Internat. J. Math. & Math. Sci.
VOL. 14 NO. 2 (1991) 407-409

ON THE ARENS PRODUCTS AND REFLEXIVE BANACH ALGEBRAS

PAK-KEN WONG

Department of Mathematics
Seton Hall University
South Orange, New Jersey 07079

(Received March 29, 1990)

ABSTRACT. We give a characterization of reflexive Banach algebras involving the Arens product.

KEY WORDS AND PHRASES. Arens products, Arena regularity, conjugate space, weakly completely continuous (w.c.c.) algebra.
1980 AMS SUBJECT CLASSIFICATION CODES. Primary 46H10; Secondary 46H99.

1. INTRODUCTION.

Let A be a semisimple Banach algebra and A^{**} the second conjugate space of A with the Arena product o. If (A^{**},o) is semisimple and it has a dense socle, then we show that the following statements are equivalent: (1) A is reflexive. (2) A^{**} is w.c.c. (3) A is w.c.c. (4) A and A^{**} have the same socle. This is a generalization of a result by Duncan and Hosseinuim [1, p.319, Theorem 6(ii)]. We also show that if $A^{**},o)$ is semisimple and A is l.w.c.c., then A is Arens regular.

2. NOTATION AND PRELIMINARIES.

Definitions not explicitly given are taken from Rickart's book [2].

Let A be a Banach algebra. Then A^* and A^{**} will denote the first and second conjugate spaces of A, and π the canonical map of A into A^{**}. The two Arens products on A^{**} are defined in stages according to the following rules (see [3] and [4]). Let $x,y \in A, f \in A^*$, and $F,G \in A^{**}$.
Define fox by (fox)(y) = f(xy). Then fox $\in A^{**}$.
Define Gof by (Gof)(x) = G(fox). Then Gof $\in A^*$.
Define FoG by (FoG)(f) = F(Gof). Then FoG $\in A^{**}$.
Define xo′f by (xo′f)(y) = f(yx). Then xo′f $\in A^*$.
Define fo′F by (fo′F)(x) = F(xo′f). Then fo′F $\in A^*$.
Define Fo′G by (Fo′G)(f) = G(fo′F). Then Fo′G $\in A^{**}$.

A^{**} is a Banach algebra under the products FoG and Fo′G and π is an algebra isomorphism of A into (A^{**},o) and (A^{**},o'). In general, o and o′ are distinct on A^{**}. If they agree on A^{**}, then A is called Arens regular.

LEMMA 2.1. Let A be a Banach algebra. Then, for all $x \in A, f \in A^*$, and $F, G \in A^{**}$, we have

(1) $\pi(x)oF = \pi(x)o'F$ and $Fo\pi(x) = Fo'\pi(x)$.

(2) If $\{F_t\} \subset A^{**}$ and $F_t \to F$ weakly in A^{**}, then $F_toG \to FoG$ and $Go'F_t \to Go'F$ weakly.

PROOF. See [3, p.842 and p. 843].

Let A be a Banach algebra. An element a $\in$ A is called left weakly completely continuous (l.w.c.c.) if the mapping L_a defined by $L_a(x) = ax(X \in A)$ is weakly completely continuous. We say that A is l.w.c.c. if each $a \in A$ is l.w.c.c. If A is both l.w.c.c. and r.w.c.c., then A is called w.c.c.

In this paper, all algebras and linear spaces under consideration are over the field C of complex numbers.

3. THE MAIN RESULT.

LEMMA 3.1. Let A be a Banach algebra. Then A is l.w.c.c. (resp. r.w.c.c.) if and only $\pi(A)$ is a right (resp. left) ideal of (A^{**}, o).

PROOF. This result is well known (see [1, p.318, Lemma 3] or [2, p.443, Lemma]).

In the rest of this section, we shall assume that A and (A^{**}, o) are semisimple Banach algebras.

THEOREM 3.3. Suppose that (A^{**}, o) has a dense socle. Then the following statements are equivalent:

(1) A is reflexive.

(2) $A^{**}$is w.c.c.

(3) A is w.c.c.

(4) $\pi(A)$ and A^{**} have the same socle.

PROOF.

(1) $\Rightarrow$ (2). Assume that A is reflexive. Then $A^{(4)} = A^{**} = A$; in particular, $\pi(A)^{**}$ is a two-sided ideal of $A^{(4)}$. Hence by Lemma 3.1, A^{**} is w.c.c.

(2) $\Rightarrow$ (3). Assume that A^{**} is w.c.c. Then $\pi(A^{**})$ is a two-sided ideal of $A^{(4)}$. As observed in [1, p.319, Theorem 6(ii)], $\pi(A)$ is a two-sided ideal of A^{**}. Hence A is w.c.c.

(3) $\Rightarrow$ (4). Assume that A is w.c.c. Then $\pi(A)$ is a two-sided ideal of A^{**}. Let E be a minimal idempotent of A^{**}. Since $EoA^{**}oE = Eo\pi(A)oE = CE$, it follows that $E \in \pi(A)$. Consequently, E is a minimal idempotent of $\pi(A)$. If e is a minimal idempotent of A, then $\pi(e)oA^{**} \subset \pi(A)$ and so $\pi(e)oA^{**} = \pi(eA)$. Hence, $\pi(e)oA^{**}o\pi(e) = \pi(eAe)' = C\pi(e)$ and so $\pi(e)$ is a minimal idempotent of A^{**}. Therefore, $\pi(A)$ and A^{**} have the same socle.

(4) $\Rightarrow$ (1). Assume that $\pi(A)$ and A^{**} have the same socle. Since $\pi(S)$ is dense in A^{**}, it follows that $\pi(A)$ is dense in A^{**} and so $\pi*(A) = A^{**}$. Therefore A is reflexive. This completes the proof of the theorem.

REMARK. It is well known that a semisimple annihilator Banach algebra A is w.c.c. (see [5]). Also, A has a dense socle. Therefore, Theorem 3.2 generalizes [1, p.319, Theorem 6(ii)].

THEOREM 3.3. If A is l.w.c.c., then A is Arens regular.

PROOF. Since A is l.w.c.c., by Lemma 3.1, $\pi(A)$ is a right ideal of A^{**}. Let F and G $\in A^{**}$ and $x \in A$. Then

$$\pi(x)o(FoG - Fo'G) = \pi(x)oFoG - \pi(x)o(Fo'G)$$

$$= \pi(x)oFoG - \pi(x)o'(fo'G) \quad \text{By Lemma 2.1(1))}$$

$$= \pi(x)oFoG - (\pi(x)oF)o'G$$

$$= \pi(x)oFoG - (\pi(x)oF)oG \quad (\text{ because } \pi(x)oF \in \pi(A))$$

$$= 0$$

Hence $\pi(A)o(FoG - Fo'G) = (0)$. Therefore, by Lemma 2.1 (2), we have $A^{**}o(FoG - Fo'G) = (0)$. Since (A^{**}, o) is semisimple, it follows that $FoG - Fo'G = 0$ and so $Fog = Fo'G$. Therefore, A is Arens regular. This completes the proof.

REFERENCES

1. DUNCAN, J. and HOSSEINIUM, S.A.R. The Second Dual of a Banach Algebra, Proc. Royal Soc. of Edinburgh, 84A (1979), 309-325.

2. WONG, P.K. Arens Produce and the Algebra of Double Multipliers, Proc. Amer. Math. Soc. 94 (1985), 441-444.

3. ARENS, R.F. The Adjoint of a Bilinear Operation, Proc. Amer. Math. Soc. 2 (1951), 839-848.

4. BONSALL, F.F. and DUNCAN, J. Complete Normed Algebras, Springer, Berlin, 1973.

5. WONG, P.K. On the Arens Product and Certain Banach Algebras, Trans. Amer. Math. Soc. 180 (1973), 437-438.

6. RICKART, C.E. General Theory of Banach Algebras, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960.

7. WONG, P.K. Arens Product and the Algebra of Double Multipliers II, Proc. Amer. Soc. 100 (1987), 447-453.