function $f \in L_\phi(\mu,X)$, $P(f, L_\phi(\mu,Y))$ is not empty.

PROOF. Let $f = \sum_{i=1}^{n} \chi_{A_i} x_i$ be a simple function in $L_\phi(\mu,X)$, where A_i are disjoint measurable sets in [0,1]. Set $g = \sum_{i=1}^{n} \chi_i y_i$, where $y_i \in P(x_i, Y)$. Let h be any element in $L_\phi(\mu,Y)$, then

$$\begin{aligned} \|f-h\|_\phi &= \inf\{c > 0: \int_0^1 \phi(\|c^{-1}(f(t)-h(t))\|)d\mu(t) \le 1\} \\ &= \inf\{c > 0: \sum_{i=1}^n \int_{A_i} \phi(\|c^{-1}(f(t) - h(t))\|)d\mu(t) \le 1\} \\ &= \inf\{c > 0: \sum_{i=1}^n \int_{A_i} \phi(\|c^{-1}(x_i - h(t))\|)d\mu(t) \le 1\} \\ &\ge \inf \{c > 0: \sum_{i=1}^n \int_{A_i} \phi(\|c^{-1}(x_i - y_i)\|)d\mu(t) \le 1\} \\ &= \inf\{c > 0: \int_0^1 \phi(\|c^{-1}(f(t) - g(t))\|)d\mu(t) \le 1\} \\ &= \|f - g\|_\phi. \end{aligned}$$

Hence $g \in P(f, L_\phi(\mu,Y))$.

THEOREM 2.3. Let Y be a closed subspace of X. If $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$, then Y is proximinal in X.

PROOF. From Theorem 2.1, $L_\phi(\mu,Y)$ proximal in $L_\phi(\mu,X)$ implies that $L^1(\mu,Y)$ is proximinal in $L^1(\mu,X)$. By Theorem 1.1 [2] this also implies that $L^\infty(\mu,Y)$ is proximinal in $L^\infty(\mu,X)$. For $x \in X$, define $f_x: \Omega \to X$ by $f_x(t) = x$ for all $t \in \Omega$. It is clear that $f_x \in L^\infty(\mu,X)$ for every $x \in X$, so there exists $h \in L^\infty(\mu,Y)$ such that

$$\|f_x - h\|_\infty \le \|f_x - w\| \text{ for every } w \in L^\infty(\mu,Y).$$

In particular take $w = f_y$, so

$$\begin{aligned} \|f_x - h\|_\infty &\le \|f_x - f_y\|_\infty \text{ for every } y \in Y \\ &= \|x - y\| \quad \text{for every } y \in Y. \end{aligned}$$

But

$$\begin{aligned} \|x - h(t)\| &= \|f_x(t) - h(t)\| \\ &\le \|f_x - h\| \\ &\le \|f_x - f_y\| \\ &= \|x - y\| \quad \text{for every } y \in Y. \end{aligned}$$

Hence every $t \in [0,1]$ gives a best approximant of x in Y. Therefore Y is proximinal in X.

The next theorem needs the following definitions:

DEFINITION 2.1. The subspace Y is called ϕ-summand of x if there is a bounded projection Q: X $\to$ Y such that

$\phi(||x||) = \phi(||(Q(x)||)+\phi(||(I-Q)(x)||)$ for all $x \in X$. Where I is the identity map on X.

DEFINITION 2.2. The subspace Y is called 1-complemented in X if there is a closed subspace Z in X that $X = Y \dotplus Z$ and the projection P: X $\to$ Z is a contractive projection.

THEOREM 2.4. If Y is 1-complemented in X, the $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$.

PROOF. Let $X = Y \dotplus Z$, P: X $\to$ Z be a contractive projection from X onto Z. Hence $x = (L-P)x + p(x)$, $||P(x)|| \leq ||x||$. For $f \in L_\phi(\mu,X)$, set $f_1 = (I-P)of$, f_2: pof. Let $\check{p}$: $L_\phi(\mu,X) \to L_\phi(\mu,Z)$ and

$$p(f) = pof = f_2 \text{ for all } f \varepsilon L_\phi(\mu, X).$$

Then $\check{p}$ is a contractive projection onto $L_\phi(\mu,Z)$ and $L_\phi(\mu,X) = L_\phi(\mu,Y) \dotplus L_\phi(\mu,Z)$. Hence $L_\phi(\mu,Y)$ is 1-complemented in $L_\phi(\mu,X)$. By Lemma 1.6 [2] $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$.

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